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ON INEXTENSIBLE FLOWS OF BINORMAL DEVELOPABLE SURFACES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

SELÇUK BAŞ, RIDVAN C. DEMIRKOL AND VEDAT ASIL

Abstract. In this study, we firstly introduce fundamental definitions belonging to a space curve in Euclidean 3-space. We also present some important structures on the surfaces in an ordinary three dimensional structure. Then we define developable surfaces by investigating the inextensible flows of binormal developable surfaces in Euclidean 3-space $\mathbb{E}^3$. Finally, we obtain results for minimal binormal developable surfaces in Euclidean 3-space $\mathbb{E}^3$.

1. Introduction

Flows of particles is an important active research field in the differential geometry studies and it is heavily studied for a long time and it is still under consideration. Therefore, it introduces the miniature, architect, simulation of kinematic motion or design of highways and mechanic tools. Park investigated inextensible flows of curves and surfaces in $\mathbb{E}^3$, [14].

One of the most effective ways of forming new surfaces is to utilize the base curve and the ruling. These ruled surface are said to be developable providing that the tangent plane of the fixed ruling is constant throughout the flow. It is formed without tearing or strecthing. Thus, it has many substantial applications on the designing and manufacturing of the products. They have also a crucial role in the study of computer aided geometric design, modelling buildings, ship hulls, airplaane wings, automobille parts, garments, etc [2, 3, 15].

Developable surfaces are also obtained by the construction of the inextensible surface and curve flows in space. These particular flows of inextensible surface and curve bring about to motions such that there is no strain energy induced [4 − 12].

In this study, we firstly introduce fundamental definitions belonging to a space curve in Euclidean 3-space. We also present some important structures on the surfaces in an ordinary three dimensional structure. Then we define developable surfaces by investigating the inextensible flows of binormal developable surfaces in Euclidean 3-space $\mathbb{E}^3$. Finally, we obtain results for minimal binormal developable surfaces in Euclidean 3-space $\mathbb{E}^3$.

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2. Preliminaries

Frenet-Serret frame is used to describe the characterization of the intrinsic geometrical features of the regular curve in space. This coordinate system is constructed by three orthonormal vectors $\mathbf{e}^{\sigma}_{(\alpha)}$, assuming the curve is sufficiently smooth at each point. In particular, $\mathbf{e}^{\sigma}_{(0)}$ is the unit tangent vector, $\mathbf{e}^{\sigma}_{(1)}$ is the unit normal vector, and $\mathbf{e}^{\sigma}_{(2)}$ is the unit binormal vector of the curve γ, respectively. Orthonormality conditions are summarized by $\mathbf{e}^{\sigma}_{(\alpha)}\mathbf{e}^{\sigma}_{(\beta)} = \rho_{\alpha\beta}$, where ρ is Euclidean metric such that: $\operatorname{diag}(1,1,1)$. For non-negative coefficients κ, τ, and vectors $\mathbf{e}^{\sigma}_{(i)}$ $(i = 0, 1, 2)$ following equations are valid [16].

$$\begin{aligned}
\nabla_{\gamma'}\mathbf{e}^{\sigma}_{(0)} &= \kappa\mathbf{e}^{\sigma}_{(1)}, \\
\nabla_{\gamma'}\mathbf{e}^{\sigma}_{(1)} &= -\kappa\mathbf{e}^{\sigma}_{(0)} + \tau\mathbf{e}^{\sigma}_{(2)}, \\
\nabla_{\gamma'}\mathbf{e}^{\sigma}_{(2)} &= -\tau\mathbf{e}^{\sigma}_{(1)}.
\end{aligned}$$

3. Inextensible Flows of Binormal Developable Surfaces

A surface $\mathcal{M}$ is ruled if through every point of $\mathcal{M}$ there is a straight line that lies on $\mathcal{M}$.

The binormal developable of γ is a ruled surface

$$\mathcal{M}(s,u) = \gamma(s) + u\mathbf{e}^{\sigma}_{(2)}(s).$$

Let ϖ be the standard unit normal vector field on a surface $\mathcal{M}$ defined by

$$\varpi = \frac{\mathcal{M}_s \wedge \mathcal{M}_u}{\left| g\left(\mathcal{M}_s \wedge \mathcal{M}_u, \mathcal{M}_s \wedge \mathcal{M}_u\right)\right|^{\frac{1}{2}}}. \tag{3.1}$$

Hence, the first fundamental form $\mathbf{I}$ and the second fundamental form $\mathbf{II}$ of a surface $\mathcal{M}$ are defined by, respectively,

$$\mathbf{I} = \mathcal{E}ds^2 + 2\mathcal{F}dsdu + \mathcal{G}du^2, \tag{3.2}$$

$$\mathbf{II} = eds^2 + 2fdsdu + gdu^2,$$

where

$$\mathcal{E} = g(\mathcal{M}_s, \mathcal{M}_s), \quad \mathcal{F} = g(\mathcal{M}_s, \mathcal{M}_u), \quad \mathcal{G} = g(\mathcal{M}_u, \mathcal{M}_u), \tag{3.4}$$

$$\begin{aligned}
e &= -g(\mathcal{M}_s, \varpi_s) = g(\mathcal{M}_{ss}, \varpi), \\
f &= -g(\mathcal{M}_s, \varpi_u) = g(\mathcal{M}_{su}, \varpi), \\
g &= -g(\mathcal{M}_u, \varpi_u) = g(\mathcal{M}_{uu}, \varpi).
\end{aligned} \tag{3.5}$$

Besides, the mean curvature $\mathbf{H}$ and the Gaussian curvature $\mathbf{K}$ are

$$\mathbf{H} = \frac{\mathcal{E}g - 2\mathcal{F}f + \mathcal{G}e}{2\left(\mathcal{E}\mathcal{G} - \mathcal{F}^2\right)}, \tag{3.7}$$

$$\mathbf{K} = \frac{eg - f^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}, \tag{3.8}$$

respectively.

Definition 3.1. *(see [14])* A surface evolution $\mathcal{M}(s,u,t)$ and its flow $\dfrac{\partial \mathcal{M}}{\partial t}$ are said to be inextensible if its first fundamental form $\{\mathcal{E},\mathcal{F},\mathcal{G}\}$ satisfies

$$\frac{\partial \mathcal{E}}{\partial t}=\frac{\partial \mathcal{F}}{\partial t}=\frac{\partial \mathcal{G}}{\partial t}=0. \tag{3.8}$$

This definition states that the surface $\mathcal{M}(s,u,t)$ is, for all time t, the isometric image of the original surface $\mathcal{M}(s,u,t_0)$ defined at some initial time t_0. For a developable surface, $\mathcal{M}(s,u,t)$ can be physically pictured as the parametrization of a waving flag. For a given surface that is rigid, there exists no nontrivial inextensible evolution.

Definition 3.2. *We can define the following one-parameter family of binormal developable ruled surface*

$$\mathcal{M}(s,u)=\gamma(s)+u\mathbf{e}^{\sigma}_{(2)}(s). \tag{3.9}$$

Theorem 3.3. *Let $\mathcal{M}$ is the binormal developable surface in $\mathbb{E}^3$. If $\dfrac{\partial \mathcal{M}}{\partial t}$ is inextensible, then*

$$\frac{\partial \tau}{\partial t}=0. \tag{3.10}$$

Proof. Assume that $\mathcal{M}(s,u,t)$ be a one-parameter family of binormal developable surface. We show that $\mathcal{M}$ is inextensible.

$$\mathcal{M}_s(s,u,t)=\mathbf{e}^{\sigma}_{(0)}-u\tau\mathbf{e}^{\sigma}_{(1)},$$
$$\mathcal{M}_u(s,u,t)=\mathbf{e}^{\sigma}_{(2)}(s).$$

If we compute first fundamental form $\{\mathcal{E},\mathcal{F},\mathcal{G}\}$, we obtain

$$\begin{aligned}\mathcal{E}&=u^2\tau^2+1,\\ \mathcal{F}&=g(\mathcal{M}_s,\mathcal{M}_u)=0,\\ \mathcal{G}&=g(\mathcal{M}_u,\mathcal{M}_u)=1.\end{aligned} \tag{3.11}$$

Using above system, we obtain

$$\frac{\partial \mathcal{E}}{\partial t}=2u^2\tau\frac{\partial \tau}{\partial t}.$$
$$\frac{\partial \mathcal{F}}{\partial t}=0,$$
$$\frac{\partial \mathcal{G}}{\partial t}=0.$$

Therefore, $\dfrac{\partial \mathcal{M}}{\partial t}$ is inextensible iff

$$\frac{\partial \tau}{\partial t}=0.$$

Corollary 3.4. *Let $\mathcal{M}$ is the binormal developable surface in $\mathbb{E}^3$. If flow of this binormal developable surface is inextensible, then this surface is minimal iff*

$$\frac{1}{\mathcal{N}}[u^2\frac{\partial\kappa}{\partial s}\tau+u\frac{\partial\tau}{\partial s}-u^2\frac{\partial\tau}{\partial s}\kappa]=0, \tag{3.12}$$

where $\mathcal{N}=\left[1+u^2\tau^2\right]^{\frac{1}{2}}$.

Proof. Using $\mathcal{M}_s$ and $\mathcal{M}_u$, we get

$$\begin{aligned}\mathcal{M}_{ss}&=u\kappa\tau\mathbf{e}^{\sigma}_{(0)}+(\kappa-u\frac{\partial\tau}{\partial s})\mathbf{e}^{\sigma}_{(1)}-u\tau^2\mathbf{e}^{\sigma}_{(2)},\\ \mathcal{M}_{su}&=-\tau\mathbf{e}^{\sigma}_{(2)},\\ \mathcal{M}_{uu}&=0.\end{aligned} \tag{3.13}$$

On the other hand, the standard unit normal vector field on a surface $\mathcal{M}$ is

$$\varpi=-\frac{1}{\wp}[\mathbf{e}^{\sigma}_{(1)}+u\tau\mathbf{e}^{\sigma}_{(0)}],$$

where $\wp=\left[1+u^2\tau^2\right]^{\frac{1}{2}}$.

Components of second fundamental form of developable surface are

$$\begin{aligned}e&=-\frac{1}{\mathcal{N}}[u^2\kappa\tau^2+\kappa-u\frac{\partial\tau}{\partial s}],\\ f&=\frac{\tau}{\mathcal{N}},\\ g&=0.\end{aligned} \tag{3.14}$$

Therefore, using above system and Eq.(3.9), we obtain

$$\mathbf{H}=-\frac{1}{2\wp^3}[u^2\kappa\tau^2+\kappa-u\frac{\partial\tau}{\partial s}], \tag{3.15}$$

where $\wp=\left[1+u^2\tau^2\right]^{\frac{1}{2}}$.

By using the relations Eq.(3.14) and Eq.(3.15), we obtain Eq.(3.12). This completes the proof.

Thus, we have the following result without proof.

Corollary 3.5. *Let $\mathcal{M}$ is the binormal developable surface in E^3. If γ is a helix, then this surface is minimal.*

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Muş Alparslan University, Department of Mathematics,49250, Muş, Turkey, Firat University, Department of Mathematics,23119, Elaziğ, Turkey
E-mail address: selcukbas79@gmail.com