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ON CHARACTERIZATION OF FERMI-WALKER DERIVATIVE FOR FOCAL CURVES ACCORDING TO RIBBON FRAME

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Abstract. In this paper, we study a new construction of curves by Fermi-Walker parallelism and derivative with Ribbon frame. Finally, we give Fermi-Walker parallelism and derivative for a focal curve according to Ribbon frame.

1. Introduction

Fermi Walker derivative in the Lorentzian manifolds is a generalization of covariant differentation. The derivatives of the spacelike unit vectors in general relativity is taken with respect to timelike unit vectors. Additionaly, this derivative is used to define non-inertial but non-rotation frames. In the special case of inertial frames, the dervative degrade covariant derivative. Fermi Walker derivative along any curve preserves the orthogonality of the tangent space parellel. A large class of reference frames in space-time experience inertial forces and therefore constitute noninertial frames. For instance, the frame adapted to an observer "at rest" on the surface of the (rotating) Earth is noninertial. Observers that follow arbitrary timelike trajectories in space-time will regard as natural a reference frame in which they are at rest, and their spatial axes do not rotate. Fermi–Walker transported frames are important in several investigations. All inner products are invariant under this transport [13]. A frame that undergoes linear and rotational acceleration may be described by the Frenet– Serret frame. In [10], authors considered tetrad fields as reference frames adapted to observers that move along arbitrary timelike trajectories in space-time. By means of a local Lorentz transformation, they transformed these frames into Fermi–Walker transported frames. Moreover, they presented a simple prescription for the construction of Fermi–Walker transported frames out of an arbitrary set of tetrad fields. Recently, Karakuş, Yaylı and Körpınar have made several studies related to Fermi-Walker derivation [6,7].

On the other hand, for any unit tangent vector of the curve , the focal curve is defined as the centers of the osculating spheres. R.U. Vargas examined the properties of the focal curve [12]. G. Öztürk and K. Arslan gave some results about the focal curve in Euclidean n-space[11-14]. N. Gürses, Ö. Bektaş and S. Yüce studied the Quaternionic Focal Curves [17]. Furthermore, authors gave differential applications of the curve and surfaces [5-15,16,15-24].

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Ribbon theory may be applicable in DNA and protein folding studies [7,8]. In addition to, the rdifferential geometry of ribbons have been considered in connection with statical models for models for polymers [9]. Bohr and Markvorsen also studied ribbon frame [4].

The main goal this article is analyzed Fermi-Walker derivations along the focal curve according to ribbon frame.

2. Preliminaries

To construct any ribbon – and hence its center curve – we will use two continuous functions, $w(s)$ and $\theta(s)$ defined in the interval $s \in [0, L]$, where L is the intrinsic length of the ribbon under construction. Throughout we assume that $\theta(s) \in]0, \pi[$ for all s so that in particular $\sin(\theta(s)) > 0$ for all s. In our construction a key role is played by a unit vector field $\mathbf{a}(s)$, which is a field tangent to the ribbon and defined by having the angle $\theta(s)$ to the center curve of the ribbon. In fact, $\mathbf{a}(s)$ will be the direction field for the Darboux vector $\mathbf{D}(s)$ with the generating function $w(s)$ as a multiplying factor, i.e. $\mathbf{D}(s) = w(s)\mathbf{a}(s)$, [4].

We let $\{e(s), f(s), g(s)\}$ denote the unique orthonormal triple of vector solutions to the following differential system:

$$\begin{aligned} e'(s) &= w(s)a(s) \times e(s), \\ f'(s) &= w(s)a(s) \times f(s), \\ g'(s) &= w(s)a(s) \times g(s), \end{aligned}$$

where the unit vector field $\mathbf{a}(s)$ is defined in terms of $e(s)$ and $g(s)$ as follows:

$$\mathbf{a}(s) = \cos(\theta(s))e(s) + \sin(\theta(s))g(s),$$

and where – for uniqueness purposes – we also apply the following arbitrary initial conditions referring to a given fixed coordinate system and basis in $\mathbb{R}^3$

$$\{e(0), f(0), g(0)\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

The above system can be written explicitly as follows:

$$\begin{aligned} e'(s) &= w(s)\sin(\theta(s))f(s), \\ f'(s) &= -w(s)\sin(\theta(s))e(s) + w(s)\cos(\theta(s))g(s), \\ g'(s) &= -w(s)\cos(\theta(s))f(s), \end{aligned}$$

Definition 2.1. *X is any vector field and $\alpha(s)$ is unit speed any curve in space.*

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - < T, X > A + < A, X > T$$

defined as $\frac{\tilde{\nabla} X}{\tilde{\nabla} s}$ derivative is called Femi-Walker derivative. Here $T = \frac{d\alpha}{ds}$, $A = \frac{dT}{ds}$ [6].

Definition 2.2. $\alpha(s)$ is unit-speed on any curve and $\{T, N, B\}$ is the Frenet Frame of $\alpha(s)$.

$$\frac{\tilde{\nabla} T}{\tilde{\nabla} s} = w^* \wedge T$$
$$\frac{\tilde{\nabla} N}{\tilde{\nabla} s} = w^* \wedge \wedge N$$
$$\frac{\tilde{\nabla} B}{\tilde{\nabla} s} = w^* \wedge B$$

w^* is Fermi-Walker terms Darboux vector according to the Frenet frame [6].

Definition 2.3. X is any vector field along the $\alpha(s)$ space curve. If the Fermi-Walker derivative of the vector field X

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = 0$$

the vector field X along the curve, parallel to the Fermi-Walker terms is called [6].

3. Ribbon Frame and Fermi-Walker Derivative

Theorem 3.1. Let $\{e, f, g\}$ and X be ribbon frame and any vector field along the $\alpha(s)$ space curve, respectively . Fermi-Walker derivative can be expressed along the curve on ribbon frame as follows:

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - w(s) \sin\theta(s)(g \wedge X).$$

Proof:

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - < e, X > \frac{de}{ds} + < \frac{de}{ds}, X > e,$$
$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - < e, X > w(s)\sin\theta(s) f + < w(s)\sin\theta(s) f, X > e,$$
$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - w(s)\sin\theta(s)[< e, X > f - < f, X > e],$$
$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - w(s)\sin\theta(s)(g \wedge X).$$

Theorem 3.2. Let $\alpha(s)$ be a unit-speed curve. Then $X = \lambda_1 e + \lambda_2 f + \lambda_3 g$ vector field according to ribbon frame along the curve is parallel to Fermi-Walker terms if and only if:

$$\lambda_1 = \text{cons.},$$
$$\lambda_2 = c_1 \cos(\int_1^s w(s)\cos\theta(s) ds) + c_2 \sin(\int_1^s w(s)\cos\theta(s) ds),$$
$$\lambda_3 = c_2 \cos(\int_1^s w(s)\cos\theta(s) ds) - c_1 \sin(\int_1^s w(s)\cos\theta(s) ds),$$

where $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions according to real parameter s.

Proof. $\Longrightarrow$:The vector field X is parallel to the Fermi-Walker terms. Then

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - w(s)\sin\theta(s)(g \wedge X),$$
$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{d\lambda_1}{ds}e + (\frac{d\lambda_2}{ds} - w(s)\cos\theta(s)\lambda_3)f + (\frac{d\lambda_3}{ds} + w(s)\cos\theta(s)\lambda_2)g$$

is obtained. X is parallel to the Fermi-Walker terms and $\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = 0$ so,

$$\frac{d\lambda_1}{ds} = 0,$$
$$\frac{d\lambda_2}{ds} - w(s)\cos\theta(s)\lambda_3 = 0,$$
$$\frac{d\lambda_3}{ds} + w(s)\cos\theta(s)\lambda_2 = 0$$

is obtained. This the solution of the equation system is given as follows:

$$\lambda_1 = \text{cons.},$$
$$\lambda_2 = c_1\cos(\int_1^s w(s)\cos\theta(s)ds) + c_2\sin(\int_1^s w(s)\cos\theta(s)ds),$$
$$\lambda_3 = c_2\cos(\int_1^s w(s)\cos\theta(s)ds) - c_1\sin(\int_1^s w(s)\cos\theta(s)ds).$$

$\Longleftarrow$: $X = \lambda_1 e + \lambda_2 f + \lambda_3 g$ vector field and $\phi = \int_1^s w(s)\cos\theta(s)ds$,

$$\lambda_1 = \text{cons.},$$
$$\lambda_2 = c_1\cos\phi + c_2\sin\phi,$$
$$\lambda_3 = c_2\cos\phi - c_1\sin\phi,$$

and from theorem 3.1,

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = 0$$

is obtained.

4. Focal Curve According to Ribbon Frame and Fermi-Walker Derivative

Denoting the focal curve by $C_\alpha^{\mathcal{R}}$ according to ribbon frame, we can write

$$C_\alpha^{\mathcal{R}}(s) = (\alpha + c_1 f + c_2 g)(s),$$

where the coefficients $c_1 = \dfrac{1}{w(s)\sin(\theta(s))}$, $c_2 = \dfrac{c_1'}{w(s)\cos(\theta(s))}$ are smooth functions of the parameter of the curve α, called the first and second focal curvatures of α, respectively. Frenet vectors of the focal curve $C_\alpha^{\mathcal{R}}$ according to ribbon frame are given as follows:

$$\begin{aligned} T_F &= (c_2' + \cot\theta(s))g \\ N_F &= (c_2' + \cot\theta(s))'g - (c_2' + \cot\theta(s))w(s)\cos\theta(s)f \\ B_F &= (c_2' + \cot\theta(s))^2 w(s)\cos\theta(s)e, \end{aligned}$$

and

$$\begin{aligned} T_F' &= \kappa_F N_F \\ N_F' &= -\kappa_F T_F + \tau_F B_F \\ B_F' &= -\tau_F N_F. \end{aligned}$$

Theorem 4.1. Let $\{T_F, N_F, B_F\}$ and X be Frenet frame and any vector field along the $C_\alpha^{\mathcal{R}}$ focal curve, respectively . Fermi-Walker derivative can be expressed along the curve on this frame as follows:

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{dX}{ds} - \kappa_F (c_2' + \cot\theta(s))^2 w(s)\cos\theta(s) e \wedge X.$$

Proof.

$$\begin{aligned} \frac{\tilde{\nabla} X}{\tilde{\nabla} s} &= \frac{dX}{ds} - < T_F, X > \frac{dT_F}{ds} + < \frac{dT_F}{ds}, X > T_F, \\ \frac{\tilde{\nabla} X}{\tilde{\nabla} s} &= \frac{dX}{ds} - < T_F, X > \kappa_F N_F + < \kappa_F N_F, X > T_F, \\ \frac{\tilde{\nabla} X}{\tilde{\nabla} s} &= \frac{dX}{ds} - \kappa_F [< T_F, X > N_F - < N_F, X > T_F], \\ \frac{\tilde{\nabla} X}{\tilde{\nabla} s} &= \frac{dX}{ds} - \kappa_F (c_2' + \cot\theta(s))^2 w(s)\cos\theta(s) e \wedge X. \end{aligned}$$

Theorem 4.2. Let $C_\alpha^{\mathcal{R}}$ be a unit-speed focal curve. Then $X = \lambda_1 e + \lambda_2 f + \lambda_3 g$ vector field according to ribbon frame along the curve is parallel to Fermi-Walker terms if and only if:

$$\begin{aligned} \lambda_1 &= \text{cons.}, \\ \lambda_2 &= c_1 \cos(\int_1^s w(s)\cos\theta(s) ds) + c_2 \sin(\int_1^s w(s)\cos\theta(s) ds), \\ \lambda_3 &= c_2 \cos(\int_1^s w(s)\cos\theta(s) ds) - c_1 \sin(\int_1^s w(s)\cos\theta(s) ds), \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions according to real parameter s.

Proof. $\Longrightarrow$:The vector field X is parallel to the Fermi-Walker terms. Then

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = \frac{d\lambda_1}{ds} T_F + (\frac{d\lambda_2}{ds} - \lambda_3 \tau_F) N_F + (\frac{d\lambda_3}{ds} + \lambda_2 \tau_F) B_F$$

is obtained. X is parallel to the Fermi-Walker terms and $\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = 0$ so,

$$\frac{d\lambda_1}{ds} = 0,$$
$$\frac{d\lambda_2}{ds} - \lambda_3 \tau_F = 0,$$
$$\frac{d\lambda_3}{ds} + \lambda_2 \tau_F = 0$$

is obtained. This the solution of the equation system is given as follows:

$$\lambda_1 = \text{cons.},$$
$$\lambda_2 = c_1 \cos(\int_1^s \tau_F ds) + c_2 \sin(\int_1^s \tau_F ds),$$
$$\lambda_3 = c_2 \cos(\int_1^s \tau_F ds) - c_1 \sin(\int_1^s \tau_F ds).$$

$\Longleftarrow$: $X = \lambda_1 e + \lambda_2 f + \lambda_3 g$ vector field and $\phi = \int_1^s \tau_F ds$,

$$\lambda_1 = \text{cons.},$$
$$\lambda_2 = c_1 \cos\phi + c_2 \sin\phi,$$
$$\lambda_3 = c_2 \cos\phi - c_1 \sin\phi,$$

and from theorem 4.1,

$$\frac{\tilde{\nabla} X}{\tilde{\nabla} s} = 0$$

is obtained.

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