Dynamic selling price and advertisement investment for deteriorating inventory with stock-dependent demand

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Abstract

This article deals with the investigation of combined pricing, replenishment cycle and the advertising effort investment problem for deteriorating items under stock dependent demand. The inclusion of effort level is considered to describe the indirect positive effect of effort investment on demand. The demand is expressed as a function of selling price, the effort level and the inventory level at time t. The expenses on advertisement depend on the effort of the firm. The analytic solution for dynamic effort investment is obtained initially by solving an optimal control problem. The objective is to maximize the total profit per unit time by finding optimal joint selling price, replenishment cycle time and investments in putting efforts. A numerical example is presented to validate the theoretical policies investigated in this paper. Further, the sensitivity analysis about the key parameters is conducted to obtain the managerial insights.

Keywords**:** Effort investment, Deterioration, Optimal control, Stock-dependent demand

1. Introduction:

In various inventory system the deterioration in products like fruits, vegetables, medicines etc. is normally observed, leading in excessive damages in quality as well as quantity of items. Many measures have been taken in order to shrink the deterioration rate. Commonly preferred measure is the preservation technology investment way. In order to raise the market demand many policies like service investment, advertising effort investment, representing all forms of demand enhancing efforts. Many literature works on inventory control are demonstrated on the basis of assuming fixed rate of demand over entire inventory cycle. Even though, in real-life situations, there are many aspects affecting the rate of demand such as the price associated with selling of the items and the obtainability of items. On the basis of facts, the models dealing with inventory includes demand rates based on selling price of items are considered by decreasing price increases sales of many products.

The rate of market demand is assumed to fluctuate as a function, based on level of stock, the price value or together, in models on inventory with rates of demand as variable one. An inventory model with EOQ concept in which the total profit is maximized on fulfilling the constraints like; budget and capacity of storage; was developed by Sana and Chaudhari (2004), with the rate of demand together based on item obtainability and expenditures on advertisements. For deteriorating of items with the concept of partial back-logging and with a level of stock based rate of demand, and a bound on the extreme level of inventory, a model on inventory was represented by Min and Zhou (2009). Other model on inventory with rate of demand based on level of stock for items which are deteriorating in nature was proposed by Yang *et al.* (2010), permitting back-logging partially and including the inflation effect. An EOQ concept consisting of back-logging partially, in which the rate of demand is based on level of stock, and a manageable rate of deterioration determines the strategies of preservation and lot order size in optimum manner to rise the total profit to maximum, was formulated by Lee *et al.* (2012).

Firstly, Silver *et al.* (1969) have attempted for time varying demand rate, then after many scholars named Silver (1979), Chung *et al.* (1993, 1994), Bose (1995), Hariga (1995), Lin, *et al.* (2000), Mehta *et al.* (2003, 2004), Kharna *et al.* (2003), Shah *et al.* (2009), and Shah *et al.* (2014) have expressed rate of demand as time varying in terms of linear, exponential or quadratic, etc. in nature.

Also demand of a product is highly affected by advertising of the products. In supply and demand theory, the investment on advertising is an uncertain area. Researchers like Bhunia *et al.* (1997), Kotler (2002), Khouja *et al.* (2003), Chowdhury *et al.* (2013), and Palanivel *et al.* (2015) have taken demand based on advertisement of products for their study.

To incorporate deterioration in inventory modeling, Ghare *et al.* (1963) utilized the effects of deterioration in their inventory model. The concept of Weibull distribution and Gamma distribution was used for deterioration by Covert *et al.* (1973). Philip (1974) presented the generalized model for the earlier one based on deterioration. Zhang *et al.* (2016) worked on policies regarding sale price, service and preservation technology for deteriorating items under common resource constraints. Here demand is price-service level sensitive. Another literature reviews on inventory modeling with items of deteriorating in nature were demonstrated by Nahmias (1982), Raafat (1991), Shah (2000), Goyal (2001), and Bakker, (2012).

In this article, the market demand depending on selling-price, advertising effort level as well as stock availability of items is considered. Constant deterioration is considered without any shortages in the proposed model. The demand is based on the efforts made by the sales team in order to boost the selling of the items. The objective is to maximize the total profit with respect to the limited resource constraints. At last, a numerical example with sensitivity analysis is considered to validate the derived model and derived theoretical concepts.

The organization of this article is consisting of the six sections stated as; section 2 represents the notations and assumptions. Introducing the model in section 3, Section 4, deals with the solution methodology adopted in this paper, also classical optimization technique is utilized to acquire the optimal policy. Section 5 consists of conducting numerical analysis and sensitivity analysis about the inventory parameters. Lastly, conclusion of the study is presented in section 6.

2. Notations and assumptions of model:

The following notations and assumptions are utilized to construct the mathematical model.

2.1 Notations

A	The cost of ordering per order
c	The cost of purchasing per unit
h	The unit inventory cost of holding products per unit time
$I(t)$	The level of inventory at time t
I_0	The quantity ordered per cycle, $I_0 = I(0)$
$E(t)$	The advertising effort level at time t

$e(t)$	The advertising effort investment rate at time t
p	The selling price per unit, where $p > c$
$Rt(p, E(t), I(t))$	The rate of market demand
θu	The co-efficient of deterioration under natural conditions
T	The replenishment cycle length, and after its completion the level of inventory moderates zero
L	The resource capacity for the advertising effort investment
$TP(p, e, T)$	The total profit per unit time

2.2 Assumptions

1. Over an unlimited scheduling horizon, only one deteriorating item without repairing or replacing is considered during each period in the system of inventory
2. There is an occurrence of instant replenishment in each cycle, resulting zero lead time.
3. No shortages are allowed.
4. Let the deterioration rate co-efficient be $\theta u = \theta_0$ where $0 \le \theta_0 < 1$, represents the deterioration under natural conditions.
5. The item starts deteriorating as soon as it enters the system.
6. The market demand is represented by $Rt(p, E(t), I(t)) = \alpha - \beta p + \gamma E(t) + \eta I(t)$, where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\eta > 0$, represents the basic market potential, price sensitivity, advertising effort level co-efficient and stock availability coefficient respectively.

3. Mathematical Model formulation

Let $[0, T]$ be the period of replenishment cycle where a firm tends to sell a single product, which is deteriorating in nature. The firm regulates the selling price p, level of inventor I and advertising effort level E to fluctuate market demand $Rt(p, E(t), I(t))$. Let the process of deterioration of product begins with the entry of the item in the system. Assume that the inventory deterioration is directly proportional to the level of inventory.

$i.e. (\theta u) I(t)$. For the period of scheduling horizon $[0, T]$, the level of inventory declines due to the collective influences of rate of market demand, and the inventory level at the end of replenishment cycle reaches zero. This inventory level scenario can be represented by the differential equation (1), with boundary condition, $I(T) = 0$

$$\frac{dI(t)}{dt} = -(\theta u) I(t) - Rt(p, E(t), I(t)) \qquad , 0 \le t \le T \tag{1}$$

Equation (1) demonstrates that the level of inventory sustains non-negative nature for all time, $i.e.$ $for\, all\, t \in [0, T]$, $I(t) \ge 0$ without backordering throughout the scheduling horizon.

Since the effort level is upgraded by effort investment rate $e(t)$ but due to inflation of effort expectations results in equal effort performance worthless tomorrow than today. Implementing this fact on the growth of effort level represented by the following equation,

$$\frac{dE(t)}{dt} = e(t) - \rho E(t) \qquad , E(0) = E_0 \tag{2}$$

By considering, the decay of effort level as $\rho > 0$ and $E_0 > 0$, as the initial effort level, one can concluded that $e(t) \ge 0$ and effort level $E(t) \ge 0$ for all $t \ge 0$ .The effort investment cost function is supposed to be quadratic in effort investment rate as,

$$C(e(t)) = c_1 e^2(t) + c_2 e(t) + c_3 \tag{3}$$

where $c_i > 0,\ i = 1, 2, 3$ implying increasing managerial costs of advertising effort investment. Along with the restricted resources, the firm takes the investment decisions with respect to advertising efforts. Let L be the upper bound of resource capacity,

$$i.e. \quad c_1 e^2(t) + c_2 e(t) + c_3 \le L \tag{4}$$

The rate of market demand as in equation (5), shows a decrement with selling price p, an increments with respect to effort level for advertising $E(t)$, as well as the displayed stock level $I(t)$.

$$i.e. \quad Rt(p, E(t), I(t)) = \alpha - \beta p + \gamma E(t) + \eta I(t) \tag{5}$$

The total profit of the system is computed with the following components:

1. Sales revenue:

The sales revenue per cycle is given by,

$$SR = \int_0^T pRt(p, E(t), I(t))dt = p\left[\alpha - \beta p\right]T + \int_0^T \left[\gamma E(t) + \eta I(t)\right]dt$$

2. Holding cost:

Suppose with the positive holding cost co-efficient h, the holding cost function is linear in level of inventory given by, $HC = h.\int_0^T I(t)\,dt$

3. Purchasing cost:

The purchasing cost with c as cost of purchasing per unit item and I_0 as the initial level of inventory is, $PC = cI_0$.

4. Ordering cost:

The ordering cost with A, as order cost per order is given by, $OC = A$

5. Effort investment cost:

The effort investment cost per order is computed as,

$$EIC = \int_0^T (c_1 e^2(t) + c_2 e(t) + c_3)dt$$

Therefore, the total profit of inventory system per unit time is calculated by subtracting the applicable costs from the sales revenue, denoted by $TP(p,e,T)$ and computed as,

$$TP(p,e,T) = \frac{1}{T}\left\{\begin{array}{l} Sales\ revenue - Holding\,cost - Effort\,investment\cos t \\ -Purchase\ \cos t - Ordering\cos t\ per\,order \end{array}\right\}$$

$$\left.\begin{array}{l} TP(p,e,T) = \dfrac{1}{T}\displaystyle\int_0^T \left\{\gamma pE(t) + (\eta p - h)I(t) - (c_1 e^2(t) + c_2 e(t) + c_3)\right\}dt \\ \qquad\qquad\qquad + p[\alpha - \beta p] - \dfrac{cI_0 + A}{T} \end{array}\right\} \tag{6}$$

The objective is to obtain the length of the replenishment cycle, the selling price per item and dynamic advertising effort investment policy by maximizing the total profit per unit time.

$$\left.\begin{aligned}
&\max_{p,T,e(.)} TP(p,e,T)=\frac{1}{T}\int_0^T\left\{\gamma pE(t)+(\eta p-h)I(t)-(c_1e^2(t)+c_2e(t)+c_3)\right\}dt\\
&\qquad\qquad + p[\alpha-\beta p]-\frac{cI_0+A}{T}\\
&s.t.\quad \dot{I}(t)=-(\theta u)I(t)-\alpha+\beta p-\gamma E(t)-\eta I(t),\\
&\qquad \dot{E}(t)=e(t)-\rho E(t),\\
&\qquad I(T)=0,\ E(0)=E_0,\\
&\qquad p\ge c,\ e(t)\ge 0,\ c_1e^2(t)+c_2e(t)+c_3\le L
\end{aligned}\right\} \quad (7)$$

In optimization problem (7), the quantity of replenishment per cycle, I_0 also represents a decision variable. Firstly, by computing the set of values $(p,T,e(.))$, the value of I_0 is calculated by $I_0=I(0)$.

4. Solution Methodology

Initially considering the dynamic advertising effort investment strategy under a specified selling price p and replenishment cycle T by solving the optimal control problem for the specified values

$$\left.\begin{aligned}
&\max_{e(.)} J=\int_0^T\left\{p\gamma E(t)+(\eta p-h)I(t)-c_1e^2(t)-c_2e(t)\right\}dt\\
&s.t.\quad \dot{I}(t)=-(\theta u)I(t)-\alpha+\beta p-\gamma E(t)-\eta I(t),\\
&\qquad \dot{E}(t)=e(t)-\rho E(t),\\
&\qquad I(T)=0,\ E(0)=E_0,\\
&\qquad 0\le e(t)\le\frac{-c_2\pm\sqrt{c_2{}^2-4c_1(c_3-L)}}{2c_1}
\end{aligned}\right\} \quad (8)$$

Applying the optimal control theory proposed by Sethi *et al.* (2000), in order to gain the dynamic optimal solution, for advertising effort investment policy $e(.)$, by defining the Hamiltonian function

$$\left.\begin{aligned}
H(e,I,E,\lambda_1,\lambda_2,t)&=\gamma pE(t)+(\eta p-h)I(t)-c_1e^2(t)-c_2e(t)\\
&\quad+\lambda_1\left(-(\theta u)I(t)-\alpha+\beta p-\gamma E(t)-\eta I(t)\right)+\lambda_2(e(t)-\rho E(t))
\end{aligned}\right\} \quad (9)$$

where, λ_1 and λ_2 represents the adjoint variables respectively of the state variables, $I(t)$ and $S(t)$, satisfying the adjoint equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial I} = h - \eta p + \lambda_1(\theta u + \eta) \tag{10}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial E} = \rho\lambda_2 + \gamma\lambda_1 - \gamma p \tag{11}$$

The terminal condition $E(T)$ and the initial condition $I(0)$ of the inventory system state variables remain free, introducing the following transversality conditions:

$$\lambda_1(0) = 0,\ \lambda_2(T) = 0 \tag{12}$$

By combining the transversality condition (12) and on integrating the adjoint equations (10) and equation (11), the adjoint variables are computed as,

$$\lambda_1(t) = \frac{(h - p\eta)}{\theta u + \eta}\left(e^{(\theta u+\eta)t} - 1\right) \tag{13}$$

$$\lambda_2(t) = \left[\frac{\gamma}{\rho}(1 - e^{\rho(t-T)})\left(\frac{h + p\theta u}{\theta u + \eta}\right)\right] + \left[\frac{\gamma(h - p\eta)}{(\theta u + \eta)(\theta u + \eta - \rho)}(e^{(\theta u+\eta)t} - e^{(\theta u+\eta-\rho)T+\rho t})\right] \tag{14}$$

Assuming $\theta u + \eta \neq \rho$ in the subsequent analysis, and if $\theta u + \eta = \rho$ is considered, then also every result holds by utilizing the mathematical concept of limit as $\theta u + \eta$ equals to ρ.

Result-1: The function $\lambda_2(t)$ is bifurcated on the basis of time period represented as:

Case-1: For $\lambda_2(t) \geq 0$, $\lambda_2(t)$ is monotonically decreasing function.

Case-2: If $\lambda_2(t_0) \leq 0$ for any time period t_0, then $\lambda_2(t) \leq 0$ for all $t \in [t_0, T]$.

Case-3: For $\lambda_2(0) > 0$, there exists a unique turning point τ_0, such that:

a) For any $t \in [0, \tau_0)$, $\lambda_2(t)$ is decreasing.

b) For any $t \in [\tau_0, T]$, $\lambda_2(t)$ is increasing.

where $$\tau_0 = \frac{\gamma}{\rho - (\theta u + \eta)} \ln\left[\frac{(\theta u + \eta)(p\eta - h)e^{\rho T}}{p(\theta u + h)(\theta u + \eta - \rho) - \rho(p\eta - h)e^{(\theta u+\eta)T}}\right] \tag{15}$$

Proof: Case-1

From equation (13), we can say that $\lambda_1(t)$ is a positive strictly increasing function. Let us assume that there exists a time period point, $\bar{t} \in [0, T)$ in such a way that, $\lambda_2(\bar{t}) \geq 0$ an $\rho\lambda_2(\bar{t}) + \gamma\lambda_1(\bar{t}) - \gamma p \geq 0$. It can be proved from equation (11) that $\lambda_2(t) > \lambda_2(\bar{t}) \geq 0$ for any $t > \bar{t}$, which shows a contradiction to the fact that $\lambda_2(T) = 0$. Thus, for any t in such way that $\lambda_2(t) \geq 0$ by concluding $\lambda_2(t)$ is monotonically decreasing with respect to t implying $\rho\lambda_2(t) + \gamma\lambda_1(t) - \gamma p = \dot{\lambda}_2(t) < 0$.

Case-2:

It can be shown easily by considering $\lambda_2(t) \leq 0$ for any time period $t \in [t_0, T]$ if $\lambda_2(t_0) \leq 0$.

Case-3:

If $\lambda_2(0) > 0$, Then $\lambda_2(t)$ will be decreasing in nature up to $\lambda_2(t) \leq 0$, which is computed by equation (14) that,

$$\dot{\lambda}_2(t) = -\gamma\left[\left\{\frac{p\theta u + h}{\theta u + \eta} + \frac{\rho e^{(\theta u+\eta)T}(h - p\eta)}{(\theta u + \eta)(\theta u + \eta - \rho)}\right\}e^{\rho(t-T)} + \frac{(p\eta - h)}{(\theta u + \eta - \rho)}e^{(\theta u+\eta)t}\right] \tag{16}$$

By utilizing equation (16), a unique turning point τ_0 as stated in equation (15) can be derived, which satisfies $\dot{\lambda}_2(\tau_0) = 0$. Using τ_0, it can be proved easily that $\lambda_2(t)$ is a decreasing function for any $t \in [0, \tau_0)$ and an increasing function for any $t \in (\tau_0, T]$. Hence, we proved the required result. Also, for $\tau_0 > T$, the case $\lambda_2(0) > 0$ exists, and as $\lambda_2(T)$ declines in $t \in [0, T]$ and extents to zero, when touches T.

The control strategy e^* for optimality, ought to make the Hamiltonian function to the maximum at each time points,

$$i.e., \; H(e^*, I, E, \lambda_1, \lambda_2, t) \geq H(e, I, E, \lambda_1, \lambda_2, t) \tag{17}$$

Let for the time period points t_1 and t_2 satisfies the following equations:

$$\lambda_2(t_1) = \pm\sqrt{c_2^{\,2} - 4c_1(c_3 - L)} \quad ; \quad \lambda_2(t_2) = c_2 \tag{18}$$

The optimal advertising effort investment strategy is demonstrated in the subsequent theorem.

Theorem 1: The optimal advertising effort investment rate with specified selling price; p and the replenishment cycle length T is given by,

$$e^*(t) = \begin{cases} \dfrac{-c_2 + \sqrt{c_2^{\,2} - 4c_1(c_3 - L)}}{2c_1} & , \quad 0 \leq t \leq t_1 \\ a_1 + a_2 e^{\rho t} + a_3 e^{(\theta u+\eta)t} & , \quad t_1 < t \leq t_2 \\ c_2 & , \quad t_2 < t \leq T \end{cases} \tag{19}$$

$$a_1 = \frac{\gamma}{2c_1\rho}.\frac{(h+p\theta u)}{(\theta u+\eta)} - \frac{c_2}{2c_1},$$

where, $$a_2 = -\frac{\gamma}{2c_1\rho}.\frac{(h+p\theta u)}{(\theta u+\eta)}e^{-\rho T} - \frac{\gamma(h-p\eta)e^{(\theta u+\eta-\rho)T}}{2c_1(\theta u+\eta)(\theta u+\eta-\rho)},$$

$$a_3 = \gamma.\left(\frac{h-p\eta}{2c_1(\theta u+\eta)(\theta u+\eta-\rho)}\right)$$

Proof: Because the Hamiltonian function H is a function with strictly concave in nature in terms of advertising effort investment rate e. In order to maximize the function H, the advertising effort investment rate can be obtained by,

$$e^*(t) = \begin{cases} \dfrac{-c_2+\sqrt{c_2{}^2-4c_1(c_3-L)}}{2c_1} & , \quad \lambda_2(t) \ge \sqrt{c_2{}^2-4c_1(c_3-L)} \\ \dfrac{\lambda_2(t)-c_2}{2c_1} & , \quad 0 \le \lambda_2(t) < \sqrt{c_2{}^2-4c_1(c_3-L)} \\ c_2 & , \quad \lambda_2(t) < 0 \end{cases} \tag{20}$$

It is assumed that $\sqrt{c_2{}^2-4c_1(c_3-L)} > 0$ and it is already proved in result-1 that the equation $\lambda_2(t) = \sqrt{c_2{}^2-4c_1(c_3-L)}$ has at the most a unique solution at $t = t_1$. Specifically, when $\lambda_2(0) \ge \sqrt{c_2{}^2-4c_1(c_3-L)}$, then by case-3 of result-1, there exists a unique solution at time point $t = t_1$. Also, for any time point, $t < t_1$ implies $\lambda_2(t) > \sqrt{c_2{}^2-4c_1(c_3-L)}$ by case-3 of result-1. Additionally, from the equation, $\lambda_2(t) = 0$ contains a unique solution at time point $t = t_2 < T$, and $0 \le \lambda_2(t) < \sqrt{c_2{}^2-4c_1(c_3-L)}$ is analogous to $t_1 < t \le t_2$. Additionally, the case $\lambda_2(t) < 0$ gives $t_2 < t \le T$. Therefore, on substitution of equation (14) into equation (20), proves the required result stated in equation (19). Hence, the theorem is proved.

Corollary-1: The optimal advertising effort investment rate $e^*(t)$ is divided as follows:

Case-1: If the selling price $p \le \overline{p}$, and for any time period point $t \in [0,T]$, then $e^*(t) \equiv c_2$.

Case-2: If the selling price $p \ge p_0$, and for any time period point $t \in [0,T)$, then $e^*(t) > 0$.

where $$\overline{p} = \frac{(1-e^{-\rho T})h(\theta u+\eta-\rho)+h\rho}{\rho\eta(1-e^{(\theta u+\eta-\rho)T})-(1-e^{-\rho T})\theta u(\theta u+\eta-\rho)}, \tag{21}$$

$$p_0 = \frac{\left((\theta u+\eta)e^{\rho T}+\rho e^{\frac{T}{\gamma}(\rho-(\theta u+\eta))}e^{(\theta u+\eta)T}\right)h}{\left((\theta u+\eta)e^{\rho T}+\rho e^{\frac{T}{\gamma}(\rho-(\theta u+\eta))}e^{(\theta u+\eta)T}\right)\eta-\left(e^{\frac{T}{\gamma}(\rho-(\theta u+\eta))}(\theta u+h)(\theta u+\eta-\rho)\right)} \quad (22)$$

Proof: From the value of $\lambda_2(t)$ stated in equation (14), it can be proved that $\lambda_2(0)\le 0$ is equivalent to $p\le \bar{p}$, where $\bar{p}$ is as shown in equation (21). By utilizing the fact proved in result-1 case-2, and from equation (20), that $e^*(t)\equiv c_2$, for any $t\in[0,T]$. Furthermore, it can be computed from equation (15) that $\tau_0\ge T$ is equivalent to $p\ge p_0$ with p_0 demonstrated in equation (22). In this case, by using the fact from result-1 that $\lambda_2(t)$ is a strictly decreasing function with $\lambda_2(T)=0$ and $\lambda_2(t)>0$ for $t<T$. Thus by utilizing equation (20), it can be proved that $e^*(t)>0$ for any $t\in[0,T)$ and $e^*(T)=c_2$ if the selling price $p\ge p_0$. Hence, the corollary is proved.

By Corollary-1, it can be shown that there occurs a selling price threshold $\bar{p}$, beneath it the market demand is comparatively larger because of the less selling price, and therefore, it does not require to invest in advertising effort to motivate market demand. Naturally, less selling price of an item do not gain any incentive for the authority to have effort investment. Alternatively, there occurs substitute threshold p_0 for the selling price, if the selling price is beyond p_0, stating a large unit selling profit, which inspires the authority for the investment further in advertising effort to grow the market demand. Therefore, it results in maintaining the effort investment rate a positive value over the entire time horizon.

By using the optimal advertising effort investment rate stated in equation (19), time path of effort level $E(t)$ can be computed from the state equation (2)

$$E(t)\begin{cases} =\dfrac{\sqrt{(c_2)^2-4c_1(c_3-L)}-c_2}{2c_1\rho}+\left[E_a-\dfrac{\sqrt{(c_2)^2-4c_1(c_3-L)}-c_2}{2c_1\rho}\right]\mathrm{e}^{-\rho t} & ,0\le t\le t_1 \\ =\begin{Bmatrix} \dfrac{a_1}{\rho}-\dfrac{a_1}{\rho}e^{\rho(t_1-t)}+\dfrac{a_2}{2\rho}\mathrm{e}^{\rho t}-\dfrac{a_2}{2\rho}\mathrm{e}^{\rho(2t_1-t)}+\dfrac{a_3}{(\theta u+\eta+\rho)}\mathrm{e}^{(\theta u+\eta)t} \\ -\dfrac{a_3}{(\rho+\theta u)}\mathrm{e}^{((\theta u+\eta+\rho)t_1-\rho t)}+E_b e^{\rho(t_1-t)} \end{Bmatrix} & ,t_1\le t\le t_2 \\ =\dfrac{c_2}{\rho}+\left[E_c-\dfrac{c_2}{\rho}\right]e^{-\rho(t-t_2)} & ,t_2\le t\le T \end{cases} \tag{23}$$

where, E_a represents the initial advertising effort level and

$$E_b=\frac{\sqrt{c_2{}^2-4c_1(c_3-L)}-c_2}{2c_1\rho}+\left(E_a-\frac{\sqrt{c_2{}^2-4c_1(c_3-L)}-c_2}{2c_1\rho}\right)e^{-\rho t_1}$$

$$E_c=\frac{a_1}{\rho}-\frac{a_1}{\rho}e^{\rho(t_1-t_2)}+\frac{a_2}{2\rho}e^{\rho t_2}-\frac{a_2}{2\rho}e^{\rho(2t_1-t_2)}+\frac{a_3}{(\theta u+\eta+\rho)}e^{(\theta u+\eta)t_2}$$
$$-\frac{a_3}{(\theta u+\rho)}e^{(\eta+\rho+\theta u)t_1-\rho t_2}+E_b e^{\rho(t_1-t_2)}$$

Using the demand function in equation (5) and the advertising effort investment level in equation (23), the inventory level $I(t)$ can be computed as follows:

$$I(t)=\begin{cases} I_1(t) & ,\ 0\le t\le t_1 \\ I_2(t) & ,\ t_1<t\le t_2 \\ I_3(t) & ,\ t_2<t\le T \end{cases} \tag{24}$$

where,

$$I_3=\frac{\left(\beta p-\alpha-\dfrac{\gamma c_2}{\rho}\right)(1-e^{-(\theta u+\eta)(t-t_2)})}{(\theta u+\eta)}-\frac{\gamma\left(E_b-\dfrac{c_2}{\rho}\right)(e^{\rho(t_2-t)}-e^{-(\theta u+\eta)(t-t_2)})}{(\theta u+\eta-\rho)}$$
$$+\frac{\gamma\left(E_b-\dfrac{c_2}{\rho}\right)(e^{\rho(t_2-T)}-e^{-(\theta u+\eta)(T-t_2)})}{(\theta u+\eta-\rho)}-\frac{\left(\beta p-\alpha-\dfrac{\gamma c_2}{\rho}\right)(1-e^{-(\theta u+\eta)(T-t_2)})}{(\theta u+\eta)}$$

$$I_2 = \frac{(1-e^{-(\theta u+\eta)(t-t_1)})\left(\beta p-\alpha-\frac{\gamma a_1}{\rho}\right)}{(\theta u+\eta)} + \frac{\left(e^{\rho(t_1-t)}-e^{-(\theta u+\eta)(t-t_1)}\right)a_1\gamma}{(\theta u+\eta-\rho)\rho}$$

$$-\frac{(e^{\rho t}-e^{-(\theta u+\eta)(t-t_1)}e^{\rho t_1})a_2\gamma}{2\rho(\theta u+\eta+\rho)} + \frac{(e^{\rho(2t_1-t)}-e^{\rho t_1}e^{-(\theta u+\eta)(t-t_1)})a_2\gamma}{2(\theta u+\eta-\rho)\rho}$$

$$-\frac{(e^{(\theta u+\eta)t}-e^{(\theta u+\eta)(2t_1-t)})a_3\gamma}{2(\theta u+\eta)(\theta u+\eta+\rho)} + \frac{e^{((\theta u+\eta+\rho)t_1-\rho t)}-e^{(\theta u+\eta)(2t_1-t)}a_3\gamma}{2(\theta u+\eta)(\rho+\theta u)}$$

$$-\frac{(e^{\rho(t_1-t)}-e^{-(\theta u+\eta)(t-t_1)})\gamma E_b}{(\theta u+\eta-\rho)}$$

$$+\frac{\gamma\left(E_b-\frac{c_2}{\rho}\right)(e^{(\theta u+\eta+\rho)t_2+2(\theta u+\eta)t_1-(\theta u+\eta)t-\rho T}-e^{(\theta u+\eta)(2t_2-T-t)})}{(\theta u+\eta-\rho)}$$

$$-\frac{\left(\beta p-\alpha-\frac{\gamma c_2}{\rho}\right)(e^{-(\theta u+\eta)(t-t_2)}-e^{((\theta u+\eta)(2t_2-T-t)})}{(\theta u+\eta)}$$

$$-\frac{(e^{-(\theta u+\eta)(t-t_2)}-e^{-(\theta u+\eta)(t-t_1)})\left(\beta p-\alpha-\frac{\gamma a_1}{\rho}\right)}{(\theta u+\eta)}$$

$$-\frac{a_1\gamma(e^{-(\theta u+\eta)t+(\theta u+\eta-\rho)t_2+\rho t_1}-e^{-(\theta u+\eta)(t-t_1)})}{\rho(\theta u+\eta-\rho)}$$

$$+\frac{a_2\gamma(e^{(\theta u+\eta+\rho)t_2-(\theta u+\eta)t}-e^{-(\theta u+\eta)(t-t_1)}e^{\rho t_1})}{2\rho(\theta u+\eta+\rho)}$$

$$-\frac{a_2\gamma(e^{-(\theta u+\eta)t+(\theta u+\eta-\rho)t_2+2\rho t_1}-e^{-(\theta u+\eta)(t-t_1)}e^{\rho t_1})}{2\rho(\theta u+\eta-\rho)}$$

$$+\frac{a_3\gamma(e^{(\theta u+\eta)(2t_2-t)}-e^{(\theta u+\eta)(2t_1-t)})}{2(\theta u+\eta)(\theta u+\eta+\rho)} - \frac{a_3\gamma e^{-(\theta u+\eta)(t-t_1)}(e^{\rho(t_1-t_2)}-e^{(\theta u+\eta)(2t_1-t)})}{2(\theta u+\eta)(\rho+\theta u)}$$

$$+\frac{\gamma E_b(e^{-(\theta u+\eta)t+(\theta u+\eta-\rho)t_2-\rho t_1}-e^{-(\theta u+\eta)(t-t_1)})}{(\theta u+\eta-\rho)}$$

$$I_1 = \left(\beta p - \alpha - \frac{\gamma\left(\sqrt{(c_2)^2 - 4c_1(c_3 - L)} - c_2\right)(1 - e^{-(\theta u+\eta)t})}{2\rho c_1(\theta u + \eta)} \right) - \frac{\gamma}{(\theta u + \eta - \rho)}\left(e^{-\rho t} - e^{-(\theta u+\eta)t}\right)$$

$$\left(E_a - \frac{\sqrt{(c_2)^2 - 4c_1(c_3 - L)} - c_2}{2\rho c_1} \right) + \frac{(1 - e^{-(\theta u+\eta)(t-t_1)})\left(\beta p - \alpha - \frac{\gamma a_1}{\rho}\right)}{(\theta u + \eta)}$$

$$+ \frac{\left(1 - e^{-(\theta u+\eta)(t-t_1)}\right)a_1\gamma}{(\theta u + \eta - \rho)\rho} - \frac{(1 - e^{-(\theta u+\eta)(t-t_1)})e^{\rho t_1}a_2\gamma}{2\rho(\theta u + \eta + \rho)} + \frac{(1 - e^{-(\theta u+\eta)(t-t_1)})e^{\rho t_1}a_2\gamma}{2\rho(\theta u + \eta - \rho)}$$

$$+ \frac{(1 - a_3\gamma)e^{(\theta u+\eta)t_1}}{2(\theta u + \eta)(\rho + \theta u)} - \frac{(1 - e^{-(\theta u+\eta)(t-t_1)})\gamma E_b}{(\theta u + \eta - \rho)}$$

$$+ \frac{\gamma\left(E_b - \frac{c_2}{\rho}\right)(e^{(\theta u+\eta+\rho)t_2 + (\theta u+\eta)t_1 - \rho T} - e^{(\theta u+\eta)(2t_2 - T - t_1)})}{(\theta u + \eta - \rho)}$$

$$- \frac{\left(\beta p - \alpha - \frac{\gamma c_2}{\rho}\right)(e^{-(\theta u+\eta)(t-t_2)} - e^{(\theta u+\eta)(2t_2 - T - t_1)})}{(\theta u + \eta)}$$

$$- \frac{(e^{-(\theta u+\eta)(t-t_2)} - e^{-(\theta u+\eta)(t-t_1)})\left(\beta p - \alpha - \frac{\gamma a_1}{\rho}\right)}{(\theta u + \eta)}$$

$$- \frac{a_1\gamma(e^{(\theta u+\eta-\rho)t_2 + (\rho - (\theta u+\eta))t_1} - e^{-(\theta u+\eta)(t-t_1)})}{\rho(\theta u + \eta - \rho)} + \frac{a_2\gamma(e^{(\theta u+\eta+\rho)t_2 - (\theta u+\eta)t_1} - e^{-(\theta u+\eta)(t-t_1)}e^{\rho t_1})}{2\rho(\theta u + \eta + \rho)}$$

$$- \frac{a_2\gamma(e^{(2\rho - (\theta u+\eta))t_1 + (\theta u+\eta-\rho)t_2} - e^{-(\theta u+\eta)(t-t_1)}e^{\rho t_1})}{2\rho(\theta u + \eta - \rho)} + \frac{a_3\gamma(e^{(\theta u+\eta)(2t_2 - t_1)} - e^{(\theta u+\eta)t_1})}{2(\theta u + \eta)(\theta u + \eta + \rho)}$$

$$- \frac{a_3\gamma e^{-(\theta u+\eta)(t-t_1)}(e^{\rho(t_1 - t_2)} - e^{(\theta u+\eta)t_1})}{2(\theta u + \eta)(\rho + \theta u)} + \frac{\gamma E_b(e^{-(\rho+\theta u+\eta)t_1 + (\theta u+\eta-\rho)t_2} - e^{-(\theta u+\eta)(t-t_1)})}{(\theta u + \eta - \rho)}$$

$$- \left(\beta p - \alpha - \frac{\gamma\left(\sqrt{(c_2)^2 - 4c_1(c_3 - L)} - c_2\right)(1 - e^{(\theta u+\eta)t_1})}{2\rho c_1(\theta u + \eta)} \right)$$

$$+ \frac{\gamma}{(\theta u + \eta - \rho)}(e^{-\rho t_1} - e^{(\theta u+\eta)t_1})\left(E_a - \frac{\sqrt{(c_2)^2 - 4c_1(c_3 - L)} - c_2}{2\rho c_1} \right)$$

Noting for the case $\lambda_2(0) > 0,$ and while $\tau_0 > T$ holds, $\lambda_2(t)$ is non-negative for any $t \in [0,T]$, Therefore, $t_2 = T$, represents that the advertising effort investment rate remains non-negative over the whole time period horizon.

The initial replenishment quantity $I_0 = I(0)$ can be computed by the inventory state equation (24) and $I(T) = 0$, the value of I_0 is based on p and T. So, initial replenishment quantity is a function $I_0(p,T)$.

Hence, for specific p,T values, on substituting the optimal advertising effort investment rate from equation (19), the effort level from equation (23) and the level of inventory from equation (24), we get the objective function of equation (8). Hence, the relevant total profit J can be obtained, which is a function of p,T, as $J(p,T)$. To obtain the optimal values of decision variables, which maximizes the optimization problem represented in equation (25) on the basis of equation (7) problem.

$$\left.\begin{aligned} &\max_{p,T} TP = \frac{1}{T} J(p,T) + p[\alpha - \beta p] - \frac{cI_0(p,T) + A}{T} \\ &\qquad s.t. \quad p \geq c,\ T \geq 0 \end{aligned}\right\} \tag{25}$$

For computing the time period points t_1 and t_2, we utilize advertising effort investment rate stated in equation (19), represented by,

$$t_1 = \frac{\left[\dfrac{-c_2 + \sqrt{c_2{}^2 - 4c_1(c_3 - L)} - 2c_1a_1}{2c_1}\right] - (a_2 + a_3)}{(a_2\rho + a_3(\theta u + \eta))}, \qquad t_2 = \frac{(c_2 - a_1) - (a_2 + a_3)}{(a_2\rho + a_3(\theta u + \eta))}$$

Now, to maximize the total profit stated in equation (25), we apply the below stated necessary and sufficient condition:

$$\frac{\partial TP(T,p)}{\partial T} = 0, \frac{\partial TP(T,p)}{\partial p} = 0 \tag{26}$$

Now in order to check the concavity of the profit function of obtained solution, adopting the below stated algorithm,

Stage 1: Assigning the inventory parameters some specific hypothetical values.

Stage2: Obtaining the solutions by solving simultaneous equations stated in equation (26) utilizing the mathematical software Maple XVIII.

Stage 3: Checking the below stated second order sufficient conditions at obtained values from equation (26)

$$\frac{\partial^2 TP(T,p)}{\partial T^2} < 0, \frac{\partial^2 TP(T,p)}{\partial p^2} < 0 \tag{27}$$

Then, on satisfying equation (27) for the obtained values, we can say profit function is concave and then stop.

4. Numerical Example and Sensitivity Analysis

4.1 Numerical example:

Considering the specified values in the appropriate units:

$\alpha = 1000, \beta = 3, \gamma = 0.4, \eta = 0.5, c = 20, A = 50, h = 0.3,$
$\rho = 0.2, E_a = 2, L = 30, \theta_0 = 0.3, c_1 = 0.5, c_2 = 1.8, c_3 = 25$

By using above algorithms, we will check concave nature of the profit function and finding the solution which is optimum also doing the sensitivity analysis of decision variables by varying the inventory parameters -20% to 20%.

On following the algorithm, below stated optimum values of decision variables are generated per cycle.

Optimum replenishment cycle length T = 0.2261.

Optimum selling price per unit item p = 168.8256.

Total Profit per unit time $TP = 84031.2233$.

Time period point $t_1 = 0.1891$.

Time period point $t_2 = 0.1897$.

Initial inventory level $I_0 = 10$ units

The advertising effort investment rate e^* at any time t is computed as,

$$e^*(t) = \begin{cases} 1.8386 & , \quad 0 \le t \le 0.1891 \\ 125.5692 - 41.4585e^{0.2t} - 70.094e^{0.8t} & , \quad 0.1891 < t \le 0.1897 \\ 1.8 & , \quad 0.1897 < t \le 0.2261 \end{cases}$$

The advertising effort investment level $E^*(t)$ is computed as,

$$E^*(t) = \begin{cases} 0.3789 & , \quad 0 \le t \le 0.1891 \\ 0.7448 & , \quad 0.1891 < t \le 0.1897 \\ 5.6529 & , \quad 0.1897 < t \le 0.2261 \end{cases}$$

The concave nature of total profit function at the optimal values of decision variables is demonstrated by figure-1. The optimal advertising effort investment rate with various resource capacities is shown in figure-2. For a specific resource capacity, it is observed from figure-2 that, the optimal advertising effort investment $e^*(t)$ maintains a high level before time point $t_1 = 0.1891$ and then after declines gradually until time $t = 0.1892$ and coincides for all resource capacities then decreases slowly up-to time point $t = 0.1897$ then after for all the resource capacities the value of $e^*(t)$ jumps to $t = 0.18$ at the end of the cycle.

Naturally, with a relatively high initial level of inventory, a high effort investment is implemented by the firm in order to motivate market demand, as a result there is a reduction in inventory loss from deterioration. Then after when there is a fall in level of inventory, the firm adopts to lower the effort investment in order to escape from the high cost burden. By using figure-3, one can observe that a higher resource capacity results a greater initial effort investment as well as the length of replenishment cycle is longer as shown in table-1

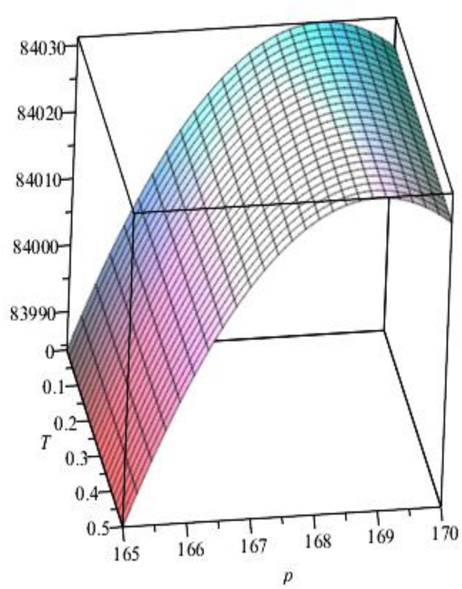

Figure 1: Concavity of Profit function

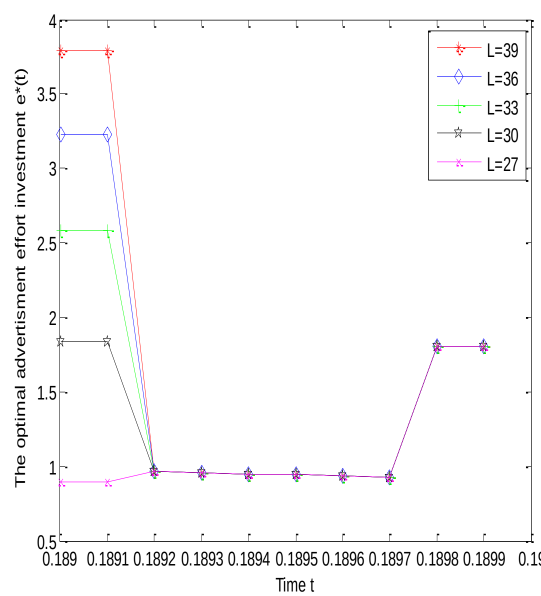

Figure 2: The optimal effort investment rate with various resource capacities

On the basis of corollary-1, by equation-22, the value of p_0 can be calculated as $p_0 = 1.2219$. Therefore, the optimal selling price, $p^* = 168.8256 \square\ p_0$ which generates firm's additional incentives for investing in advertising efforts to motivate the market demand because of the high marginal total profit per unit sale of item. As a concern, the advertising effort investment $e^*(t)$ maintains a positive value throughout the entire planning horizon. This shows the consistency with the statements represented in corollary1.

Inventory Parameters	**Decision Variables**	**Percentage variation of Decision Variables** -20%	-10%	0	10%	20%
α	T	0.27866	0.25011	0.22611	4.97618	4.9073
	p	135.0382	151.9271	168.8256	217.2553	237.9483
	TP	53724.2439	68033.6527	84031.2219	48699.2758	55522.8491
β	T	0.15845	0.19459	0.22611	0.25477	0.28139
	p	212.5783	188.0935	168.8256	153.1995	140.2492
	TP	105471.2078	93529.4581	84031.2213	76289.0451	69854.5407
γ	T	4.93289	4.99759	0.22611	0.20107	0.1759
	p	204.752	200.556	168.8256	169.5312	170.4746
	TP	39522.4506	40932.1876	84031.223	84179.0231	84382.1341
η	T	0.25427	0.23867	0.22611	4.76067	4.5124
	p	168.6323	168.7466	168.8256	204.1505	213.4584
	TP	83890.3664	83960.0902	84031.2201	36966.1813	31842.0057
c	T	0.19958	0.21334	0.22611	0.23814	0.2495
	p	169.0446	168.9269	168.8256	168.7316	168.6393
	TP	84264.14	84136.58	84031.22	83945.08	83875.9540
A	T	0.22442	0.22527	0.22611	0.22694	0.2277
	p	168.8989	168.862	168.8256	168.7896	168.7542
	TP	84075.6093	84053.3756	84031.2216	84009.1482	83987.1587
h	T	0.22597	0.22604	0.22611	0.22618	0.2262
	p	168.8274	168.8264	168.8256	168.8247	168.8238
	TP	84032.3878	84031.8038	84031.2207	84030.6365	84030.0550
ρ	T	0.22299	0.22479	0.22611	4.89259	4.7568
	p	169.0344	168.92	168.8256	193.4197	190.7255
	TP	84109.3235	84067.3747	84031.2217	44238.1101	46280.2272
	T	0.22618	0.22615	0.22611	0.22608	4.9453

E_a	p	168.7978	168.8117	168.8256	168.8395	198.0277
	TP	84006.1841	84018.7012	84031.2207	84043.7402	39102.0448
L	T	0.15909	0.20333	0.22611	0.24292	0.2566
	p	167.339	168.4325	168.8256	169.0539	169.2060
	TP	83554.6247	83830.5206	84031.2236	84199.3709	84347.5065
θ_0	T	0.18997	0.21064	0.22611	4.94578	4.8361
	p	170.8211	169.5886	168.8256	197.9535	198.906
	TP	84535.11	84234.79	84031.22	42478.69	42750.8175
c_1	T	0.19916	0.21382	0.22611	0.23668	0.2459
	p	169.7085	169.191	168.8256	168.5547	168.3466
	TP	84216.56	84106.69	84031.22	83978.2	82940.5377
c_2	T	0.22287	0.22444	0.22611	4.9898	4.9265
	p	168.7527	168.7883	168.8256	198.1895	199.7823
	TP	83961.7614	83995.4472	84031.2203	40380.565	38657.0172
c_3	T	0.25232	0.24037	0.22611	0.20781	0.1797
	p	169.1613	169.0225	168.8256	168.5179	167.9081
	TP	84304.91	84175.46	84031.22	83864.72	83657.0727

Table 1: Sensitivity analysis of decision variables w. r. to various inventory parameters

4.2 Sensitivity analysis on the optimal inventory policy

In this part, the sensitivity analysis of the decision variables with respect to various inventory parameters is carried out. Table-1 demonstrates the values of decision variables on varying the various inventory parameters $(\alpha,\beta,\gamma,\eta,c,A,h,\rho,E_a,L,\theta_0,c_1,c_2,c_3)$ in the range -20% to 20%. From table-1, the following observations are extracted;

(a) Sensitivity analysis of basic market potential demand α

As the product starts deteriorating the firm tends to sell the products rapidly therefore showing the decrement in replenishment cycle length and increment in the selling price resulting in increment in profit margin for the firm.

(b) Sensitivity analysis of price sensitive co-efficient of demand β

With respect to the price sensitive co-efficient of demand β, there is a reduction in sale price implying decrement in total profit gain with longer replenishment cycle length.

(c) Sensitivity analysis of advertising effort level co-efficient γ

Cycle length decreases, raising total profit by showing increment in selling price with γ, which is obvious that, if advertising efforts are grown then the sale would be increased by and by hiking the selling price, marginal profit is increased.

(d) Sensitivity analysis of stock availability coefficient η

Due to deterioration of items, the firm chooses to sell items quickly with decreasing the selling price shortening the replenishment cycle and hence, growing the total profit.

(e) Sensitivity analysis of purchasing cost c

The firm's total profit is damaged with respect to variation in purchase cost, it shows decrement in selling price due to deterioration of items, resulting in the corresponding longer replenishment cycle length relatively.

(f) Sensitivity analysis of ordering cost A:

With increased ordering cost, the firm choses longer replenishment cycle length, and due to deterioration effect the firm decrease the selling price resulting in total profit loss.

(g) Sensitivity analysis of holding cost co-efficient h:

Total profit decreases with lowering selling price and high replenishment cycle length, with respect to holding cost co-efficient.

(h) Sensitivity analysis of decay of effort level ρ:

With respect to the effort level decay co-efficient, the fall in total profit occurs with lowering selling price and increasing in length of replenishment cycle.

(i) Sensitivity analysis of initial advertising effort level E_a

The variation in initial effort level shows shortening in the cycle length, and by increasing the sales price a gradual increase in total profit is seen.

(j) Sensitivity analysis of resource capacity of advertising effort level L:

It is established that with a higher resource capacity, the selling price and replenishment cycle increases. With a high resource capacity, the firm decides to order more quantities and invest more in advertising effort, which subsequently resulting in a longer replenishment cycle. Also, the firm is inspired to raise effort investment, permitting a higher selling price. As per the expectation the total profit of the firm increases with resource capacity.

(k) Sensitivity analysis of basic deterioration rate coefficient $\left(\theta_0\right)$:

The selling price p, decreases, this is because the high deterioration rate requires the firm to decrease the selling price of item in such a way that the products are traded out to escape the deterioration loss with longer length of replenishment cycle. Intuitively, a low profit is generated with respect to high basic deterioration rate.

(l) Sensitivity analysis of costs associated with advertising effort:

With respect to c_1, the co-efficient associated with quadratic nature of advertising effort, the selling price falls with loss in total profit by lengthening the cycle length. With respect to c_2, the co-efficient associated with linear nature of advertising effort, indicates a hike in selling price and resulting total profit by lengthening the length of replenishment cycle. With respect to the fixed constant cost c_3, the decrease in selling price results in reduction of total profit gain with lengthening the cycle length.

6. Conclusions:

As such any firm's resource capacity is always restricted, by compelling the firm to have a worthy deal for the allocation of resources in a rational manner. This article deals with a decision making problem consists of deterioration of items in association with computing the selling price, advertising effort investment rate and the replenishment cycle length under resource constraints. By using classical method in optimization, the optimal value of each decision variable is computed jointly. The numerical example demonstrates the validity of derived theoretical concepts and the solution methodology. In order to gain some managerial insights, the sensitivity analysis of various inventory parameters is done.

It is suggested to encourage in advertising effort investments, in order to raise the profit margin of the firm. Also, to deal with deterioration of items, further the work can be extended by utilizing the concept of preservation technology. Instead of single product, a multi-product situation with common resource constraints can be stretched in future work.

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