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THE OBESITY PREVENTION STRATEGIES IN POPULATION DYNAMICS

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Abstract. Obesity is one of the serious health problems in USA and among many countries all over the world. It is the root cause of many health issues which also include the risk of fall and the resultant injury. To prevent obesity, there are many weight loss and strength training programs that can potentially minimize several health problems. This paper focuses the dynamical behavior and optimal control of obesity. In order to do this, first the reproductive number of obesity transmission model is investigated by using next generation matrix method. To find local stability of the model, we use stability analysis theory, while for global stability we use Lyapunov function. To minimize the attitude towards obesity, three possible control variables in the form of education and treatment campaigns are used for the control of obesity in the community. We also show the existence of an optimal control for the control problem and than derive the optimality system by using the Pontryagin maximum principle. Finally, we present and discuss results of numerical simulation.

Key words: Obesity; Mathematical modeling; Population dynamics; Optimal control.

1. Introduction

Obesity is one of the serious health problems in USA and among many countries all over the world [1]. More than 1 billion people around the globe are considered overweight and of those, 300 million are considered obese [2] and the high prevalence of obesity is continually increasing. United States has the highest obesity rates among developed countries. obesity rates doubled and reaching the current rate of 32 % of the adult population. Recent research suggest that, the prevalence of obesity among US adult men increased from 27.5% in 1999 to 31.1% in 2004 [3]. Obesity is the root cause of many health problems which also include the risk of fall and the resultant injury. There are many weight loss and strength training programmes that prevent obesity and can potentially minimize several health problems.

Several epidemic models and methods have been used to better understand different feature of communicable diseases like AIDS, influenza, dengue, rubella, malaria, measles, and others [18]. Several mathematical models have been applied to study infectious disease by Hethcote [9]. These models have proved important agents for giving conceptual results like basic reproduction numbers, contact numbers and replacement numbers. However, application of such mathematical models for non contagious diseases is rather rare. That is why few authors have suggested epidemiological models for diabetes and obesity [4, 10, 11].

In this research work, we consider the model presented by Jódara, at el. [4]. In their work, they used two different transmission rates one transit from normal to latent and other transit from normal to overweight and obese.

In order to decrease the attitude toward obesity, we use optimal prevention strategies in this work to reduce transmission rate, rate at which latent individuals moves to the overweight and rate at which an overweight individual becomes obese by using three control variables in the form of education and treatment campaigns. [1] Our goal is to minimize overweight and obese and maximize normal individuals in a community. We first reveal the presence of an optimal control for the control problem and then we deduce the optimality system.

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The technical tool used in this work to know the optimal strategy is the Pontryagin Maximum Principle. We deduce the optimality system consisting of the state and adjoint equations and solve numerically the system by using an iterative method.

The structure of this paper is organized as follow. In the next section, we introduce the obesity epidemic model. Then, in Section 3, we find the basic reproductive number through Next Generation method and show local stability. In Section 4, we find global stability by using Lyapunov function theory. In Section 5, we derive a control system for the optimal control problem. The existence of proposed optimal control problem is studied in Section 6. In Section 7, we solve numerically the optimality system. Finally, we give conclusion.

2. Model formulation

In the general $SEIR$ model, every individual in the population belongs to one of these four classes: susceptible, exposed (infected but not yet infectious), infectious and recovered [6,17]. The general hypothesis is that the exposed and infectious periods are exponentially dispersed. Jódara, at el. [4] considered the same idea and introduced a deterministic mathematical model to study the dynamics of overweight and obesity in childhood population. A population size T is divided into subclasses of individuals which are normal weight, latent, overweight, obese, overweight on diet and obese on deit individuals, with sizes represented as $N(t)$, $L(t)$, $F(t)$, $O(t)$, $W(t)$ and $D(t)$, respectively. The obesity transmission model is given by

$$
\begin{aligned}
\frac{dN(t)}{dt} &= \mu + bW(t) - \beta_1 N(t)L(t) - \beta_2 N(t)(F(t)+O(t)) - \mu N(t),\\
\frac{dL(t)}{dt} &= \beta_1 N(t)L(t) + \beta_2 N(t)(F(t)+O(t))) - (\gamma_1+\mu)L(t),\\
\frac{dF(t)}{dt} &= \gamma_1 L(t) + \eta_2 W(t) - (\gamma_2+\eta_1+\mu)F(t),\\
\frac{dO(t)}{dt} &= \gamma_2 F(t) + \rho_2 D(t) - (\rho_1+\mu)O(t),\\
\frac{dW(t)}{dt} &= \rho_3 D(t) + \eta_1 F(t) - (\eta_2+b+\mu)W(t),\\
\frac{dD(t)}{dt} &= \rho_1 O(t) - (\mu+\rho_2+\rho_3)D(t),
\end{aligned}
\tag{1}
$$

subject to the conditions

$$N(0)\geq 0, L(0)\geq 0, F(0)\geq 0, O(0)\geq 0, W(0)\geq 0, D(0)\geq 0.$$

The parameter μ represents inflow rate to the system population, b is the moving rate of over weight on diet individuals to become normal individuals. γ_1 is the moving rate at which latent individuals become over weight individuals. Over weight individuals become obese individuals at the rate γ_2, while the over weight individuals move to over weight on diet individuals at the rate η_1. η_2 and ρ_2 represent the overweight and obese on diet fail, respectively. ρ_3 is the moving rate at which obese on diet individual moves to overweight on diet individual. The parameter β_1 is the average number of interaction with latent and normal weight individuals, while β_2 represents contact of overweight and obese with normal weight individuals.

We assume that the total population at time t is $T(t)$ that satisfied

$$T(t) = N(t) + L(t) + F(t) + O(t) + W(t) + D(t).$$

Summing up the equations in the model (1) gives

$$\frac{dT}{dt} = \mu - \mu(N(t) + L(t) + F(t) + O(t) + W(t) + D(t)).$$

Therefore, from biological consideration we study the system (1) in the closed set is given by

$$\Omega = \{(N, L, F, O, W, D)R_+^6, 0 < N + L + F + O + W + D \leq \mu - \mu(N(t))\}.$$

To find the unique positive endemic equilibrium of the obesity model (1), we equate the right hand sides of all equations equal to zero and rearrange to get

$$
\begin{aligned}
N^\star &= \frac{\mu^2\gamma_2+\mu\gamma_2\rho_2+\mu\gamma_2\rho_3+b\rho_1\rho_3\gamma_1+\eta_1\rho_1\mu+\beta_1\chi_1\chi_2+\beta_2\chi_2\gamma_1\eta_1\mu\rho_3+\eta_1\mu\rho_3}{\beta_1\gamma_2\rho_1\rho_2+\beta_1\gamma_2\rho_1\rho_2+\beta_1\gamma_2\rho_2\mu+\beta_1\rho_1\rho_3+(1-R_0)\beta_1\mu^2\eta_1-\eta_1\eta_2\mu^2-\eta_1\rho_1\mu}O^\star,\\
L^\star &= \frac{\chi_1\chi_2\chi_3-\rho_1\rho_3-\eta_2(\rho_1\rho_3\gamma_2+\eta_1\chi_2\chi_3)-\eta_2(\rho_1\rho_3\gamma_2+\eta_1\chi_2\chi_3)+\gamma_1\eta_2\rho_1\rho_3}{\gamma_1\gamma_2\chi_3}O^\star,\\
F^\star &= \frac{\gamma_1(\chi_1\chi_2\chi_3-\rho_1\rho_3)-\eta_2(\rho_1\rho_3\gamma_2+\eta_1\chi_2\chi_3)(R_0-1)+\gamma_1\eta_2\rho_1\rho_3+\eta_1\chi_2\chi_3}{\gamma_1\gamma_2\chi_3\chi_1}O^\star,\\
O^\star &= \frac{\gamma_1\gamma_2(\chi_1\chi_2\chi_3-\rho_1\rho_3)(R_0-1)-\beta_1\chi_1\chi_2+\beta_2\chi_2\gamma_1-\eta_1\chi_2\chi_3}{\gamma_1\gamma_2\chi_1\chi_3},\\
W^\star &= \frac{\rho_3\rho_1\gamma_1\gamma_2\chi_3+\eta_1\gamma_1\gamma_2+\gamma_1(\chi_1\chi_2\chi_3-\rho_1\rho_3)-\eta_2(\rho_1\rho_3\gamma_2\eta_2\gamma_1\rho_1)\rho_3\eta_1}{\gamma_1\gamma_2\chi_1\chi_3}O^\star,\\
D^\star &= \frac{\rho_1}{\mu+\rho_2+\rho_3}O^\star,
\end{aligned}
\tag{2}
$$

where
$\chi_1 = (\gamma_2+\eta_1+\mu), \chi_2 = (\rho_1+\mu), \chi_3 = (\mu+\rho_2+\rho_3)$

The disease free equilibrium (DFE) point is represented as $S_0 = (N_0, 0, 0, 0, 0, 0)$ and the endemic equilibrium point is $S_* = (N^*, L^*, F^*, O^*, W^*, D^*)$

The point $E_0 = (0, 0, 0, 0)$ and $E_s = (1, 0, 0, 0)$ are trivial and the obesity-free equilibrium of the system (1). There are four more different equilibria of the system (1).

- Normal weight and laten equilibrium state: $E_2 = (N_2, L_2, 0, 0, 0, 0)$, where $N_2 = (\gamma_1+\mu)/\beta_1$, $L_2 = \mu/(\mu+\gamma_1)[1-(\gamma_1+\mu)/\beta_1]$.
- Obesity and overweight on diet free equilibrium state: $E_3 = (N_3, L_3, F_3, 0, 0, 0)$, where

$$
\begin{aligned}
N_3 &= \frac{\gamma_1+\mu}{\beta_1(\gamma_2+\eta_1+\mu)+\beta_2\gamma_1},\\
L_3 &= \frac{\mu(\gamma_2+\eta_1+\mu)}{\beta_1(\gamma_2+\eta_1+\mu)+\beta_2\gamma_1}[K_1-1],\\
F_3 &= \frac{\gamma_1}{(\gamma_2+\eta_1+\mu)}L_3,
\end{aligned}
$$

  where $K_1 = \beta_1(\gamma_2+\eta_1+\mu)/\gamma_1+\mu$ show us the overweight generation number also called basic reproductive number in absence of obesity individuals.
- Hospitalized free equilibrium state: $E_4 = (N_4, L_4, F_4, O_4, 0, 0)$, where

$$
\begin{aligned}
N_4 &= \frac{(\gamma_1+\mu)(\rho_1+\mu)(\gamma_2+\eta_1+\mu)}{\beta_1(\rho_1+\mu)(\gamma_2+\eta_1+\mu)+\beta_2(\gamma_1(\rho_1+\mu)+\gamma_1\gamma_2)},\\
L_4 &= \frac{\mu(\rho_1+\mu)(\gamma_2+\eta_1+\mu)}{\beta_1(\rho_1+\mu)(\gamma_2+\eta_1+\mu)+\beta_2(\gamma_1(\rho_1+\mu)+\gamma_1\gamma_2)}[K_2-1],\\
F_4 &= \frac{\gamma_1}{(\gamma_2+\eta_1+\mu)}L_4,\\
O_4 &= \frac{\gamma_1\gamma_2}{(\rho_1+\mu)(\gamma_2+\eta_1+\mu)}L_4,
\end{aligned}
$$

  $K_2 = (\beta_1(\rho_1+\mu)(\gamma_2+\eta_1+\mu)+\beta_2(\gamma_1(\rho_1+\mu)+\gamma_1\gamma_2))/(\gamma_1+\mu)(\rho_1+\mu)(\gamma_2+\eta_1+\mu)$ show the obesity generation number in absence of overweight and obesity on diet individuals.

3. Reproductive Number and Local Stability Analysis

In epidemiological models, the basic reproduction number is one of the key values that can predict whether the infectious disease will spread in a population or dies out. The basic reproduction number is the average rate of secondary infectious cases when one infectious individual is introduced in a susceptible population.

We use the Next Generation matrix method developed by Driessche et al. [5] to find the basic reproduction number. In order to do this, we divide the system as follow

$$F = \begin{bmatrix} \beta_1 N(t) & \beta_2 N(t) & \beta_2 N(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} (\gamma_1+\mu) & 0 & 0 \\ -\gamma_1 & (\gamma_2+\eta_1+\mu) & 0 \\ 0 & -\gamma_2 & (\rho_1+\mu) \end{bmatrix}, \tag{3}$$

$$FV^{-1} = \begin{pmatrix} \frac{\beta_1 N(t)}{\gamma_1+\mu} + \frac{\beta_2 N(t)}{(\gamma_1+\mu)(\gamma_2+\eta_1+\mu)} + \frac{\beta_2 N(t)\gamma_1\gamma_2}{(\gamma_1+\mu)(\gamma_2+\eta_1+\mu)} & \frac{\beta_2 N(t)}{(\gamma_2+\eta_1+\mu)} + \frac{\beta_2 N(t)\gamma_2}{(\gamma_2+\eta_1+\mu)} & \frac{\beta_2 N(t)}{\rho_1+\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the reproductive number R_0 is given by

$$R_0 = \frac{\beta_1(\eta_1+\gamma_2+\mu)(\rho_1+\mu) + \beta_2(\rho_1+\mu)\gamma_1 + \beta_2\gamma_1\gamma_2}{(\gamma_1+\mu)(\eta_1+\gamma_2+\mu)(\rho_1+\mu)}.$$

Next we find local stability of the disease free and endemic equilibria. To show the local stability of obesity transmission model (1) regarding $S_0 = (N_0, 0, 0, 0, 0, 0)$, and $S_* = (N^*, L^*, F^*, O^*, W^*, D^*)$ we state following results.

Theorem 1. *The DFE point $(N_0, 0, 0, 0, 0, 0)$ is locally stable if $R_0 \leq 1$, otherwise unstable when $R_0 > 1$.*

Proof. To discuss the local stability regarding the point $(N_0, 0, 0, 0, 0, 0)$ we find the Jacobian of the model (1) is given by

$$J_0 = \begin{pmatrix} -Q_1 & -\beta_1 & -\beta_2 & -\beta_2 & b & 0 \\ 0 & -Q_2 & \beta_2 & \beta_2 & 0 & 0 \\ 0 & \gamma_1 & Q_3 & 0 & \eta_2 & 0 \\ 0 & 0 & \gamma_2 & -(\rho_1+\mu) & 0 & \rho_2 \\ 0 & 0 & \eta_1 & 0 & -Q_4 & \rho_3 \\ 0 & 0 & 0 & \rho_1 & 0 & -Q_5 \end{pmatrix}, \tag{4}$$

where $Q_1 = \beta_2 - \mu, Q_2 = \beta_1 - (\gamma_1+\mu), Q_3 = -(\gamma_2+\eta_1+\mu), Q_4 = (\eta_2+b+\mu), Q_5 = (\mu+\rho_2+\rho_3)$.

After some matrix operations, we obtained the following matrix,

$$J_0 = \begin{pmatrix} -Q_1 & -\beta_1 & -\beta_2 & -\beta_2 & b & 0 \\ 0 & Q_2 & \beta_2 & \beta_2 & 0 & 0 \\ 0 & 0 & -T_1 & \beta_2\gamma_1 & \beta_2\gamma_1 & Q_2 \\ 0 & 0 & \gamma_2 & -(\rho_1+\mu) & 0 & \rho_2 \\ 0 & 0 & \eta_1 & 0 & -Q_4 & \rho_3 \\ 0 & 0 & 0 & \rho_1 & 0 & -Q_5 \end{pmatrix}, \tag{5}$$

with $T_1 = (\gamma_2+\eta_1+\mu)(\beta_1) - (\gamma_1+\mu) + \beta_2\gamma_1$.
The characteristics equation of the above Jacobian matrix becomes

$$(\zeta+Q_1)(\zeta+Q_5)(\zeta^4 + a_1\zeta^3 + a_2\zeta^2 + a_3\zeta + a_4) = 0, \tag{6}$$

where

$$\begin{array}{rcl} a_1 & = & \eta_2 + b + 3\mu + \rho_1 + \gamma_1 - \beta_1, \\ a_2 & = & \gamma_2\eta_2\beta_1 + \gamma_2\beta_1 b + \gamma_2\beta_1 b + \gamma_2\beta_1\mu + \gamma_1\gamma_2\eta_2 + \gamma_1\gamma_2\mu \\ & + & \gamma_2\mu\eta_2 + \gamma_2\mu b + \eta_1\eta a_2\beta_1 + \eta_1\beta_1\mu + \eta_1\gamma_1 b + \eta_1\gamma_1 b + \eta_1\eta_2\mu \\ & + & \mu^2\beta_1\mu + \mu\gamma_1 b + \mu^2\gamma_1 + \mu^2\eta_2 + \beta_2\gamma_1\eta_2 + \beta_2\gamma_1 b + (1 - R_0) + \mu\eta_2 + \mu b + \mu^2 \\ & + & \rho_1\gamma_1 + \gamma_2\beta_1 + \gamma_1\gamma_2 + \mu\beta_1, \\ a_3 & = & \beta_2\gamma_1 + \beta_2\gamma_1\gamma_2 + T_1\mu Q_4 + T_1\rho_1 Q_4 + \mu(\gamma_1 + \mu)Q_4 \\ & + & (1 - R_0)T_1 Q_4(\gamma_1 + \mu), \\ a_4 & = & \beta_2\eta_1\rho_1 + \beta_2\gamma_1\mu + \beta_2\gamma_1\gamma_2 + (1 - R_0). \end{array}$$

The equation (6), having six roots, in which two of them having negative roots and for the rest, we can write

$$\delta(\zeta) = (\zeta^4 + a_1\zeta^3 + a_2\zeta^2 + a_3\zeta + a_4) = 0.$$

The Routh Hurwitz criteria [21], satisfied if the roots of the above equation having negative real parts, such that, for Routh Hurwitz criteria, $a_i > 0$ where $i = 1, 2, 3, 4$ and $a_1a_2a_3 > a_3^2 + a_1^2 a_4$. Thus the system is locally asymptotically stable at DFE.

□

Now to discuss the Endemic equilibrium regarding the model (1) is stable locally if $R_0 > 1$.

Theorem 2. *If $R_0 \geq 1$ then the endemic equilibrium is locally asymptotically stable.*

Proof. The Jacobian at $S_* = (N^*, L^*, F^*, O^*, W^*, D^*)$ is

$$J_0 = \begin{pmatrix} -M_1 & -\beta_1 N^*(t) & -\beta_2 N^*(t) & -\beta_2 N^*(t) & b & 0 \\ M2 & M_3 & \beta_2 N^*(t) & \beta_2 N^*(t) & 0 & 0 \\ 0 & \gamma_1 & -M_4 & 0 & \eta_2 & 0 \\ 0 & 0 & \gamma_2 & -(\rho_1 + \mu) & 0 & \rho_2 \\ 0 & 0 & \eta_1 & 0 & -M_5 & \rho_3 \\ 0 & 0 & 0 & \rho_1 & 0 & -M_6 \end{pmatrix}, \tag{7}$$

where

$$\begin{array}{rcl} M_1 & = & \beta_1 L^*(t) + \beta_2 F^*(t) + \beta_2 O^*(t) - \mu, \\ M_2 & = & \beta_1 L^*(t) + \beta_2 F^*(t) + \beta_2 O^*(t), \\ M_3 & = & \beta_1 N^*(t) - (\gamma_1 + \mu), M_4 = (\gamma_2 + \eta_1 + \mu), M_5 = (\eta_2 + b + \mu), M_6 = (\mu + \rho_2 + \rho_3). \end{array}$$

The characteristic equation of the above Jacobian matrix is

$$(\zeta + M_1)(\zeta + M_6)(\zeta^4 + a_1\zeta^3 + a_2\zeta^2 + a_3\zeta + a_4) = 0 \tag{8}$$

with

$$\begin{aligned}
a_1 &= 4\mu + b + \eta_2 + \rho_1 + \eta_1 + \gamma_2 + \gamma_1 - \beta_1 N^*(t),\\
a_2 &= 4\mu^2 + 2\mu b + \mu\eta_2 + \rho_1 b + \rho_1\eta_1 + \mu\gamma - 2 + b\gamma_2 + \eta_2\gamma_2 + \mu\gamma_2 + \rho_1\gamma_2 + \mu\eta_2\\
&+ \rho_\mu + \mu\eta_1 + \gamma_2\mu + \mu\gamma_1 + \eta_2\gamma_1 + \rho_1\gamma_1 + \mu\gamma_1 + \eta_1\gamma_1 + \gamma_1\gamma_2 + \gamma_1\beta_2 N^*(t)\\
&+ \beta_1 N^*(t)(2\mu + b + \eta_2 + \mu\eta_2 - \rho_1 - \eta_1 + \gamma_2),\\
a_3 &= 3\mu^3 + 3\mu^2 b + 3\mu^2\eta_2 + 3\rho_1\mu^2 + 3\rho_1 b\mu + 2\rho_1\mu\eta_2 + 3\mu^2\eta_1 + 2\mu\eta_1\eta_2 + \rho_1\mu\eta_1 + \rho_1 b\eta_1\\
&+ 2\mu\eta_1\eta_2 - \mu^2\gamma_2 + 2\mu b\gamma_2 + 2\mu\eta_2\gamma_2 + \rho_1\mu\gamma_2 + \rho_1 b\gamma_2 + \gamma_2\rho_1\eta_2 + \mu^2 b\\
&+ \rho_1\mu^2 + \mu^2\eta_1 + \rho_1\eta_1\mu + \mu^2\gamma_1 + \mu\gamma_1 b + \mu\eta_2\gamma_1 + 2\rho_1\mu\gamma_1 + \rho_1 b\gamma_1 + \rho_1\eta_2\gamma_1\\
&+ 2\mu^2\gamma_1 + \mu b\gamma_1 + \mu\eta_2\gamma_2 + \mu\eta_1\gamma_1 + b\eta_1\gamma_1 + \mu\eta_1\gamma_1 + \rho_1\eta_1\gamma_1 + \mu\gamma_1\gamma_2\\
&+ (R_0 - 1)D + b\gamma_1\gamma_2 + \rho_1\gamma_1\gamma_2,\\
a_4 &= \mu^4 + \mu^3 b + \mu^3\eta_2 + \rho_1 b\mu^2 + \rho_1\eta_2\mu^2 + \mu^2 b\eta_1 + \mu^2\eta_1\eta_2 + \rho_1\mu^2\eta_1 + \rho_1 b\eta_1\mu\\
&+ \eta_1\eta_2\rho_1\mu + \mu^2 b\gamma_2 + \mu^2\eta_2\gamma_2 + \rho_1 b\gamma_2\mu + \gamma_2\rho_1\eta_2\mu + \mu^2 b\gamma_1 + \rho\eta_2\mu\gamma_1\\
&+ \mu b\eta_1\gamma_1 + \mu\eta_1\eta_2\gamma_1 + \rho_1 b\eta_1\gamma_1 + \eta_1\eta_2\gamma_1\rho_1 + \mu\gamma_1\gamma_2 b + \mu\eta_2\gamma_1\gamma_2\\
&+ \mu(R_0 - 1)D + \mu\eta_2\gamma_1\gamma_2 + \gamma_1\rho_1\mu\gamma_2 + \gamma_1\rho_1 b\gamma_2 + \gamma_1\gamma_2\rho_1\eta_2,
\end{aligned}$$

where

$$D = ((\gamma_1 + \mu)(\eta_1 + \gamma_2 + \mu)(\rho_1 + \mu)).$$

There are six eigen values corresponding to equation (8), two of the eigen values having already negative real part. For other four eigen values we have

$$\delta(\zeta) = \zeta^4 + a_1\zeta^3 + a_2\zeta^2 + a_3\zeta + a_4.$$

The Routh Hurwitz criteria [21], satisfied if the roots of the above equation having negative real parts such that $a_i > 0$ for $i = 1, 2, 3, 4$ and $a_1a_2a_3 > a_3^2 + a_1^2 a_4$. Here $a_i > 0$ when $R_0 \geq 1$. Thus all the eigen values of the characteristic equation (8) have negative real part if $R_0 \geq 1$ which shows that the endemic equilibrium is locally asymptotically stable. □

4. Global Stability Of Disease Free Equilibrium

To discuss the global stability regarding the DFE, we use the Lyapunov function theory. For this we develop the following Lyapunov function regarding the global stability of disease free equilibrium.

Theorem 3. *For $R_0 < 1$ the disease free equilibrium regarding the obesity transmission model* (1) *is stable globally, if $N = N_0$ other wise unstable if $R_0 > 1$.*

Proof. Let us define Lyapunov function is given by

$$U(t) = (N - N_0) + L + E + F + O + W + D. \tag{9}$$

By differentiating equation (9) with respect to time t, we have

$$\frac{dU}{dt} = \frac{dN}{dt} + \frac{dL}{dt} + \frac{dF}{dt} + \frac{dO}{dt} + \frac{dW}{dt} + \frac{dD}{dt}. \tag{10}$$

By using the system (1) and after some simplification, we get

$$\frac{dU}{dt} = -\mu(N - N_0)$$

$\frac{dU}{dt}$ is negative if $N > N_0$ and $R_0 < 1$ and $\frac{dU}{dt} = 0$ if and only if $N = N_0$. By Lasala Principle [16, 17], and $L = F = O = W = D = 0$.
Thus the disease free equilibrium is globally stable in S_0. □

Next to show the global behavior of the system (1) at endemic equilibrium (EE). To do this we develop the following Lyapunov function.

Theorem 4. *For $R_0 > 1$, the EE of the obesity transmission model* (1) *is stable globally , if $S = S_*$ otherwise unstable if $R_0 < 1$.*

Proof. First, we define the Lyapunov function for the global stability of EE is given by

$$U(t) = \frac{1}{2}[(N - N^*) + (L - L^*) + (F - F^*) + (O - O^*) + (D - D^*)]^2. \tag{11}$$

By differentiating equation (11) with respect to time, we obtain

$$\begin{aligned} \frac{dU}{dt} &= [(N - N^*) + (L - L^*) + (F - F^*) + (O - O^*) \\ &+ (D - D^*)][(\frac{dN}{dt}) + (\frac{dL}{dt}) + (\frac{dF}{dt}) + (\frac{dO}{dt}) + (\frac{dD}{dt})]. \end{aligned}$$

By using the system (1) we have

$$\begin{aligned} \frac{dU}{dt} &= [(N - N^*) + (L - L^*) + (F - F^*) + (O - O^*) + (D - D^*)]((\mu + bW(t) \\ &- \mu N(t)) - \mu L(t) + \eta_2 W(t) - \mu F(t) - \eta_1 F(t) \\ &- \mu O(t) - \rho_3 D(t) - \mu D(t). \end{aligned}$$

After some re arrangement we get, and using (2) we have

$$\begin{aligned} \frac{dU}{dt} &= [(N - N^*) + (L - L^*) + (F - F^*) + (O - O^*) + (D - D^*)][(\rho_1 + \mu)(\mu + \rho_2 + \rho_3)O^* + (\gamma_2 + \eta_1 \\ &+ \mu)(\rho_1 + \mu)L^* + (\mu + \rho_2 + \rho_3)D^* - (\mu + \rho_2 + \rho_3)D - (\rho_1 + \mu)(\mu + \rho_2 + \rho_3)O \\ &- (\gamma_2\eta_1 + \mu)(\rho_1 + \mu)L + \mu^2\gamma_2 + \mu\gamma_2\rho_2 + \mu\gamma_2\rho_3 + b\rho_1\rho_3(R_0 - 1)] \end{aligned}$$

After some rearrangement, we have

$$\begin{aligned} \frac{dU}{dt} &= -[(N - N^*) + (L - L^*) + (F - F^*) + (O - O^*) + (D - D^*)][(O - O^*)(\rho_1 + \mu)(\mu + \rho_2 + \rho_3) \\ &+ (D - D^*)(\mu + \rho_2 + \rho_3) + (L - L^*)(\gamma_2\eta_1 + \mu)(\rho_1 + \mu) + \mu^2\gamma_2 + \mu\gamma_2\rho_2 \\ &+ \mu\gamma_2\rho_3 + b\rho_1\rho_3(R_0 - 1)]. \end{aligned}$$

Hence $\frac{dU}{dt} \leq 0$ for all $S = S_*$, which proved that the EE of the obesity transmission model (1) is stable globally. □

5. Optimal control problem

In this section, we develop obesity prevention strategies to control the spread of obesity in population. To do this, we use optimal control theory to develop the control strategies [15]. Our purpose here is to minimize obesity in the population through increasing the number of normal individuals $N(t)$ and decreasing the number of latent $L(t)$, overweight $F(t)$ obese $O(t)$ individuals by using the time dependent variables, called control variables, such as diet $v_1(t)$, exercise $v_2(t)$ and treatment $v_3(t)$.

In the obesity transmission model (1) there are six state variables $N(t)$, $L(t)$, $F(t)$, $O(t)$, $W(t)$ and $D(t)$. For optimal control problem, we adjust the three control variables namely diet $v_1(t)$, exercise $v_2(t)$ and treatment $v_3(t)$ in our optimal problem. To get our above goal, we construct the objective function

$$J(v_1, v_2, v_3) = \int_0^T [A_1N(t) + A_2L(t) + A_3F(t) + A_4O(t) + A_5W(t) + A_6D(t) + \frac{1}{2}(B_1v_1^2(t) + B_2v_2^2(t) + B_3v_3^2)]dt \tag{12}$$

In equation (12) A_1, A_2, A_3, A_4, A_5 and A_6 represents the weight constants of normal individuals $N(t)$ latent individuals $L(t)$, overweight individuals $F(t)$, obese individuals O(t), overweight on diet individuals $W(t)$ and obese on diet individuals $D(t)$ in the objective functional B_1, B_2 and B_3. The terms $\frac{1}{2}B_1v_1^2$, $\frac{1}{2}B_2v_2^2$ and $\frac{1}{2}B_3v_3^2$ describes the cost associated with diet, exercise and treatment.

The three control variables can be adjusted in the optimal control system is given by

$$
\begin{aligned}
\frac{dN(t)}{dt} &= \mu + bW(t) - \beta_1 N(t)L(t) - \beta_2 N(t)(F(t)+O(t)) - \mu N(t) + v_1(t)L(t) + v_2(t)F(t) \\
&+ (1 - v_3(t))O(t), \\
\frac{dL(t)}{dt} &= \beta_1 N(t)L(t) + \beta_2 N(t)(F(t)+O(t))) - (\gamma_1+\mu)L(t) - v_1(t)L(t), \\
\frac{dF(t)}{dt} &= \gamma_1 L(t) + \eta_2 W(t) - (\gamma_2+\eta_1+\mu)F(t) - v_2(t)F(t), \\
\frac{dO(t)}{dt} &= \gamma_2 F(t) + \rho_2 D(t) - (\rho_1+\mu)O(t) - (1-v_3(t))O(t), \\
\frac{dW(t)}{dt} &= \rho_3 D(t) + \eta_1 F(t) - (\eta_2+b+\mu)W(t), \\
\frac{dD(t)}{dt} &= \rho_1 O(t) - (\mu+\rho_2+\rho_3)D(t).
\end{aligned} \tag{13}
$$

subject to the conditions

$$N(0) \geq 0, L(0) \geq 0, F(0) \geq 0, O(0) \geq 0, W(0) \geq 0, D(0) \geq 0.$$

Our goal is to find the control function, such that,

$$J(v_1^*, v_2^*, v_3^*) = min\{J(v_1, v_2, v_3), v_1, v_2, v_3 \in V\} \tag{14}$$

subject to the system (13), where the control set is defined as,

$$V = \{(v_1, v_2, v_3)/v_i(t) \text{ is lebesgue measurable on } [0,1], 0 \leq v_i(t) \leq 1, i = 1,2,3\}. \tag{15}$$

6. Existence of the problem

To discuss the existence of the control problem, we consider the system (13) with initial conditions. For bounded Lebesgue measurable controls, positive initial conditions and positive bounded solutions to the state system exist [19]. Going back to the optimal control problem (12) − (13) for finding the optimal solution. So first we need to define the Lagrangian and Hamiltonian for the optimal control problem (12) and (13). In fact the Lagrangian for the optimal control problem is given by the following equation,

$$
\begin{aligned}
L(N, L, F, O, W, D, v_1, v_2, v_3) &= A_1 N(t) + A_2 L(t) + A_3 F(t) + A_4 O(t) \\
&+ A_5 W(t) + A_6 D(t) + \frac{1}{2}(B_1 v_1^2(t) + B_2 v_2^2(t) + B_3 v_3^2(t)).
\end{aligned}
$$

In order to find the least value of Lagrangian, we define the Hamiltonian H for our proposed problem is

$$H = L(N, L, F, O, W, D, v_1, v_2, v_3) + \lambda_1(t)\frac{dN(t)}{dt} + \lambda_2(t)\frac{dL(t)}{dt} + \lambda_3(t)\frac{dF(t)}{dt} + \lambda_4(t)\frac{dO(t)}{dt} + \lambda_5(t)\frac{dW(t)}{dt} + \lambda_6(t)\frac{dD(t)}{dt}. \tag{16}$$

Now we state the following results regarding the existence of our control problem.

Theorem 5. *There exist an optimal control $v^* = (v_1^*, v_2^*, v_3^*) \in V$, such that,*

$$J(v_1^*, v_2^*, v_3^*) = minJ(v_1, v_2, v_3),$$

subject to the control system (15) *with initial condition.*

Proof. Here, we use the result presented in [20] to show the existence. Since all the optimal variables and the state variables are positive values. So in this problem, the necessary convexity of the objective functional define at equation (12) in $v_1(t)$, $v_2(t)$ and $v_3(t)$ are satisfied. The control variables set $v_1, v_2, v_3 \in V$ is also convex and closed. Also the control system is bounded, which guarantee the compactness needed for the existence. Further the integrand in the objective functional (12), $A_1 N(t) + A_2 L(t) + A_3 F(t) + A_4 O(t) + A_5(t) + A_6(t) + \frac{1}{2}(B_1 v_1^2(t) + B_2 v_2^2(t) + B_3 v_3^2(t))$ is convex on the control set V. Which ensure about the existence of the variables (optimal control) (v_1^*, v_2^*, v_3^*), which minimize (13). Hence the theorem.

To find the optimal solution for our the proposed control problem, by using the Pontryagin'n Principle [15], we use the Hamiltonian

$$H(t, x(t), v(t), \lambda(t)) = f(t, x(t), v(t) + \lambda(t)g(x(t), v(t)). \tag{17}$$

If $(x^*, v_1^*, v_2^*, v_3^*)$ is an optimal solution of our proposed problem, then there exist a non-trivial vector function $\lambda(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t))$, such that,

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H(t, x(t), v(t), \lambda(t))}{\partial \lambda},\\ 0 &= \frac{\partial H(t, x(t), v(t), \lambda(t))}{\partial v},\\ \lambda(t)' &= -\frac{\partial H(t, x(t), v(t), \lambda(t))}{\partial x}.\end{aligned} \tag{18}$$

□

The application of necessary condition to the Hamiltonian, yields the following results.

Theorem 6. *Let N^*, L^*, F^*, O^*, W^* and D^* be optimal solution and (v_1^*, v_2^*, v_3^*) be the optimal control variables for the proposed control problem (12) – (13). Then the adjoint variables exist, such that $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and $\lambda_4(t)$, $\lambda_5(t)$, $\lambda_6(t)$, which satisfying*

$$\begin{aligned}\lambda_1'(t) &= -A_1 + (\lambda_1 - \lambda_2)\beta_1 L^*(t) + (\lambda_1 - \lambda_2)\beta_2(F^*(t) + O^*(t)) + \mu\lambda_1,\\ \lambda_2'(t) &= -A_2 + (\lambda_1 - \lambda_2)\beta_1 N^*(t) + (\gamma_1 + \mu)\lambda_2 - \lambda_1 v_1^* - \lambda_3\gamma_1,\\ \lambda_3'(t) &= -A_3 + (\lambda_1 - \lambda_2)N^*(t) + (\lambda_3 - \lambda_1)v_2^* + (\lambda_3 - \lambda_4)\gamma_2 + (\lambda_3 - \lambda_5)\eta_1 - \mu\lambda_3,\\ \lambda_4'(t) &= -A_4 + (\lambda_1 - \lambda_2)\beta_2 N^*(t) + (\lambda_4 - \lambda_1)(1 - v_3^*) + (\lambda_4 - \lambda_6)\rho_1 - \lambda_4\mu,\\ {\lambda_5}'(t) &= -A_5 - (\lambda_1 + \lambda_5)b + (\lambda_5 - \lambda_3)\eta_2 - \mu\lambda_5,\\ {\lambda_6}'(t) &= -A_6 - (\lambda_5 + \lambda_6)\rho_3 - \lambda_6\rho_2 + \mu\lambda_6,\end{aligned} \tag{19}$$

with transversality conditions (Boundary conditions)

$$\lambda_i(t) = 0 \text{ for } i = 1, 2, 3, 4. \tag{20}$$

Furthermore, the optimal controls variables $v_1^*(t)$, $v_2^*(t)$ and $v_3^*(t)$ are given by

$$v_1^*(t) = max\{min\{\frac{(\lambda_1 - \lambda_2)L^*}{B_1}, 1\}, 0\} \tag{21}$$

$$v_2^*(t) = max\{min\{\frac{(\lambda_3 - \lambda_1)F^*}{B_2}, 1\}, 0\} \tag{22}$$

$$v_3^*(t) = max\{min\{\frac{(\lambda_1 - \lambda_3)O^*}{B_2}, 1\}, 0\} \tag{23}$$

Proof: To find equation (19), the transversality/boundery condition (20), here we use the Hamiltonian (17). By putting $N(t) = N^*(t)$, $L(t) = L^*(t)$, $F(t) = F^*(t)$, $O(t) = O^*(t)$, $D(t) = D^*(t)$, $W(t) = W^*$ then taking the derivative of Hamiltonian with respect to $N(t)$, $L(t)$, $F(t)$, $O(t)$, $W(t)$, and $D(t)$, respectively, we will get the required system (19). Further for obtaining v_1^*, v_2^*, and v_3^* differentiating Hamiltonian with respect to v_1, v_2 and v_3, respectively and then solving $\frac{\partial H}{\partial v_1} = 0$, $\frac{\partial H}{\partial v_2} = 0$ and $\frac{\partial H}{\partial v_3} = 0$ and by applying the optimality conditions. Finally with the application of the control space property, we get (21) – (22) and (23).

Here, we call the formulas (21) – (22) and (23) for the optimal control v^* the optimal control characterization. Here the state variables and the optimal control variables are originate by solving the optimal system, which encompass the state system (13), the system (19), conditions (20), together with the description of the optimum controls (v_1^*, v_2^*, v_3^*) given by equation (21) – (22) and (23). Moreover, the second derivative of the Lagrangian corresponding to control variables v_1, v_2 and v_3 are positive, which show that the proposed optimal problem is least at control. By putting the values of v_1^*, v_2^* and v_3^* in the system (13), which yields the following

$$\begin{aligned}\frac{dN^*(t)}{dt} &= \mu + bW^*(t) - \beta_2 N^*(t)(F^*(t) + O^*(t)) - \mu N^*(t)\\ &+ max\{min\{\frac{(\lambda_1 - \lambda_2)L^*}{B_1}, 1\}, 0\}L^*(t) + max\{min\{\frac{(\lambda_3 - \lambda_1)F^*}{B_2}, 1\}, 0\}\\ &+ (1 - max\{min\{\frac{(\lambda_1 - \lambda_3)O^*}{B_2}, 1\}, 0\})O^*,\end{aligned}$$

$$
\begin{aligned}
\frac{dL^*(t)}{dt} &= \beta_1 N^*(t)L^*(t) + \beta_2 N^*(t)(F^*(t) + O^*(t))) - (\gamma_1 + \mu)L^*(t) \\
&- max\{min\{\frac{(\lambda_1 - \lambda_2)L^*}{B_1}, 1\}, 0\}, \\
\frac{dF^*(t)}{dt} &= \gamma_1 L(t) + \eta_2 W^*(t) - (\gamma_2 + \eta_1 + \mu)F^*(t) \\
&- max\{min\{\frac{(\lambda_1 - \lambda_2)L^*}{B_1}, 1\}, 0\}F^*(t), \\
\frac{dO^*(t)}{dt} &= \gamma_2 F^*(t) + \rho_2 D(t) - (\rho_1 + \mu)O^*(t) \\
&- (1 - max\{min\{\frac{(\lambda_1 - \lambda_3)O^*}{B_2}, 1\}, 0\}), \\
\frac{dW^*(t)}{dt} &= \rho_3 D(t) + \eta_1 F^*(t) - (\eta_2 + b + \mu)W^*(t), \\
\frac{dD^*(t)}{dt} &= \rho_1 O^*(t) - (\mu + \rho_2 + \rho_3)D^*(t).
\end{aligned}
$$

with H^* at $(t, N^*, L^*, F^*, O^*, W^*, D^*, v_1^*, v_2^*, v_3^*, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$, such that,

$$
\begin{aligned}
H^* &= A_1 N^* + A_2 L^* + A_3 F^* + A_4 O^* + A_5 W^* + A_6 D^* \\
&+ \frac{1}{2} B_1 [max\{min\{\frac{(\lambda_1 - \lambda_2)L^*}{B_1}, 1\}, 0\}]^2 \\
&+ \frac{1}{2} B_2 [max\{min\{\frac{(\lambda_3 - \lambda_1)F^*}{B_2}, 1\}, 0\}]^2 \\
&+ \frac{1}{2} B_3 [max\{min\{\frac{(\lambda_1 - \lambda_3)O^*}{B_2}, 1\}, 0\}]^2 \\
&+ \frac{dN^*}{dt} + \lambda_2 \frac{dL^*}{dt} + \lambda_3 \frac{dF^*}{dt} + \lambda_4 \frac{dO^*}{dt} + \lambda_5 \frac{dW^*(t)}{dt} + \lambda_6 \frac{dD^*}{dt}.
\end{aligned}
$$

(24)

Further solving numerically the optimal control system 24 to show our goal graphically.

7. Numerical results of the control problem

In this section, we solve the optimal control system (13) by using the Runge-Kutta order four procedure. Here, first we solve the state system (13) by Runge-Kutta of fourth order scheme with initial condition forward in time and then solving the adjoint system (19) by the backward Runge-Kutta order four in the same interval of time with the help of transversality condition and the solution of the state system. For the simulation purpose, we use the parameters value as follows: $\mu = 0.089$, $b = 0.91$, $\beta_1 = 0.00021$, $\beta_2 = 0.0031$, $\gamma_1 = 0.08$, $\gamma_2 = 0.00007$, $\eta_1 = 0.5$ and $\eta_2 = 0.4$, $\rho_1 = 0.093$, $\rho_2 = 0.052$, $\rho_3 = 0.06$. Furthermore the weight constants in the objective functional (12) are assumed to be $A_1 = 5$, $A_2 = 10$, $A_3 = 15$, $A_4 = 20$, $A_5 = 25$, $A_6 = 30$, $B_1 = 100$, $B_2 = 120$ and $B_3 = 130$, to obtain the following results presented from Fig.1 to Fig.6.

Fig.1, Fig.2, Fig.3 and Fig.4, Fig.5, Fig.6 represents the graphs of normal, latent, overweight, obese, overweight on diet and obese on diet individuals, respectively. Since here in this work our main objective of applying the optimal control is to minimize the number of overweight, obese individuals and to maximize the number of normal individuals, which are shown clearly in the simulation results if Fig 1 - 6.

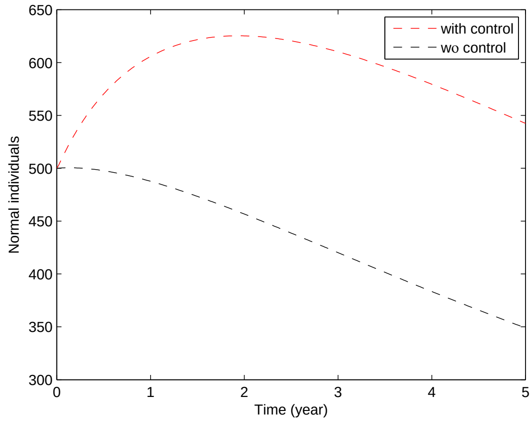

FIGURE 1. The plot shows the dynamic of normal individuals with and without control.

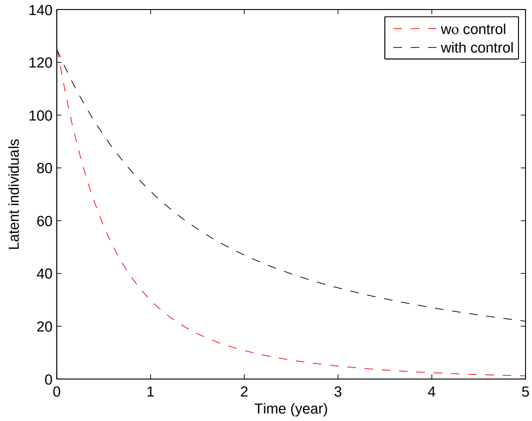

FIGURE 2. The plot shows the dynamic of latent individuals with and without control.

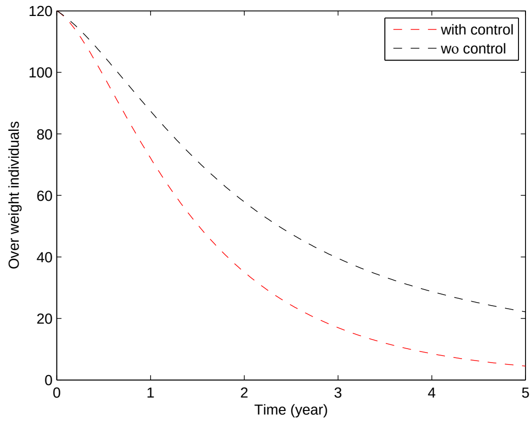

FIGURE 3. The plot shows the dynamic of overweight individuals with and without control.

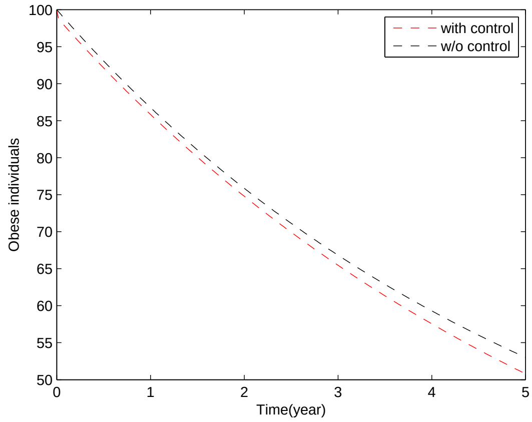

FIGURE 4. The plot shows the dynamic of obese individuals with and without control.

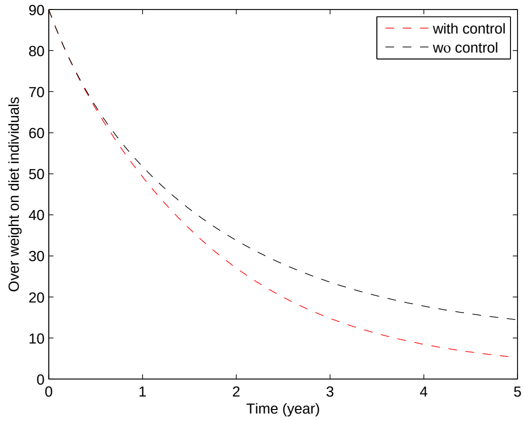

FIGURE 5. The plot shows the dynamic of overweight on diet individuals with and without control.

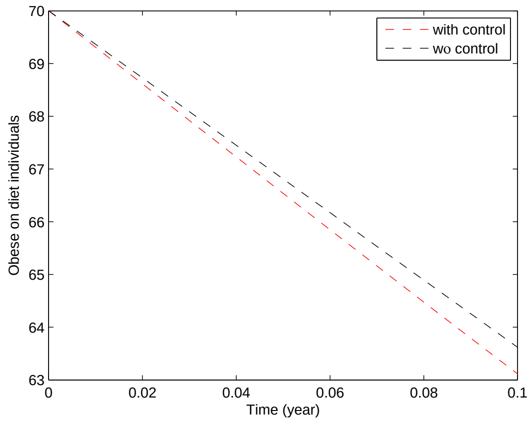

FIGURE 6. The plot shows the dynamic of obese on diet individuals with and without control.

8. Conclusion

In this paper we developed mathematical model and optimal strategies to control obesity in population. The paper focused on mathematical model of obesity. To find out the reproductive number of obesity model we used next generation matrix method. We used stability analysis theory for local stability analysis and Lyapunov function for global stability. In order to decrease the attitude towards obesity we suggested three possible control variables that are diet, exercise $v_2(t)$ and treatment $v_3(t)$ to minimize the number of obesity individuals in the community. In this paper, we have shown the existence of an optimal control for the control problem and then derived the optimality system by using the pontryagin maximum principle. Finally we presented and discussed results of numerical simulation.

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