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NEW REPRESENTATION OF THE SURFACE PENCIL ACCORDING TO THE MODIFIED ORTHOGONAL FRAME WITH CURVATURE IN EUCLIDEAN 3-SPACE

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Abstract. This paper presents new characterizations of line of curvature on a surface family in $\mathbb{E}^3$. For different normal surfaces of the curve, it is shown that the curve is line of curvature on this surface family by using the modified frame with curvature in Euclidean 3-space. Then, we obtain some special characterizations by writing marchingscale functions in the form of product of two functions.

1. Introduction

Papers that address their design are relatively limited, in contrast to the extensive literature on the differential geometry of developable surfaces which is called a surface with zero Gaussian curvature [8]. Most of the existing methods exploit the fact that a developable surface can be characterized as the envelope of a one-parameter family of planes in space. In this approach one first designs a curve in three-dimensional projective space using existing CAD techniques. Then, Park examined design of developable surfaces using uptimal control in [13] and Azariadis and Elber presented methods for refining developable surfaces to approximate arbitrary surfaces in [1,5]. Sun interested in modelling and animating objects seen in everyday life, and many objects can be approximated by piecewise continuous developable surfaces in [14]. Izumiya defined new special curves in Euclidean 3-space in [9].Thus, developable surface pencil have been used in many works. For example, Kobayashi expressed as surface pencil bu using a linear combination in [11], Wanga studyed developable surface in [15].

Line of curvature which is always tangent to a principal direction and geodesic which is defined to be a curve whose tangent vectors remain parallel if they are transported along it on a surface plays an important role in practical applications, [6,10,11]. Thus in [12], Li studyed the moduly space of surfaces with one family of spherical lines of curvature in $\mathbb{R}^3$ by use Certan's theory of exterior differential system. Further in [7], Gutierrez established the differential equation of the lines of curvature for immersions of surfaces into $\mathbb{R}^4$.

In this paper, we study new parametric representation of a surface pencil by using Modified frame. The first, we tersely summarize properties modified frame

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which is parameterized by arc-length parameter s and the basic concepts on curves and surfaces. The second, we obtain general conditions of each other different Q_1, Q_2 ve Q_3 normal surface with line of curvature of the surface $\mathbf{P}(s,t)$. In Finally, we give some characterizations for the curve $\alpha(s)$ as lines of curvature.

2. Preliminaries

Let $\mathcal{S}$ be a surface in three-dimensional Euclidean space $\mathbb{E}^3$ and α be a curve lying on the surface $\mathcal{S}$. In this paper, all curves are assumed to be regular. For given a spatial curve $\alpha : s \to \alpha(s)$, $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative of the Frenet frame according to arc-length parameter is written as

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}.$$

$(\dot{\alpha}(s), \ddot{\alpha}(s), \dddot{\alpha}(s))$ represents the mixed product of vectors $\dot{\alpha}(s)$, $\ddot{\alpha}(s)$, $\dddot{\alpha}(s)$. We assume that the curvature $k(s)$ of α is not identically zero. Now we define an orthogonal frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ as follows:

$$\begin{aligned} &\mathbf{t}{=}\mathbf{T}, \\ &\mathbf{n} = k\mathbf{N}, \\ &\mathbf{b} = k\mathbf{B}. \end{aligned} \tag{2.1}$$

Thus, $\mathbf{n}(s_0) = \mathbf{b}(s_0) = 0$ when $k(s_0) = 0$ and squares of the length of $\mathbf{n}$ and $\mathbf{b}$ vary analitically in s. By the definition of $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ or eq (2.1), a simple calculation show that

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -k^2(s) & \frac{\dot{k}}{k} & \tau \\ 0 & -\tau & \frac{\dot{k}}{k} \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}, \tag{2.2}$$

where a dash denotes the differentation with respect to arc length s and

$$\tau(s) = \frac{(\dot{\alpha}(s), \ \ddot{\alpha}(s), \ \dddot{\alpha}(s))}{k^2(s)}$$

is torsion of α. Moreover, $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ satisfies:

$$\begin{aligned} &\langle \mathbf{t},\mathbf{t}\rangle = 1, \langle \mathbf{n},\mathbf{n}\rangle = \langle \mathbf{b},\mathbf{b}\rangle = k^2, \\ &\langle \mathbf{t},\mathbf{n}\rangle = \langle \mathbf{t},\mathbf{b}\rangle = \langle \mathbf{n},\mathbf{b}\rangle = 0, \end{aligned}$$

where $\langle , \rangle$ denotes the inner product of $\mathbb{E}^3$, [2].

Theorem 2.1. *A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable surface,* [11].

3. On Surface Pencil by using Modified Frame

In this paper, our goal is to find some conditions for the given spatial curve $\alpha(s)$ as a line of curvature on the surface family $\mathcal{P}(s,t)$. Given a spatial curve $\alpha : s \to \alpha(s)$. In [15], Wanga construct a surface pencil that possess $\alpha(s)$ as a common geodesic, they gave the parametric form of the surface $\mathcal{P}(s,t)$ as follows

$$\mathcal{P}(s,t) = \alpha(s) + u(s,t)\,\mathbf{t}(s) + v(s,t)\,\mathbf{n}(s) + w(s,t)\,\mathbf{b}(s), \tag{3.1}$$

where $0 \le s \le L,\ 0 \le t \le T$, $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions.

Case 1. Assume that the normal surface of $\alpha(s)$ is

$$\mathcal{G}_1(s,t) = \alpha(s) + t\mathbf{m}_1(s), \tag{3.2}$$

where $\mathbf{m}_1(s) = \mathbf{n}(s)\cos\theta_1 + \mathbf{b}(s)\sin\theta_1$, the vector functions $\mathbf{n}(s)$ is the unit normal of the surface $\mathcal{P}$. According to Theorem 2.1, $\alpha(s)$ is the line of curvature of the surface $\mathcal{P}(s,t)$ iff $\mathcal{G}_1(s,t)$ is developable. Moreover, the normal vector $\mathbf{m}_1$ is parallel to the normal vector $\mathbf{m}(s,t)$. However, from [15], the surface $\mathcal{G}_1(s,t)$ is developable iff $(\dot{\alpha}(s), \mathbf{m}_1(s), \dot{\mathbf{m}}_1(s)) = 0$. After simple computation, we have

$$\begin{aligned}(\dot{\alpha}(s), \mathbf{m}_1(s), \dot{\mathbf{m}}_1(s)) &= 0,\\ &\Leftrightarrow \dot{\theta}_1(s) + \tau(s) = 0\\ &\Leftrightarrow \dot{\theta}_1(s) = -\tau(s).\end{aligned}$$

That is,

$$\theta_1(s) = -\int_{s_0}^{s} \tau(s)\,ds + \theta_0. \tag{3.3}$$

In the rest of the article, assume that $s_0 = 0$. Substituting θ into the normal vector $\mathbf{m}_1(s)$, if $\mathbf{m}_1(s)$ is parallel to $\mathbf{m}(s)$, the curve $\alpha(s)$ is the line of curvature of the surface $\mathcal{P}(s,t)$. According to Theorem 2.1, the partial differentials of the surface $\mathcal{P}(s,t)$ can simply be written as

$$\frac{\partial \mathcal{P}}{\partial s} = [1 - vk^2 + \frac{\partial u}{\partial s}]\mathbf{t} + [uk - w\tau + v\frac{\dot{k}}{k} + \frac{\partial v}{\partial s}]\mathbf{n} + [w\frac{\dot{k}}{k} + v\tau + \frac{\partial w}{\partial s}]\mathbf{b}$$

and

$$\frac{\partial \mathcal{P}}{\partial t} = \frac{\partial u}{\partial t}\mathbf{t} + \frac{\partial v}{\partial t}\mathbf{n} + \frac{\partial w}{\partial t}\mathbf{b}.$$

The normal vector which occured vector product of $\frac{\partial \mathcal{P}(s,t)}{\partial s}$ and $\frac{\partial \mathcal{P}(s,t)}{\partial t}$ can be expressed as

$$\begin{aligned}\mathbf{m} &= \frac{\partial \mathcal{P}}{\partial s} \times \frac{\partial \mathcal{P}}{\partial t}\\ &= \phi_1(s,t)\,\mathbf{t}(s) + \phi_2(s,t)\,\mathbf{n}(s) + \phi_3(s,t)\,\mathbf{b}(s),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\phi_1(s,t) &= uk\frac{\partial w}{\partial t} - w\tau\frac{\partial w}{\partial t} + \frac{\partial v}{\partial s}\frac{\partial w}{\partial t} + v\frac{\dot{k}}{k}\frac{\partial w}{\partial t} - w\frac{\dot{k}}{k}\frac{\partial v}{\partial t} - v\tau\frac{\partial v}{\partial t} - \frac{\partial w}{\partial s}\frac{\partial v}{\partial t},\\ \phi_2(s,t) &= \frac{\partial w}{\partial t} - vk^2\frac{\partial w}{\partial t} - w\frac{\dot{k}}{k}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial w}{\partial t} - v\tau\frac{\partial u}{\partial t} - \frac{\partial w}{\partial s}\frac{\partial u}{\partial t},\\ \phi_3(s,t) &= \frac{\partial v}{\partial t} - vk^2\frac{\partial v}{\partial t} - uk\frac{\partial u}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial v}{\partial t} + w\tau\frac{\partial u}{\partial t} - v\frac{\dot{k}}{k}\frac{\partial u}{\partial t} - \frac{\partial v}{\partial s}\frac{\partial u}{\partial t}.\end{aligned}$$

This follows that $\mathbf{m}_1(s) \parallel \mathbf{m}(s,t),\ \ 0 \leq s \leq L$, if and only if there exists a function $\lambda_1(s)$, where

$$\phi_1(s,t_0) = 0,\ \ \phi_2(s,t_0) = \lambda_1(s)\cos\theta_1,\ \ \phi_3(s,t_0) = \lambda_1(s)\sin\theta_1$$

and $\lambda_1(s) \neq 0$.

Corollary 3.1. *Let $\alpha(s)$ be curve on the surface family $\mathcal{P}(s,t)$.$\alpha(s)$ is a line of curvature iff*

$$u(s,t_0) = v(s,t_0) = w(s,t_0) = 0,$$

$$\phi_1(s,t_0) = 0,\ \ \phi_2(s,t_0) = \lambda_1(s)\cos\theta_1,\ \ \phi_3(s,t_0) = \lambda_1(s)\sin\theta_1, \tag{3.5}$$

where $0 \leq t_0 \leq T,\ \ 0 \leq s \leq L,\ \ \lambda_1(s) \neq 0$.

In the light of the above case 1, the following cases will be given without proof

Case 2. Assume that the normal surface of $\alpha(s)$ be

$$\mathcal{G}_2(s,t) = \alpha(s) + t\mathbf{m}_2(s), \tag{3.6}$$

where $\mathbf{m}_2(s) = \mathbf{t}(s)\cos\theta_2 + \mathbf{n}(s)\sin\theta_2$. The normal vector $\mathbf{m}(s,t)$ of the regular surface $\mathcal{P}(s,t)$. According to Theorem 2.1., $\alpha(s)$ is the line of curvature of the surface $\mathcal{P}(s,t)$ iff $\mathcal{G}_2(s,t)$ is developable. Moreover, the normal vector $\mathbf{m}_2$ is parallel to the normal vector $\mathbf{m}(s,t)$. However, from [15], the surface $\mathcal{G}_2(s,t)$ is developable iff $(\dot{\alpha}(s), \mathbf{m}_2(s), \dot{\mathbf{m}}_2(s)) = 0$. After simple computation, we have

$$\begin{aligned} (\dot{\alpha}(s), \mathbf{m}_2(s), \dot{\mathbf{m}}_2(s)) &= 0 \\ &\Leftrightarrow\ \ \tau(s)\sin\theta_2 = 0. \end{aligned} \tag{3.7}$$

Substituting θ into the normal vector $\mathbf{m}_2(s)$, if $\mathbf{m}_2(s)$ is parallel to $\mathbf{m}(s)$, the curve $\alpha(s)$ is the line of curvature of the surface $\mathcal{P}(s,t)$.

This follows that $\mathbf{m}_2(s) \parallel \mathbf{m}(s,t),\ \ 0 \leq s \leq L$, if and only if there exists a function $\lambda_2(s)$, where

$$\phi_1(s,t_0) = \lambda_2(s)\cos\theta_2,\ \ \phi_2(s,t_0) = \lambda_2(s)\sin\theta_2,\ \ \phi_3(s,t_0) = 0$$

and $\lambda_2(s) \neq 0$.

Corollary 3.2. *Let $\alpha(s)$ be curve on the surface family $\mathcal{P}(s,t)$.$\alpha(s)$ is a line of curvature iff*

$$u(s,t_0) = v(s,t_0) = w(s,t_0) = 0,$$

$$\phi_1(s,t_0) = \lambda_2(s)\cos\theta_2,\ \ \phi_2(s,t_0) = \lambda_2(s)\sin\theta_2,\ \ \phi_3(s,t_0) = 0, \tag{3.8}$$

where $0 \leq t_0 \leq T,\ \ 0 \leq s \leq L,\ \ \lambda_2(s) \neq 0$.

Case 3. Assume that the normal surface of $\alpha(s)$ be

$$\mathcal{G}_3(s,t) = \alpha(s) + t\mathbf{m}_3(s), \tag{3.9}$$

where $\mathbf{m}_3(s) = \mathbf{t}(s)\cos\theta_3 + \mathbf{b}(s)\sin\theta_3$. The normal vector $\mathbf{m}(s,t)$ of the regular surface $\mathcal{P}(s,t)$. According to Theorem 2.1, $\alpha(s)$ is the line of curvature of the surface $\mathcal{P}(s,t)$ iff $\mathcal{G}_3(s,t)$ is developable. Moreover the normal vector $\mathbf{m}_3$ is parallel to the normal vector $\mathbf{m}(s,t)$. However, from [15], the surface $\mathcal{G}_3(s,t)$ is developable iff $(\dot{\alpha}(s), \mathbf{m}_3(s), \dot{\mathbf{m}}_3(s)) = 0$. After simple computation, we have

$$(\dot{\alpha}(s), \mathbf{m}_3(s), \dot{\mathbf{m}}_3(s)) = 0$$
$$\Leftrightarrow \tan\theta_3 = \frac{k(s)}{\tau(s)}.$$

Substituting θ into the normal vector $\mathbf{m}_3(s)$, if $\mathbf{m}_3(s)$ is parallel to $\mathbf{m}(s)$, the curve $\alpha(s)$ is the line of curvature of the surface $\mathcal{P}(s,t)$.

This follows that $\mathbf{m}_3(s) \parallel \mathbf{m}(s,t)$, $0 \le s \le L$, iff there exists a function $\lambda_3(s)$, where

$$\phi_1(s,t_0) = \lambda_3(s)\cos\theta_3,\ \phi_2(s,t_0) = 0,\ \phi_3(s,t_0) = \lambda_3(s)\sin\theta_3$$

and $\lambda_3(s) \neq 0$.

Corollary 3.3. *Let $\alpha(s)$ be curve on the surface family $\mathcal{P}(s,t)$.$\alpha(s)$ is a line of curvature iff*

$$u(s,t_0) = v(s,t_0) = w(s,t_0) = 0,$$

$$\phi_1(s,t_0) = \lambda_3(s)\cos\theta_3,\ \phi_2(s,t_0) = 0,\ \phi_3(s,t_0) = \lambda_3(s)\sin\theta_3, \tag{3.10}$$

where $0 \le t_0 \le T,\ 0 \le s \le L,\ \lambda_3(s) \neq 0$.

4. Special Cases of the Surface Pencil

In this section, we study special cases of parametric representations of a surface pencil.

The functions $\theta_1(s)$, $\theta_2(s)$, $\theta_3(s)$ and $\lambda_1(s)$, $\lambda_2(s)$, $\lambda_3(s)$ are called controlling functions. Now, we also consider the case when the marching-scale functions $u(s,t)$, $v(s,t)$ and $w(s,t)$ can be decomposed into two factors:

$$\begin{aligned} u(s,t) &= l(s)\,U(t), \\ v(s,t) &= m(s)\,V(t), \\ w(s,t) &= n(s)\,W(t), \end{aligned} \tag{4.1}$$

were $0 \le t \le T,\ 0 \le s \le L$ and $l(s),\ m(s),\ n(s),\ U(t),\ V(t)$ and $W(t)$ are C^1 functions and $l(s),\ m(s)$ and $n(s)$ are not identically zero.

Thus, by using corollary 3.1, corollary 3.2 and corollary 3.3, we can get the following theorems.

Theorem 4.1. *The necessary and sufficient condition of the curve* $\alpha(s)$ and $\mathcal{G}_1(s,t) = \alpha(s) + t\mathbf{m}_1(s)$, *where* $\mathbf{m}_1(s) = \mathbf{n}(s)\cos\theta_1 + \mathbf{b}(s)\sin\theta_1$ *is the normal surface of* $\alpha(s)$ be *being a line of curvature on the surface* $\mathcal{P}(s,t)$ *is*

$$U(t_0) = V(t_0) = W(t_0) = 0,$$

$$m(s)\,V'(t_0) = \lambda_1(s)\sin\theta_1, \tag{4.2}$$

$$-n(s)\,W'(t_0) = \lambda_1(s)\cos\theta_1, \tag{4.3}$$

where $0 \le t_0 \le T$, $\lambda_1(s) \neq 0$.

Theorem 4.2. *The necessary and sufficient condition of the curve* $\alpha(s)$ and $\mathcal{G}_2(s,t) = \alpha(s) + t\mathbf{m}_2(s)$, *where* $\mathbf{m}_2(s) = \mathbf{t}(s)\cos\theta_2 + \mathbf{n}(s)\sin\theta_2$ *is the normal surface of* $\alpha(s)$ be *being a line of curvature on the surface* $\mathcal{P}(s,t)$ *is*

$$U(t_0) = V(t_0) = W(t_0) = 0,$$

$$V'(t_0) = 0, \tag{4.4}$$

$$-n(s)\,W'(t_0) = \lambda_2(s)\sin\theta_2, \tag{4.5}$$

where $0 \leq t_0 \leq T$, $\lambda_2(s) \neq 0$.

Theorem 4.3. *The necessary and sufficient condition of the curve* $\alpha(s)$ and $\mathcal{G}_3(s,t) = \alpha(s) + t\mathbf{m}_3(s)$, *where* $\mathbf{m}_3(s) = \mathbf{t}(s)\cos\theta_3 + \mathbf{b}(s)\sin\theta_3$ *is the normal surface of* $\alpha(s)$ be *being a line of curvature on the surface* $\mathcal{P}(s,t)$ *is*

$$U(t_0) = V(t_0) = W(t_0) = 0,$$

$$m(s)\,V'(t_0) = \lambda_3(s)\sin\theta_3, \tag{4.6}$$

$$W'(t_0) = 0, \tag{4.7}$$

where $0 \leq t_0 \leq T$, $\lambda_3(s) \neq 0$.

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