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On the dynamics of a delayed SEIR epidemic model

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Abstract

In this work, we propose a delayed SEIR epidemic model. The time delay, τ, is introduced to model the latent period. The resulting model has two possible equilibria (free disease equilibrium and endemic equilibrium). Our main contribution affirms the existence of non constant periodic solutions which bifurcate from the endemic equilibrium when the delay crosses some critical values. Also we propose a comparison of a delayed SEIR model and its corresponding delayed SIR model. Furthermore, some numerical simulations are presented to illustrate our theoretical results.
Keywords. *SEIR epidemic model; generalized incidence rates; global asymptotic stability; Lyapunov-LaSalle's principle; Hopf bifurcation.*

1 Introduction

The spread of an infectious disease in a population is a dynamic phenomenon. Such a phenomenon can be modeled by differential systems [1, 2, 3, 4, 6, 11, 15, 16, 18, 5]. The behavior of the resulting systems is very rich: stability, fluctuations, periodic oscillations or chaos. In particular, the existence of non constant periodic solution which bifurcates from an equilibrium can be guaranteed by the Hopf bifurcation theorem. This problem in epidemic models are analyzed in literature (see [8, 10, 12, 14, 17] and the references therein).
In this paper, we propose the following SEIR epidemic model with a discrete time delay and a general nonlinear incidence function [17]:

$$\begin{cases} \frac{dS}{dt} = A - \mu S - f(S,I), \\ \frac{dE}{dt} = f(S,I) - \mu E - \sigma E_\tau, \\ \frac{dI}{dt} = \sigma E_\tau - (\mu+\gamma) I, \\ \frac{dR}{dt} = \gamma I - \mu R. \end{cases} \tag{1}$$

Here $A = \mu N$, where $N = S + E + I + R$ is the total number of population, S is the number of susceptible individuals, I is the number of infectious individuals, E is the number of exposed individuals, R is the number of recovered individuals, μ is the natural death of the population, f is the incidence function, γ is the recovery rate of the infectious individuals, σ is the rate at which exposed individuals become infectious and τ is the time delay.

The first three equation in system (1) do not depend on the fourth equation, and therefore this equation can be omitted without loss of generality. System (1) can be rewritten as

$$\begin{cases} \frac{dS}{dt} = A - \mu S - f(S, I), \\ \frac{dE}{dt} = f(S, I) - \mu E - \sigma E_\tau, \\ \frac{dI}{dt} = \sigma E_\tau - (\mu + \gamma) I. \end{cases} \tag{2}$$

With the initial conditions

$$S(\theta) = \varphi_1(\theta), \quad E(\theta) = \varphi_2(\theta), \quad I(\theta) = \varphi_3(\theta),$$

where $\varphi_i \in C$ such that $\varphi_i(\theta) \geq 0$, for $i = 1, 2, 3$. Here C denotes the Banach space $C([-\tau, 0], \mathbb{R})$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ with the supremum norm. From a biological meaning, we assume that f is twice continuously differentiable function satisfaying:

(H_0) $f(S, 0) = f(0, I) = 0$;

(H_1) f is a strictly monotone increasing function of $S \geq 0$, for any fixed $I > 0$, and f is a monotone increasing function of $I \geq 0$, for any fixed $S \geq 0$;

(H_2) $\phi(S, I) = \frac{f(S,I)}{I}$ is a bounded and monotone decreasing function of $I > 0$, for any fixed $S \geq 0$, and $K(S) = \lim_{I \to 0^+} \phi(S, I)$ is a continuous and monotone increasing function on $S \geq 0$.

Xu and Liao [17] analyzed the stability and the local Hopf bifurcation for system (2) with bilinear incidence function($f(S, I) = \beta SI$). The aim of this paper is to use model (2) with a generalized incidence function to investigate the global stability of the disease-free equilibrium and the existence of periodic solutions bifurcating from the endemic equilibrium. The rest of the paper is organized as follows: In Section 2, we prove the existence and uniqueness of the endemic equilibrium and by the Lyapunov-LaSalle invariance principle to prove the global stability of the disease-free equilibrium. In Section 3, by analyzing the characteristic equation, the local stability and the Hopf bifurcation of the endemic equilibrium is established for some sufficient conditions. In section 4, we present some application of the main result with particular incidence function, and some numerical simulations are given. In Section 5, we present some concluding remarks.

2 Equilibria and global stability analysis of disease-free equilibrium

2.1 Equilibria

Note that the system (2) always has a disease-free equilibrium $P_0 = (\frac{A}{\mu}, 0, 0)$. On the other hand, The existence of endemic equilibrium is determined by the following proposition:

Proposition 1 *Under the hypotheses* $(H_0) - (H_2)$*, if*

$$R_0 = \frac{\sigma K(\frac{A}{\mu})}{(\mu+\sigma)(\mu+\gamma)} > 1,$$

then system (2) *admits a unique endemic equilibrium* $P^* = (S^*, E^*, I^*)$*, with*

$$S^* = \frac{A}{\mu} - \frac{(\mu+\sigma)(\mu+\gamma)I^*}{\mu\sigma} =, \quad E^* = \frac{(\mu+\gamma)I^*}{\sigma},$$

and I^* *is the unique solution of the following equation:*

$$\frac{f(\frac{A}{\mu} - \frac{(\mu+\sigma)(\mu+\gamma)I^*}{\mu\sigma}, I)}{I} - \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} = 0.$$

Proof. We prove the existence and the uniqueness of the endemic equilibrium P^*. At a fixed point (S, E, I) of system (2), the following equations hold.

$$A - \mu S - f(S,I) = 0, \quad f(S,I) - (\mu+\sigma)E = 0, \quad \sigma E - (\mu+\gamma)I = 0.$$

A simple calculation gives the following system:

$$\begin{cases} S = \frac{A}{\mu} - \frac{(\mu+\sigma)(\mu+\gamma)I}{\mu\sigma}, \\ E = \frac{(\mu+\gamma)I}{\sigma}, \\ \frac{f(\frac{A}{\mu} - \frac{(\mu+\sigma)(\mu+\gamma)I}{\mu\sigma}, I)}{I} - \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} = 0. \end{cases} \tag{3}$$

using the three equation in (3), we set

$$g(I) := \frac{f(\frac{A}{\mu} - \frac{(\mu+\sigma)(\mu+\gamma)I}{\mu\sigma}, I)}{I} - \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} = 0.$$

By the hypotheses (H_0), (H_1), (H_2) and $R_0 > 1$, g is strictly monotone decreasing on $]0, \frac{\sigma A}{(\mu+\sigma)(\mu+\gamma)}]$ satisfying:

$$\begin{aligned} \lim_{I \to 0^+} g(I) &= K(\frac{A}{\mu}) - \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} \\ &= \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma}\left(\frac{K(\frac{A}{\mu})}{\frac{(\mu+\sigma)(\mu+\gamma)}{\sigma}} - 1\right) \\ &= \frac{(\mu+\sigma)(\mu+\gamma)}{\sigma}(R_0 - 1) \ > 0 \end{aligned}$$

and

$$g(\frac{\sigma A}{(\mu+\sigma)(\mu+\gamma)}) = -\frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} < 0.$$

Thus, there exists a unique I^* such that $g(I^*) = 0$.
Hence, we conclude the existence and uniqueness of the endemic equilibrium P^*. ■

2.2 Global stability analysis of the disease-free equilibrium

Now, we discuss the global stability of the disease-free equilibrium P_0 of system (2).

Proposition 2 *If $R_0 \leq 1$, then the disease-free equilibrium P_0 is globally asymptotically stable.*

Proof. Define a Lyapunov functional

$$W_0(t) = V_0(t) + U_0(t),$$

where

$$V_0(t) = \int_{\frac{A}{\mu}}^{S} (1 - \frac{K(\frac{A}{\mu})}{K(u)})du,$$

and

$$U_0(t) = E + \frac{\sigma + \mu}{\sigma} I + \mu \int_{t-\tau}^{t} E(u)du.$$

We will show that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have :

$$\begin{aligned}
\frac{dV_0(t)}{dt} &= (1 - \frac{K(\frac{A}{\mu})}{K(S)})\dot{S} \\
&= (1 - \frac{K(\frac{A}{\mu})}{K(S)})(A - \mu S - f(S, I)) \\
&= \mu(1 - \frac{K(\frac{A}{\mu})}{K(S)})(\frac{A}{\mu} - S) - f(S, I) + \frac{K(\frac{A}{\mu})}{K(S)} f(S, I),
\end{aligned}$$

and

$$\begin{aligned}
\frac{dU_0(t)}{dt} &= f(S, I) - \mu E - \sigma E_\tau + (\sigma + \mu)E_\tau \\
&\quad - \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma} I + \mu E - \mu E_\tau \\
&= f(S, I) - \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma} I.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dW_0(t)}{dt} &= \mu(1 - \frac{K(\frac{A}{\mu})}{K(S)})(\frac{A}{\mu} - S) \\
&\quad + \frac{K(\frac{A}{\mu})}{K(S)} f(S, I) - \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma} I \\
&= \mu(1 - \frac{K(\frac{A}{\mu})}{K(S)})(\frac{A}{\mu} - S) \\
&\quad + \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma} I[\frac{\phi(S, I)}{K(S)} R_0 - 1].
\end{aligned}$$

By the hypothesis (H_2), we obtain that

$$(1-\frac{K(\frac{A}{\mu})}{K(S)})(\frac{A}{\mu}-S)\leq 0,$$

Where equality holds if and only if $S=\frac{A}{\mu}$.
Furthermore, it follows from the hypothesis (H_2) that

$$\frac{\phi(S,I)}{K(S)}R_0\leq R_0.$$

Therefore, $R_0\leq 1$ ensures that $\frac{dW_0(t)}{dt}\leq 0$ for all $t\geq 0$, where $\frac{dW_0(t)}{dt}=0$ holds if $(S,E,I)=(\frac{A}{\mu},0,0)$. Hence, it follows from system (2) that $\{P_0\}$ is the largest invariant set in $\Big\{(S,E,I)|\frac{dW_0(t)}{dt}=0\Big\}$. From the Lyapunov-LaSalle theorem, we obtain that P_0 is globally asymptotically stable. This completes the proof. ■

3 Stability and Hopf bifurcation of the endemic equilibrium

In this section, we discuss the local asymptotic stability of the endemic equilibrium P^* and we use the Hopf Bifurcation theorem to derive sufficient conditions for the bifurcation of nonconstant periodic solutions from this nontrivial equilibrium.
Let $x=S-S^*$, $y=E-E^*$ and $z=I-I^*$. Then the linearized equation around the equilibrium point P^* is given as follows:

$$\begin{cases}\frac{dx}{dt}=(-\mu-\frac{\partial f}{\partial S}(S^*,I^*))x(t)-\frac{\partial f}{\partial I}(S^*,I^*)z(t),\\ \frac{dy}{dt}=\frac{\partial f}{\partial S}(S^*,I^*)x(t)+\frac{\partial f}{\partial I}(S^*,I^*)z(t)-\mu y(t)-\sigma y(t-\tau),\\ \frac{dz}{dt}=\sigma y(t-\tau)-(\mu+\gamma)z(t).\end{cases}\tag{4}$$

The Jacobian matrix $M(\lambda)$ of equation (4) is defined as follows:

$$\mathcal{M}(\lambda)=\begin{pmatrix}\lambda+\mu+\frac{\partial f}{\partial S}(S^*,I^*) & 0 & \frac{\partial f}{\partial I}(S^*,I^*)\\ -\frac{\partial f}{\partial S}(S^*,I^*) & \lambda+\mu+\sigma e^{-\lambda\tau} & -\frac{\partial f}{\partial I}(S^*,I^*)\\ 0 & -\sigma e^{-\lambda\tau} & \lambda+(\mu+\gamma)\end{pmatrix}$$

The characteristic equation of the equation (4) takes the general form:

$$\Delta(\lambda,\tau)=\lambda^3+A\lambda^2+B\lambda+C+[D\lambda^2+E\lambda+F]e^{-\lambda\tau}=0,\tag{5}$$

where

$$A=(2\mu+\gamma)+(\mu+\frac{\partial f}{\partial S}(S^*,I^*)),\quad B=\mu(\mu+\gamma)+(2\mu+\gamma)(\mu+\frac{\partial f}{\partial S}(S^*,I^*)),$$

$$C=\mu(\mu+\gamma)(\mu+\frac{\partial f}{\partial S}(S^*,I^*)),\quad D=\sigma,\quad E=\sigma(\mu+\gamma)-\sigma\frac{\partial f}{\partial I}(S^*,I^*)+\sigma(\mu+\frac{\partial f}{\partial S}(S^*,I^*)),$$

$$F=(\mu+\frac{\partial f}{\partial S}(S^*,I^*))(\mu+\gamma)\sigma-\sigma(\mu+\frac{\partial f}{\partial S}(S^*,I^*))\frac{\partial f}{\partial I}(S^*,I^*)+\sigma\frac{\partial f}{\partial S}(S^*,I^*)\frac{\partial f}{\partial I}(S^*,I^*).$$

In order to investigate the local stability of the endemic equilibrium P^*, we begin by considering the case without delay $\tau=0$.

Lemma 3 *Suppose that the hypotheses* $(H_0) - (H_2)$ *hold.*
For $\tau = 0$, *if* $R_0 > 1$, *then the endemic equilibrium* P^* *is locally asymptotically stable.*

Proof. When $\tau = 0$ the characteristic equation (5) reads as

$$\lambda^3 + (A+D)\lambda^2 + (B+E)\lambda + (C+F) = 0. \tag{6}$$

Firstly, from hypothesis (H_1), we have $\frac{\partial f}{\partial S}(S^*, I^*) \geq 0$, which implies that $(A+D) > 0$. Secondly, from the formula $\frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} = \frac{f(S^*,I^*)}{I^*}$, we find that

$$\begin{aligned} C+F &= \mu(\mu+\gamma)(\mu + \frac{\partial f}{\partial S}(S^*, I^*)) + \sigma(\mu+\gamma)(\mu + \frac{\partial f}{\partial S}(S^*, I^*)) - \mu\sigma\frac{\partial f}{\partial I}(S^*, I^*) \\ &= \mu\sigma\Big(\frac{(\mu+\sigma)(\mu+\gamma)}{\sigma} - \frac{\partial f}{\partial I}(S^*, I^*)\Big) + (\mu+\gamma)(\mu+\sigma)\frac{\partial f}{\partial S}(S^*, I^*) \\ &= \mu\sigma\Big(\frac{f(S^*, I^*)}{I^*} - \frac{\partial f}{\partial I}(S^*, I^*)\Big) + (\mu+\gamma)(\mu+\sigma)\frac{\partial f}{\partial S}(S^*, I^*), \end{aligned}$$

and

$$\begin{aligned} (A+D)(B+E) - (C+F) &= (2\mu+\gamma)\Big((2\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \sigma(\mu + \frac{\partial f(S^*, I^*)}{\partial S})\Big) \\ &+ (\mu + \frac{\partial f(S^*, I^*)}{\partial S})(2\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \sigma(\mu + \frac{\partial f(S^*, I^*)}{\partial S})^2 \\ &+ \sigma(2\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \sigma^2(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) \\ &+ \sigma\Big((2\mu+\gamma) + (\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \sigma\Big)\Big(\frac{(\mu+\gamma)(\mu+\sigma)}{\sigma} - \frac{\partial f(S^*, I^*)}{\partial I}\Big) \\ &+ \sigma\mu\frac{\partial f(S^*, I^*)}{\partial I} - \mu(\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) - \sigma(\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) \\ &= (2\mu+\gamma)\Big((2\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \sigma(\mu + \frac{\partial f(S^*, I^*)}{\partial S})\Big) \\ &+ \sigma(\mu + \frac{\partial f(S^*, I^*)}{\partial S})^2 + \sigma^2(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) \\ &+ \frac{\partial f(S^*, I^*)}{\partial S}(2\mu+\gamma)(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) \\ &+ \sigma\mu\frac{\partial f(S^*, I^*)}{\partial I} + \mu^2(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \mu\sigma(\mu + \frac{\partial f(S^*, I^*)}{\partial S}) \\ &+ \sigma\Big((2\mu+\gamma) + (\mu + \frac{\partial f(S^*, I^*)}{\partial S}) + \sigma\Big)\Big(\frac{f(S^*, I^*)}{I^*} - \frac{\partial f(S^*, I^*)}{\partial I}\Big). \end{aligned}$$

Hence, by hypothesis (H_1) and Kaddar's lemma (see Lemma 4.1 in [7]), we have $C + F \geq 0$ and $(A+D)(B+E) - (C+F) \geq 0$. Thus, according to the Routh-Hurwitz criterion, the endemic equilibrium P^* is locally asymptotically stable. ■

We assume in the sequel that hypotheses $(H_0) - (H_2)$ are true and we return to the study of equation (5) with $\tau > 0$.
Equation (5) has a purely imaginary root $i\omega$ $(\omega > 0)$ if and only if

$$-A\omega^2 + C = (D\omega^2 - F)\cos(\omega\tau) - E\omega\sin(\omega\tau) \tag{7}$$

$$\omega^3 - B\omega = (D\omega^2 - F)\sin(\omega\tau) + E\omega\cos(\omega\tau) \tag{8}$$

Squaring and adding the squares together, we obtain

$$\omega^6 + a\omega^4 + b\omega^2 + c = 0, \tag{9}$$

with $a = A^2 - D^2 - 2B,\ \ b = B^2 - 2AC - E^2 + 2DF,\ \ c = C^2 - F^2$, where $A, B, C, D, E\ and\ F$ are given by (5).
Letting $z = \omega^2$, equation (9) becomes the following cubic equation

$$h(z) = z^3 + az^2 + bz + c = 0. \tag{10}$$

Lemma 4 *[13] Define*

$$\Delta = a^2 - 3b$$

(i) *If $c < 0$, then equation* (10) *has at least one positive root.*

(ii) *If $c \geq 0$ and $\Delta \leq 0$, then equation* (10) *has no positive roots.*

(iii) *If $c \geq 0$ and $\Delta > 0$, then equation* (10) *has positive roots if and only if $\overline{z} := \frac{1}{3}(-a + \sqrt{\Delta}) > 0$ and $h(\overline{z}) \leq 0$.*

Suppose that equation (10) *has positive roots. Without loss of generality, we assume that equation* (10) *has three positive roots, denoted by $z_1,\ z_2\ and\ z_3$, respectively. Then equation* (9) *has three positive roots, say*

$$\omega_1 = \sqrt{z_1},\ \omega_2 = \sqrt{z_2},\ \omega_3 = \sqrt{z_3}$$

Let

$$\tau_l = \frac{1}{\omega_l}[\arccos\Big(\frac{(A\omega_l^2 - C)(F - D\omega_l^2) + (\omega_l^3 - B\omega_l)E\omega_l}{(D\omega_l^2 - F)^2 + E^2\omega_l^2}\Big)], l = 1, 2, 3.$$

Then $\pm i\omega_l$ is a pair of purely imaginary roots of equation (5) *with $\tau = \tau_l,\ l = 1, 2, 3$. Thus, we can define*

$$\tau_0 = \tau_{l_0} = \min_{l=1,2,3}(\tau_l),\ \omega_0 = \omega_{l_0}. \tag{11}$$

To investigate the local stability and the existence of periodic solutions bifurcating from the endemic equilibrium, we need the following results:

Lemma 5 *Suppose that $(H_0) - (H_2)$ hold.*

(i) *If one of the following:*

(N_1) $c \geq 0$ *and* $\Delta \leq 0$;

(N_2) $c \geq 0$ $\Delta > 0$ *and* $\overline{z} \leq 0$;

(N_3) $c \geq 0$ *and* $\Delta > 0$, *and* $h(\overline{z}) \leq 0$.

is true, then all roots of equation (5) *have negative real parts for all* $\tau \geq 0$.

(ii) *If* $c < 0$, *or* $c \geq 0$, $\Delta > 0$, $\overline{z} > 0$, *and* $h(\overline{z}) \leq 0$, *then all roots of equation* (5) *have negative real parts when* $\tau \in [0, \tau_0)$,

where Δ *and* $\overline{z}$ *are defined in lemma 4 and* τ_0 *is defined by* (11).

Proof. *The proof follows from the lemmas 3 and 4.* ■

To guarantee the transversality condition of the Hopf bifurcation theorem [9], we establish the following result:

Lemma 6 *Suppose that* $(H_0) - (H_2)$ *re satisfied and that* $\lambda(\tau) = u(\tau) + i\omega(\tau)$ *is a root of equation* (5).
If one of the following:

(S_1) $c < 0$ *and* $h'(\omega_0^2) \neq 0$,

(S_2) $c \geq 0$, $\Delta > 0$, $\overline{z} > 0$ *and* $h(\overline{z}) < 0$,

is true, then

$$\frac{du}{d\tau}(\tau_0) \neq 0,$$

where τ_0, *and* ω_0 *are defined in* (11).

Proof. $\lambda(\tau) = u(\tau) + i\omega(\tau)$, *is a root of equation* (5) *if and only if*

$$u^3 - 3u\omega^2 + Au^2 - A\omega^2 + Bu + C = -e^{-u\tau}\Big(Du^2\cos(\omega\tau) - D\omega^2\cos(\omega\tau) \tag{12}$$

$$+Eu\cos(\omega\tau) + F\cos(\omega\tau) + 2Du\omega\sin(\omega\tau) + E\omega\sin(\omega\tau)\Big)$$

and

$$3u^2\omega - \omega^3 + 2Au\omega + B\omega = -e^{-u\tau}\Big(- Du^2\sin(\omega\tau) + D\omega^2\sin(\omega\tau) \tag{13}$$

$$-Eu\sin(\omega\tau) - F\sin(\omega\tau) + 2Du\omega\cos(\omega\tau) + E\omega\cos(\omega\tau)\Big)$$

Let $u(\tau)$ *and* $\omega(\tau)$ *satisfying* $u(\tau_0) = 0$, *and* $\omega(\tau_0) = \omega_0$.
By differentiating equations (12) *and* (13) *with respect to* τ *in* $\tau = \tau_0$, *we get*

$$g_1\frac{du(\tau_0)}{d\tau} + g_2\frac{d\omega(\tau_0)}{d\tau} = g_3, \tag{14}$$

$$-g_2\frac{du(\tau_0)}{d\tau} + g_1\frac{d\omega(\tau_0)}{d\tau} = g_4, \tag{15}$$

where

$$g_1 = -3\omega_0^2 + B + (E + D\omega_0^2\tau_0 - F\tau_0)\cos(\omega_0\tau_0) + (2D\omega_0 - E\omega_0\tau_0)\sin(\omega_0\tau_0),$$

$$g_2 = -2A\omega_0 + (E + D\omega_0^2\tau_0 - F\tau_0)\sin(\omega_0\tau_0) + (-2D\omega_0 + E\omega_0\tau_0)\cos(\omega_0\tau_0),$$
$$g_3 = (-D\omega_0^3 + F\omega_0)\sin(\omega_0\tau_0) - E\omega_0^2\cos(\omega_0\tau_0),$$

and

$$g_4 = (-D\omega_0^3 + F\omega_0)\cos(\omega_0\tau_0) - E\omega_0^2\sin(\omega_0\tau_0).$$

Solving for $\frac{du(\tau_0)}{d\tau}$ *we get*

$$\frac{du(\tau_0)}{d\tau} = \frac{g_1g_3 - g_2g_4}{g_1^2 + g_2^2}, \tag{16}$$

Therefore, we have

$$\frac{du(\tau_0)}{d\tau} = \frac{\omega_0^2 h'(\omega_0^2)}{g_1^2 + g_2^2} \tag{17}$$

Thus, if $h'(\omega_0^2) \neq 0$ *we have the transversally condition:*

$$\frac{du(\tau_0)}{d\tau} \neq 0.$$

If $\frac{du(\tau_0)}{d\tau} < 0$ *for* $\tau < \tau_0$ *and close to* τ_0, *then equation* (5) *has a root* $\lambda(\tau) = u(\tau) + i\omega(\tau)$ *satisfying* $u(\tau) > 0$, *which contradicts (ii) of Lemma 4. This completes the proof.* ■

From lemmas 3, 5 and 6, we obtain the following theorem.

Theorem 7 *Assume that* $(H_0) - (H_2)$ *hold and* $R_0 > 1$.

(a) *If (i) of lemma 5 holds, then the equilibrium* P^* *of system* (2) *is locally asymptotically stable for all* $\tau \geq 0$.

(b) *If* (S_1) *or* (S_2) *of lemma 6 holds, then there exists a positive* τ_0 *such that, when* $\tau \in [0, \tau_0)$ *the endemic equilibrium* P^* *is locally asymptotically stable, and when* $\tau = \tau_0$, *a non constant periodic solution bifurcates from this equilibrium, where* τ_0 *is given by*

$$\tau_0 = \frac{1}{\omega_0}\arccos\Big(\frac{(A\omega_0^2 - C)(F - D\omega_0^2) + (\omega_0^3 - B\omega_0)E\omega_0}{(D\omega_0^2 - F)^2 + E^2\omega_0^2}\Big), \tag{18}$$

and ω_0 *is the leat simple positive root of equation* (9), *with* A, B, C, D, E *and* F *are defined in* (5).

4 Numerical simulations

In this section, we give some numerical simulations to illustrate the theoretical analysis. Let

$$f(S, I) = \frac{\beta SI}{1 + \alpha I}.$$

We take the parameters of the system (1) *as follows:*

$$A = 10, \quad \alpha = 0.9, \quad \mu = 0.005, \quad \gamma = 0.02, \quad \beta = 0.1 \; and \quad \sigma = 0.5.$$

By theorem 7, we have if $\tau < \tau_0 = 3.1621$, *then* P_1^* *is locally asymptotically stable (see Figure 1). If we increase the value of* τ, *then a periodic solution occurs at* $\tau_0 = 3.1621$, *(see Figure 2), with the initial condition* $(S(0), E(0), I(0)) = (86.36577, 16.2469, 381.7174)$.

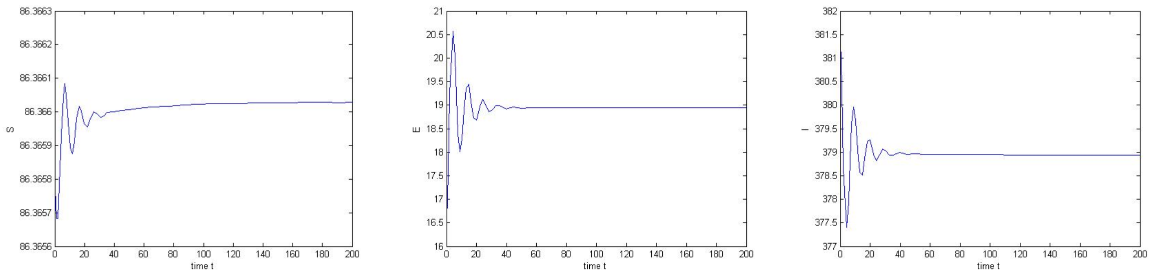

Figure 1: For $\tau = 2.2182$ Solutions (S, E, I) of a discrete SEIR epidemic model (1) are locally asymptotically stable and converge to the endemic equilibrium P_1^*

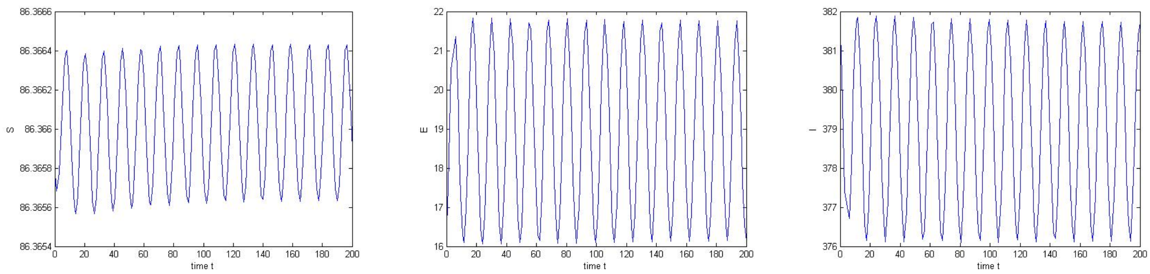

Figure 2: For $\tau = \tau_0 = 3.1621$ a Hopf bifurcation occurs and periodic solutions appean of a discrete SEIR epidemic model (1).

5 Concluding remarks

In this work, we have proposed a generalization of the delayed SEIR epidemiological model set forth by Xu et al. [17]. Our contribution consists to consider this model with a generalized incidence function.

By the Hopf bifurcation theorem, we have proved that the introduction of time delay in the SEIR model can destabilize this system, giving rise to a branch of periodic solutions bifurcated from the endemic equilibrium. This result confirms the result obtained by Xu et al. in the particular case of the incidence function $(f(S, I) = \alpha SI)$ *[17].*

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