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SOLVING THE PROBLEM OF INDUSTRY BY FINDING PARADOX IN FRACTIONAL PLUS FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM

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Abstract. This paper discusses a paradox in a capacitated transportation problem where the objective function is the sum of two fractional functions consisting of variable costs only. A paradoxical situation arises in a transportation problem when value of the objective function falls below the optimal value and this lower value is attainable by shipping larger quantities of goods over the same routes that were previously designated as optimal. Firstly, optimality condition at which a feasible solution of fractional plus fractional capacitated transportation problem will be an optimal solution is established. Then a sufficient condition for the existence of paradox is found. If paradox exists, then the procedure for finding the best paradoxical pair is proposed which ultimately gives a paradoxical range of flows. Moreover, a method is proposed to find the paradoxical solution for a specified flow . Developed algorithm is applied on the real data taken from the account keeping books of the firm D.M Chemicals, Delhi. The solution so obtained by using the developed algorithm is compared with the existing data. Moreover, the solution obtained is verified by a computing software Excel Solver.

1. Introduction

Capacitated transportation problem are bounded variable transportation problem where the decision variables such as number of goods shipped from various sources to different destinations are bounded. Many researchers like Verma and Puri [10], Misra and Das[8], Gupta and Arora [5, 6], Bit and Biswal[2] have contributed a lot in this field.

Sometimes, in a transportation problem, it is possible to find a cheaper solution than the optimal one by shipping more along the optimal routes in such a way that no destination will receive less and no origin will ship less of the product. This phenomenon is called paradox. The source of so-called transportation paradox is unclear. This usual phenomenon was first observed by Szwarc [9] in 1971. Later on, many researchers studied paradox in different types of objective functions under different set of constraints. In 2000, Arora and Ahuja [1] have studied the paradoxical situation in a fixed charge transportation problem. Dahiya and

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Verma [3] had studied the paradox in a non-linear capacitated transportation problem. Joshi and Gupta [7] developed a heuristic for finding the initial basic feasible solution for linear plus linear fractional transportation problem and also established the sufficient condition for the existence of a paradoxical solution in 2010. Gupta and Arora [4] studied paradox in a fractional capacitated transportation problem.

Another class of transportation problem is a fractional transportation problem where the objective function to be optimized is a ratio of two linear functions. Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system . Fractional programs finds its application in a variety of real world problems such as stock cutting problem , resource allocation problems , routing problem for ships and planes , cargo - loading problem , inventory problem and many other problems.

The extensive literature on paradox and capacitated transportation problem motivated us to study paradox in a fractional plus fractional capacitated transportation problem where the objective function is the sum of linear and linear fractional functions. We apply the developed algorithm on data taken from the account keeping books of a trading firm D.M Chemicals, Delhi. This firm deals in the trading of soap stone across various states in India. We contacted the manager of the firm and asked him about the business transactions, sellers, buyers, cartage , cost price per unit, selling price per unit etc. The manager told us that he wishes to determine the quantity (in tons) of soap stone that the firm should purchase from different sellers and sell it to the different buyers such that the ratio of actual cartage to standard cartage plus ratio of purchasing cost to profit is minimized provided the demand and supply conditions are satisfied. Surely, the firm would be benefited if it is possible to supply more number of goods at a cost lesser than the optimal one. Conversation that we had with the manager and the data that we obtained from the books of the firm motivated us to study paradox in a fractional plus fractional capacitated transportation problem.

This paper is organized as follows: In section 2, fractional plus fractional capacitated transportation problem is formulated. In section 3 optimality criterion for the solution of fractional plus fractional capacitated transportation problem is developed . In section 4, theory is developed for the existence of paradox. In section 5,sufficient condition for the existence of a paradox is developed. In section 6, method to determine the best paradoxical pair is proposed and in section 7, method to get a paradoxical solution for a specified flow within the paradoxical range of flows is developed . In section 8, an algorithm to find paradoxical situation, best paradoxical pair, paradoxical range of flows and all possible paradoxical pairs within this range is presented. In section 9, data taken from the account keeping books of the firm D.M Chemicals, Delhi is presented. In section 10, the developed algorithm is applied on the data of the firm and the solution so obtained is then compared with the existing data.

2. Problem Formulation

Let $I = \{1, 2, ..., m\}$ be the index set of m sellers.
$J = \{1, 2, ..., n\}$ is the index set of n buyers.
x_{ij} =the number of units purchased from the i^{th} seller and sold to the j^{th} buyer.
c_{ij} = cost of purchasing one unit of a commodity from the i^{th} seller and selling it to the j^{th} buyer.
d_{ij} = profit per unit earned from the j^{th} buyer when the goods purchased from the i^{th} seller are supplied.
e_{ij} = actual cartage of transporting one unit of a commodity from the i^{th} seller to the j^{th} buyer.
f_{ij} = standard cartage of transporting one unit of a commodity from the i^{th} seller to the j^{th} buyer.
l_{ij} and u_{ij} are the lower and upper bounds on number of units to be transported from the i^{th} seller to the j^{th} buyer.
a_i = number of units available at seller i
b_j = number of units demanded by the buyer j

Consider a fractional plus fractional capacitated transportation problem given by :

$$(P_0) : \min\{\frac{\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} d_{ij}x_{ij}} + \frac{\sum\limits_{i\in I}\sum\limits_{j\in J} e_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j\in J} x_{ij} \leq a_i, \forall i \in I$$

$$\sum_{i\in I} x_{ij} = b_j, \forall j \in J$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ and integers } \forall i \in I, \forall j \in J$$

It can be easily seen that the problem (P_0) is equivalent to the following balanced transportation problem.

$$(P_0^{'}) : \min\{\frac{\sum\limits_{i\in I}\sum\limits_{j\in J'} c_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J'} d_{ij}x_{ij}} + \frac{\sum\limits_{i\in I}\sum\limits_{j\in J'} e_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J'} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j\in J'} x_{ij} = a_i, \forall i \in I$$

$$\sum_{i\in I} x_{ij} = b_j, \forall j \in J'$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J$$

$$b_{n+1} = \sum_{i\in I} a_i - \sum_{j\in J} b_j$$

$$l_{i,n+1} = 0; u_{i,n+1} \geq 0; c_{i,n+1} = 0; \forall i \in I$$

$$d_{i,n+1} = 0; e_{i,n+1} = 0; f_{i,n+1} = 0; \forall i \in I$$

where $J = \{1, 2, ..., n, n+1\}$

3. Optimality criteria for a fractional plus fractional capacitated transportation problem

Theorem 3.1. *Let $X^0 = \{x^0_{ij}\}_{I\times J}$ be a feasible solution of problem (P_0). Let $C^0 = \sum_{i\in I}\sum_{j\in J} c_{ij}x^0_{ij}; D^0 = \sum_{i\in I}\sum_{j\in J} d_{ij}x^0_{ij}; E^0 = \sum_{i\in I}\sum_{j\in J} e_{ij}x^0_{ij}; F^0 = \sum_{i\in I}\sum_{j\in J} e_{ij}x^0_{ij}$. Let B be the set of cells (i,j) which are basic and N_1 and N_2 denotes the set of non-basic cells (i,j) which are at their lower bounds and upper bounds respectively. Let $u^1_i, u^2_i, u^3_i, u^4_i, v^1_j, v^2_j, v^3_j, v^4_j; i \in I, j \in J$ be the dual variables such that $u^1_i + v^1_j = c_{ij}, \forall (i,j) \in B; u^2_i + v^2_j = d_{ij}, \forall (i,j) \in B; u^3_i + v^3_j = e_{ij}, \forall (i,j) \in B; u^4_i + v^4_j = f_{ij}, \forall (i,j) \in B; u^1_i + v^1_j = z^1_{ij}, \forall (i,j) \notin B; u^2_i + v^2_j = z^2_{ij}, \forall (i,j) \notin B; u^3_i + v^3_j = z^3_{ij}, \forall (i,j) \notin B; u^4_i + v^4_j = z^4_{ij}, \forall (i,j) \notin B$. Then a feasible solution $X^0 = \{x^0_{ij}\}_{I\times J}$ of problem (P_0) with objective function value $\frac{C^0}{D^0} + \frac{E^0}{F^0}$ will be an optimal solution if and only if the following conditions holds.*

$$\delta^1_{ij} = \frac{\theta_{ij}[D^0(c_{ij} - z^1_{ij}) - C^0(d_{ij} - z^2_{ij})]}{D^0[D^0 + \theta_{ij}(d_{ij} - z^2_{ij})]} + \frac{\theta_{ij}[F^0(e_{ij} - z^3_{ij}) - E^0(f_{ij} - z^4_{ij})]}{F^0[F^0 + \theta_{ij}(f_{ij} - z^4_{ij})]} \geq 0; \forall (i,j) \in N_1$$

$$\delta^2_{ij} = \frac{-\theta_{ij}[D^0(c_{ij} - z^1_{ij}) - C^0(d_{ij} - z^2_{ij})]}{D^0[D^0 - \theta_{ij}(d_{ij} - z^2_{ij})]} - \frac{\theta_{ij}[F^0(e_{ij} - z^3_{ij}) - E^0(f_{ij} - z^4_{ij})]}{F^0[F^0 - \theta_{ij}(f_{ij} - z^4_{ij})]} \geq 0; \forall (i,j) \in N_2$$

Proof. Let $X^0 = \{x^0_{ij}\}_{I\times J}$ be a feasible solution of problem (P_0) with equality constraints. Let z^0 be the corresponding value of objective function. Then

$$z = [\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x^0_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x^0_{ij}} + \frac{\sum_{i\in I}\sum_{j\in J} e_{ij}x^0_{ij}}{\sum_{i\in I}\sum_{j\in J} f_{ij}x^0_{ij}}] = \frac{C^0}{D^0} + \frac{E^0}{F^0}$$

$$= \frac{\sum_{i\in I}\sum_{j\in J}(c_{ij} - u^1_i - v^1_j)x^0_{ij} + \sum_{i\in I}\sum_{j\in J}(u^1_i + v^1_j)x^0_{ij}}{\sum_{i\in I}\sum_{j\in J}(d_{ij} - u^2_i - v^2_j)x^0_{ij} + \sum_{i\in I}\sum_{j\in J}(u^2_i + v^2_j)x^0_{ij}}$$
$$+ \frac{\sum_{i\in I}\sum_{j\in J}(e_{ij} - u^3_i - v^3_j)x^0_{ij} + \sum_{i\in I}\sum_{j\in J}(u^3_i + v^3_j)x^0_{ij}}{\sum_{i\in I}\sum_{j\in J}(f_{ij} - u^4_i - v^4_j)x^0_{ij} + \sum_{i\in I}\sum_{j\in J}(u^4_i + v^4_j)x^0_{ij}}$$

$$= \frac{\sum\sum_{(i,j)\in N_1}(c_{ij}-z^1_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(c_{ij}-z^1_{ij})u_{ij}+\sum_{i\in I}\sum_{j\in J}(u^1_i+v^1_j)x^0_{ij}}{\sum\sum_{(i,j)\in N_1}(d_{ij}-z^2_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(d_{ij}-z^2_{ij})u_{ij}+\sum_{i\in I}\sum_{j\in J}(u^2_i+v^2_j)x^0_{ij}}$$
$$+\frac{\sum\sum_{(i,j)\in N_1}(e_{ij}-z^3_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(e_{ij}-z^3_{ij})u_{ij}+\sum_{i\in I}\sum_{j\in J}(u^3_i+v^3_j)x^0_{ij}}{\sum\sum_{(i,j)\in N_1}(f_{ij}-z^4_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(f_{ij}-z^4_{ij})u_{ij}+\sum_{i\in I}\sum_{j\in J}(u^4_i+v^4_j)x^0_{ij}}$$

$$= \frac{\sum\sum_{(i,j)\in N_1}(c_{ij}-z^1_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(c_{ij}-z^1_{ij})u_{ij}+\sum_{i\in I}a_iu^1_i+\sum_{j\in J}b_jv^1_j}{\sum\sum_{(i,j)\in N_1}(d_{ij}-z^2_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(d_{ij}-z^2_{ij})u_{ij}+\sum_{i\in I}a_iu^2_i+\sum_{j\in J}b_jv^2_j}$$
$$+\frac{\sum\sum_{(i,j)\in N_1}(e_{ij}-z^3_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(e_{ij}-z^3_{ij})u_{ij}+\sum_{i\in I}a_iu^3_i+\sum_{j\in J}b_jv^3_j}{\sum\sum_{(i,j)\in N_1}(f_{ij}-z^4_{ij})l_{ij}+\sum\sum_{(i,j)\in N_2}(f_{ij}-z^4_{ij})u_{ij}+\sum_{i\in I}a_iu^4_i+\sum_{j\in J}b_jv^4_j}$$

Let some non-basic variable $x_{ij}\in N_1$ undergoes change by an amount θ_{rs} where θ_{rs} is given by $\min\{u_{rs}-l_{rs}; x^0_{ij}-l_{ij}$ for all basic cells (i,j) with a $(-\theta)$ entry in θ-loop;$u_{ij}-x^0_{ij}$ for all basic cells (i,j) with a $(+\theta)$ entry in θ-loop. Then the new value of the objective function $\widehat{z}$ will be given by

$$\widehat{z}=\frac{C^0+\theta_{rs}(c_{rs}-z^1_{rs})}{D^0+\theta_{rs}(d_{rs}-z^2_{rs})}+\frac{E^0+\theta_{rs}(e_{rs}-z^3_{rs})}{F^0+\theta_{rs}(f_{rs}-z^4_{rs})}$$

$$\widehat{z}-z^0=\frac{C^0+\theta_{rs}(c_{rs}-z^1_{rs})}{D^0+\theta_{rs}(d_{rs}-z^2_{rs})}-\frac{C^0}{D^0}+\frac{E^0+\theta_{rs}(e_{rs}-z^3_{rs})}{F^0+\theta_{rs}(f_{rs}-z^4_{rs})}-\frac{E^0}{F^0}$$

$$=\frac{\theta_{rs}[D^0(c_{rs}-z^1_{rs})-C^0(d_{rs}-z^2_{rs})]}{D^0[D^0+\theta_{rs}(d_{rs}-z^2_{rs})]}+\frac{\theta_{rs}[F^0(e_{rs}-z^3_{rs})-E^0(f_{rs}-z^4_{rs})]}{F^0[F^0+\theta_{ij}(f_{rs}-z^4_{rs})]}=\delta^1_{rs}(say)$$

Similarly, when some non- basic variable $x_{pq}\in N_2$ undergoes change by an amount θ_{pq} then

$$\widehat{z}-z^0=\frac{-\theta_{pq}[D^0(c_{pq}-z^1_{pq})-C^0(d_{pq}-z^2_{pq})]}{D^0[D^0-\theta_{pq}(d_{pq}-z^2_{pq})]}-\frac{\theta_{pq}[F^0(e_{pq}-z^3_{pq})-E^0(f_{pq}-z^4_{pq})]}{F^0[F^0-\theta_{pq}(f_{pq}-z^4_{pq})]}=\delta^2_{pq}(say)$$

Converse can be proved similarly. Hence X^0 will be an optimal solution if and only if $\delta^1_{ij}\geq 0;(i,j)\in N_1$ and $\delta^2_{ij}\geq 0;(i,j)\in N_2$. □

4. Theoretical Development

Let an optimal solution of (P_0) yield value z^0 of the objective function and $F^0=\sum_{i\in I}a^{'}_i=\sum_{j\in J}b^{'}_j$ be the corresponding flow where $a^{'}_i\leq a_i, i\in I; b^{'}_j=b_j, j\in J$.A paradox exists if more than F^0 is flown at an objective function value less than z^0 . It may be observed that flow can be increased by an increase of a certain $a_i^{'}$ and $b_j^{'}$.This gives rise to the following problem (P_1).

$$(P_1): \min\{\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}} + \frac{\sum_{i\in I}\sum_{j\in J} e_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i^{'}, \forall i \in I$$

$$\sum_{i\in I} x_{ij} \geq b_j^{'}, \forall j \in J$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J$$

The feasible region of (P_1) being larger than that of (P_0) implies that the minimum objective function value z^1 of (P_1) is not greater than z^0 i.e $z^1 < z^0$. So more goods may be flown than that in (P_0) at an objective function value less than that of (P_0). Hence a paradox may arise in this case. To solve (P_1) , we consider the related transportation problem (RP_1) with an additional supply point and an additional destination given by

$$(RP_1): \min\{\frac{\sum_{i\in I'}\sum_{j\in J'} c_{ij}^{'}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d_{ij}^{'}y_{ij}} + \frac{\sum_{i\in I'}\sum_{j\in J'} e_{ij}^{'}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} f_{ij}^{'}y_{ij}}\}$$

subject to

$$\sum_{j\in J'} y_{ij} = A_i^{'}, \forall i \in I^{'}$$

$$\sum_{i\in I'} y_{ij} = B_j^{'}, \forall j \in J^{'}$$

$$l_{ij} \leq y_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J$$

$$0 \leq y_{m+1,j} \leq \sum_{i\in I} u_{ij} - b_j^{'}, \forall j \in J$$

$$0 \leq y_{i,n+1} \leq \sum_{j\in J} u_{ij} - a_i^{'}, \forall i \in I$$

$$y_{m+1,n+1} \geq 0 \text{ and integers}$$

$$A_i^{'} = \sum_{j\in J} u_{ij}, \forall i \in I,\ A_{m+1}^{'} = \sum_{i\in I}\sum_{j\in J} u_{ij} = B_{n+1}^{'}; B_j^{'} = \sum_{i\in I} u_{ij},\ \forall j \in J$$

$$c_{ij}^{'} = c_{ij}, c_{m+1,j}^{'} = c_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ c_{m+1,n+1}^{'} = 0$$

$$d_{ij}^{'} = d_{ij}, d_{m+1,j}^{'} = d_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ d_{m+1,n+1}^{'} = 0$$

$$e_{ij}^{'} = e_{ij}, e_{m+1,j}^{'} = e_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ e_{m+1,n+1}^{'} = 0$$

$$f_{ij}^{'} = f_{ij}, f_{m+1,j}^{'} = f_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ f_{m+1,n+1}^{'} = 0$$

$I^{'}=\{1,2,\cdots,m,m+1\}, J^{'}=\{1,2,\cdots,n,n+1\}$

It can be easily shown that (P_1) and (RP_1) are equivalent .

Theorem 4.1. *There is a one -to-one correspondence between the feasible solution to problem (P_1) and the feasible solution to problem (RP_1).* [4]

Theorem 4.2. *The value of the objective function of problem (P_1) at a feasible solution $\{x_{ij}\}_{I\times J}$ is equal to the value of the objective function of (RP_1) at its corresponding feasible solution $\{y_{ij}\}_{I^{'}\times J^{'}}$ and conversely.*

Proof. The value of the objective function of(RP_1) at a feasible solution $\{y_{ij}\}_{I^{'}\times J^{'}}$ is

$$z=[\frac{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}}c^{'}_{ij}y_{ij}}{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}}d^{'}_{ij}y_{ij}}+\frac{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}}e^{'}_{ij}y_{ij}}{\sum\limits_{i\in I^{'}}\sum\limits_{j\in J^{'}}f^{'}_{ij}y_{ij}}]$$

$$=[\frac{\sum\limits_{i\in I}\sum\limits_{j\in J}c^{'}_{ij}y_{ij}+\sum\limits_{j\in J}c^{'}_{m+1,j}y_{m+1,j}+\sum\limits_{j\in J}c^{'}_{i,n+1}y_{i,n+1}+c^{'}_{m+1,n+1}y_{m+1,n+1}}{\sum\limits_{i\in I}\sum\limits_{j\in J}d^{'}_{ij}y_{ij}+\sum\limits_{j\in J}d^{'}_{m+1,j}y_{m+1,j}+\sum\limits_{j\in J}d^{'}_{i,n+1}y_{i,n+1}+d^{'}_{m+1,n+1}y_{m+1,n+1}}]$$
$$+[\frac{\sum\limits_{i\in I}\sum\limits_{j\in J}e^{'}_{ij}y_{ij}+\sum\limits_{j\in J}e^{'}_{m+1,j}y_{m+1,j}+\sum\limits_{j\in J}e^{'}_{i,n+1}y_{i,n+1}+e^{'}_{m+1,n+1}y_{m+1,n+1}}{\sum\limits_{i\in I}\sum\limits_{j\in J}f^{'}_{ij}y_{ij}+\sum\limits_{j\in J}f^{'}_{m+1,j}y_{m+1,j}+\sum\limits_{j\in J}f^{'}_{i,n+1}y_{i,n+1}+f^{'}_{m+1,n+1}y_{m+1,n+1}}]$$

$$=[\frac{\sum\limits_{i\in I}\sum\limits_{j\in J}c_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J}d_{ij}x_{ij}}+\frac{\sum\limits_{i\in I}\sum\limits_{j\in J}e_{ij}x_{ij}}{\sum\limits_{i\in I}\sum\limits_{j\in J}f_{ij}x_{ij}}]$$

= value of objective function of problem(P_1)at its corresponding feasible solution$\{x_{ij}\}_{I\times J}$.

because$\forall i\in I, j\in J$

$c^{'}_{ij}=c_{ij}, d^{'}_{ij}=d_{ij}, e^{'}_{ij}=e_{ij}, f^{'}_{ij}=f_{ij}$

$x_{ij}=y_{ij}$

$c^{'}_{i,n+1}=c^{'}_{m+1,j}=d^{'}_{i,n+1}=d^{'}_{m+1,j}=0$

$e^{'}_{i,n+1}=e^{'}_{m+1,j}=f^{'}_{i,n+1}=f^{'}_{m+1,j}=0$

$c^{'}_{m+1,n+1}=d^{'}_{m+1,n+1}=e^{'}_{m+1,n+1}=f^{'}_{m+1,n+1}=0$

The converse can be proved in a similar way. □

Theorem 4.3. *There is a one -to-one correspondence between the optimal solution to (P_1) and optimal solution to (RP_1).*[4]

Definition 4.4. Paradoxical Pair : An objective function flow pair (Z, F) of problem (P_1) is called a paradoxical pair if $Z < Z^0$ and $F > F^0$.

Definition 4.5. Best Paradoxical Pair : The paradoxical pair (Z^1, F^1) is called the best paradoxical pair if $Z^1 = \min\{Z : (Z, F) \textit{ is a paradoxical pair}\}$ and $F^1 = \max\{F : (Z, F) \textit{ is a paradoxical pair}\}$

Definition 4.6. Paradoxical Range of Flows: Paradoxical range of flows is $[F^0, F^1]$ where F^1 is the flow corresponding to the best paradoxical pair. All objective function flow pairs in this range are paradoxical pairs.

5. Sufficient condition for the existence of a paradoxical solution

Let X^0 be the basic feasible solution of problem (P_0) . Let B denotes the set of cells (i, j) which are basic and N_1 and N_2 denotes the set of non- basic cells (i, j) which are at their lower bounds and upper bounds respectively. Let z^0 be the corresponding value of the objective function and $F^0 = \sum_{i\in I} a_i' = \sum_{j\in J} b_j'$ be the corresponding flow , where $a_i' \leq a_i, i \in I; b_j' = b_j, j \in J$. Let $u_i^1, u_i^2, u_i^3, u_i^4, v_j^1, v_j^2, v_j^3, v_j^4; i \in I, j \in J$ be the dual variables such that $u_i^1 + v_j^1 = c_{ij}, \forall (i, j) \in B; u_i^2 + v_j^2 = d_{ij}, \forall (i, j) \in B; u_i^3 + v_j^3 = e_{ij}, \forall (i, j) \in B; u_i^4 + v_j^4 = f_{ij}, \forall (i, j) \in B; u_i^1 + v_j^1 = z_{ij}^1, \forall (i, j) \notin B; u_i^2 + v_j^2 = z_{ij}^2, \forall (i, j) \notin B; u_i^3 + v_j^3 = z_{ij}^3, \forall (i, j) \notin B; u_i^4 + v_j^4 = z_{ij}^4, \forall (i, j) \notin B$. Then as in Theorem 1,

$$
\begin{aligned}
z^0 &= \frac{\sum\sum_{(i,j)\in N_1}(c_{ij} - z_{ij}^1)l_{ij} + \sum\sum_{(i,j)\in N_2}(c_{ij} - z_{ij}^1)u_{ij} + \sum_{i\in I} a_i' u_i^1 + \sum_{j\in J} b_j' v_j^1}{\sum\sum_{(i,j)\in N_1}(d_{ij} - z_{ij}^2)l_{ij} + \sum\sum_{(i,j)\in N_2}(d_{ij} - z_{ij}^2)u_{ij} + \sum_{i\in I'} a_i' u_i^2 + \sum_{j\in J'} b_j' v_j^2} \\
&+ \frac{\sum\sum_{(i,j)\in N_1}(e_{ij} - z_{ij}^3)l_{ij} + \sum\sum_{(i,j)\in N_2}(e_{ij} - z_{ij}^3)u_{ij} + \sum_{i\in I'} a_i' u_i^3 + \sum_{j\in J'} b_j' v_j^3}{\sum\sum_{(i,j)\in N_1}(f_{ij} - z_{ij}^4)l_{ij} + \sum\sum_{(i,j)\in N_2}(f_{ij} - z_{ij}^4)u_{ij} + \sum_{i\in I'} a_i' u_i^4 + \sum_{j\in J'} b_j' v_j^4} \\
&= \frac{C^0}{D^0} + \frac{E^0}{F^0}
\end{aligned}
$$

Now suppose that a_p' is replaced by $a_p' + \lambda$ and b_q' by $b_q' + \lambda$ where $\lambda > 0$ is such that same basis B remains optimal after replacement. Then the new value z^1 of the objective function is given by

$$
z^1 = \frac{C^0 + \lambda(u_p^1 + v_q^1)}{D^0 + \lambda(u_p^2 + v_q^2)} + \frac{E^0 + \lambda(u_p^3 + v_q^3)}{F^0 + \lambda(u_p^4 + v_q^4)}
$$

$$
z^1 - z^0 = \frac{C^0 + \lambda(u_p^1 + v_q^1)}{D^0 + \lambda(u_p^2 + v_q^2)} - \frac{C^0}{D^0} + \frac{E^0 + \lambda(u_p^3 + v_q^3)}{F^0 + \lambda(u_p^4 + v_q^4)} - \frac{E^0}{F^0}
$$

$$= \frac{\lambda[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)]}{D^0(D^0 + \lambda(u_p^2 + v_q^2))} + \frac{\lambda[F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)]}{F^0(F^0 + \lambda(u_p^4 + v_q^4))}$$

On simplifying the above equation, we get the denominator as -

$$D^0(D^0 + \lambda(u_p^2 + v_q^2))F^0(F^0 + \lambda(u_p^4 + v_q^4))$$

and numerator as-

$$\lambda[[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)]F^0(F^0 + \lambda(u_p^4 + v_q^4))$$
$$+[F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)]D^0(D^0 + \lambda(u_p^2 + v_q^2))]$$

Now, the denominator is always positive and $\lambda > 0$, therefore,$z^1 < z^0$ if
Condition (1) : $[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)]F^0(F^0 + \lambda(u_p^4 + v_q^4))$
$+[F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)]D^0(D^0 + \lambda(u_p^2 + v_q^2)) < 0$
As $\lambda > 0, D^0, (D^0 + \lambda(u_p^2 + v_q^2), F^0, (F^0 + \lambda(u_p^4 + v_q^4)) > 0$, condition (1) implies that to obtain paradoxical solution , we consider only those cells (p, q) for which any of the following conditions hold.
Condition(a) : $[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)] < 0$ and $[F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)] < 0$
Condition(b) : If any one of the expressions $[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)]$ and $[F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)]$ are negative then condition(1)should hold for some $\lambda > 0$. For simplicity, take $\lambda = 1$.
Thus if there exists a non- basic cell (p, q) for which either of the above two conditions (a) or (b) holds true, then the new value z^1 of the objective function is less than z^0. Hence the flow is increased by λ but the objective function value is reduced i.e. a paradox exists. This result can be stated as :

Theorem 5.1. *Let X^0 be an optimal solution of the problem (P_0) with the objective function value $z^0 = \frac{C^0}{D^0} + \frac{E^0}{F^0}$. If there exists a non- basic cell (p, q) such that on changing a_p by $a_p + \lambda$ and b_q by $b_q + \lambda$ where $\lambda > 0$ and the basis remaining the same , the condition (a) or (b) is satisfied , then there exists a paradox.*

6. Best Paradoxical Pair

If a paradox exist, one would obviously be interested in the best paradoxical pair. Theorem 6 below proves that the optimal solution of the problem (P_1) yields the best paradoxical pair.

Theorem 6.1. *Global optimal solution of the problem (P_1) yields the best paradoxical pair.* [4]

7. Paradoxical solution for a specified flow in $[F^0, F^1]$

Let $F^0 = \sum_{i \in I} a_i^{'} = \sum_{j \in J} b_j^{'}$ be the flow corresponding to the optimal solution X^0 of the problem (P_0) where $a_i^{'} \leq a_i; i \in I; b_j^{'} = b_j; j \in J$.Also let F^1 be the flow corresponding to the optimal solution X^1 of the problem (P_1). Then $[F^0, F^1]$ is the paradoxical range of flows. Let the specified flow be $P \in [F^0, F^1]$. The paradoxical solution for P is given by the optimal solution of the problem (P_2) given below.

$$(P_2): \min\{\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}} + \frac{\sum_{i\in I}\sum_{j\in J} e_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j\in J} x_{ij} \geq a_i^{'}, \forall i \in I$$

$$\sum_{i\in I} x_{ij} \geq b_j^{'}, \forall j \in J$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J$$

The problem (P_2) is different from the problem (P_1) because of the flow constraint. To solve the problem (P_2) , we consider the following related transportation problem (RP_2) with an additional origin and an additional destination.

$$(RP_2): \min\{\frac{\sum_{i\in I'}\sum_{j\in J'} c_{ij}^{'}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d_{ij}^{'}y_{ij}} + \frac{\sum_{i\in I'}\sum_{j\in J'} e_{ij}^{'}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} f_{ij}^{'}y_{ij}}\}$$

subject to

$$\sum_{j\in J'} y_{ij} = a_i^{''}, \forall i \in I^{'}$$

$$\sum_{i\in I'} y_{ij} = b_j^{''}, \forall j \in J^{'}$$

$$l_{ij} \leq y_{ij} \leq u_{ij} \text{ and integers}, \forall i \in I, \forall j \in J$$

$$0 \leq y_{m+1,j} \leq \sum_{i\in I} u_{ij} - b_j^{'}, \forall j \in J$$

$$0 \leq y_{i,n+1} \leq \sum_{j\in J} u_{ij} - a_i^{'}, \forall i \in I$$

$$y_{m+1,n+1} \geq 0$$

$$a_i^{''} = \sum_{j\in J} u_{ij}, \forall i \in I,\ a_{m+1}^{''} = \sum_{i\in I}\sum_{j\in J} u_{ij} - P = b_{n+1}^{''}; b_j^{''} = \sum_{i\in I} u_{ij},\ \forall j \in J$$

$$c_{ij}^{'} = c_{ij}, c_{m+1,j}^{'} = c_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ c_{m+1,n+1}^{'} = M$$

$$d_{ij}^{'} = d_{ij}, d_{m+1,j}^{'} = d_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ d_{m+1,n+1}^{'} = M$$

$$e_{ij}^{'} = e_{ij}, e_{m+1,j}^{'} = e_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ e_{m+1,n+1}^{'} = M$$

$$f_{ij}^{'} = f_{ij}, f_{m+1,j}^{'} = f_{i,n+1}^{'} = 0; \forall i \in I, \forall j \in J,\ f_{m+1,n+1}^{'} = M$$

where M is a large positive number.

$$I^{'} = \{1, 2, \cdots, m, m+1\}, J^{'} = \{1, 2, \cdots, n, n+1\}$$

Definition 7.1. Corner Feasible SolutionA basic feasible solution $\{y_{ij}\}_{I^{'}\times J^{'}}$ of the problem (RP_2) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$.

Theorem 7.2. *A non- corner feasible solution of the problem (RP_2) cannot provide a basic feasible solution of the problem (P_2).* [4]

Theorem 7.3. *There is a one -to-one correspondence between the feasible solution of the problem (P_2) and the corner feasible solution of the problem (RP_2).* [4]

Remark 7.4. If the problem (RP_2) has a cfs ,then since $c^{'}_{m+1,n+1} = M = d^{'}_{m+1,n+1} = e^{'}_{m+1,n+1} = f^{'}_{m+1,n+1}$, it follows that the non corner feasible solution cannot be an optimal solution of the problem (P_2) .

Theorem 7.5. *The value of the objective function of the problem (P_2) at a feasible solution $\{x_{ij}\}_{I\times J}$ is equal to the value of the objective function of problem (RP_2) at its corresponding cfs $\{y_{ij}\}_{I^{'}\times J^{'}}$ and conversely.*[4]

Theorem 7.6. *There is a one -to-one correspondence between the optimal solution of the problem (P_2) and an optimal solution among the corner feasible solution of the problem (RP_2).*[4]

Theorem 7.7. *Optimizing the problem (RP_2) is equivalent to optimizing the problem (P_2) provided the problem (P_2) has a feasible solution.*[4]

8. Algorithm to find a paradoxical solution, best paradoxical pair and paradoxical solution for specified flow

Step 1: Find a basic feasible solution of the problem (P_0) by converting it in to the problem $(P_0^{'})$. Let B be its corresponding basis.
Step 2: Calculate dual variables $u_i^1, u_i^2, u_i^3, u_i^4, v_j^1, v_j^2, v_j^3, v_j^4; i \in I, j \in J$ by using the equations given below and taking one of the $u_i^{'}s$ *or* $v_j^{'}s$ as zero.
$u_i^1 + v_j^1 = c_{ij}; u_i^2 + v_j^2 = d_{ij}; u_i^3 + v_j^3 = e_{ij}; u_i^4 + v_j^4 = f_{ij}, \forall (i,j) \in B$
$u_i^1 + v_j^1 = z_{ij}^1; u_i^2 + v_j^2 = z_{ij}^2; u_i^3 + v_j^3 = z_{ij}^3; u_i^4 + v_j^4 = z_{ij}^4, \forall (i,j) \in N_1 and N_2$.
N_1 *and* N_2 denotes the set of all non- basic cells (i,j) which are at their lower bounds and upper bounds respectively.
Step 3: Calculate $\theta_{ij}, c_{ij} - z_{ij}^1; d_{ij} - z_{ij}^2; e_{ij} - z_{ij}^3; f_{ij} - z_{ij}^4; \forall i \in I, j \in J^{'}$ for all non- basic cells and also calculate $C^0 = \sum\limits_{i\in I}\sum\limits_{j\in J^{'}} c_{ij}x_{ij}; D^0 = \sum\limits_{i\in I}\sum\limits_{j\in J^{'}} d_{ij}x_{ij}; E^0 = \sum\limits_{i\in I}\sum\limits_{j\in J^{'}} e_{ij}x_{ij}; F^0 = \sum\limits_{i\in I}\sum\limits_{j\in J^{'}} f_{ij}x_{ij}$.
Step 4: Calculate δ_{ij}^1 and δ_{ij}^2 where

$$\delta_{ij}^1 = \frac{\theta_{ij}[D^0(c_{ij} - z_{ij}^1) - C^0(d_{ij} - z_{ij}^2)]}{D^0[D^0 + \theta_{ij}(d_{ij} - z_{ij}^2)]} + \frac{\theta_{ij}[F^0(e_{ij} - z_{ij}^3) - E^0(f_{ij} - z_{ij}^4)]}{F^0[F^0 + \theta_{ij}(f_{ij} - z_{ij}^4)]}; \forall (i,j) \in N_1$$

$$\delta_{ij}^2 = \frac{-\theta_{ij}[D^0(c_{ij}-z_{ij}^1) - C^0(d_{ij}-z_{ij}^2)]}{D^0[D^0-\theta_{ij}(d_{ij}-z_{ij}^2)]} - \frac{\theta_{ij}[F^0(e_{ij}-z_{ij}^3) - E^0(f_{ij}-z_{ij}^4)]}{F^0[F^0-\theta_{ij}(f_{ij}-z_{ij}^4)]}; \forall (i,j) \in N_2$$

If $\delta_{ij}^1 \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall (i,j) \in N_2$, then the current solution so obtained is the optimal solution of the problem $(P_0^{'})$ and subsequently of the problem (P_0).Then go to step 4. Otherwise some $(i,j) \in N_1$ for which$\delta_{ij}^1 \leq 0$ or some $(i,j) \in N_2$ for which$\delta_{ij}^2 \leq 0$ will enter the basis. Go to step 3.

Step 5:Find the optimal cost $z^0 = \frac{C^0}{D^0} + \frac{E^0}{F^0}$ and flow $F^0 = \sum_{i \in I} \sum_{j \in J} x_{ij}$

Step 6:Let $F^0 = \sum_{i \in I} a_i^{'} = \sum_{j \in J} b_j^{'}$ be the optimal flow where $a_i^{'} \leq a_i, i \in I; b_j^{'} = b_j, j \in J$. Choose a non- basic cell (p,q) for which either of the following two conditions holds,

$(a)[D^0(u_p^1+v_q^1) - C^0(u_p^2+v_q^2)] < 0$ and $[F^0(u_p^3+v_q^3) - E^0(u_p^4+v_q^4)] < 0$

(b)If any one of the expressions $[D^0(u_p^1+v_q^1) - C^0(u_p^2+v_q^2)]$ and $[F^0(u_p^3+v_q^3) - E^0(u_p^4+v_q^4)]$ are negative then the following condition should hold for some $\lambda > 0$. For simplicity, take $\lambda = 1$.

$[D^0(u_p^1+v_q^1) - C^0(u_p^2+v_q^2)]F^0(F^0 + \lambda(u_p^4+v_q^4))$
$+[F^0(u_p^3+v_q^3) - E^0(u_p^4+v_q^4)]D^0(D^0 + \lambda(u_p^2+v_q^2)) < 0$

Then paradox exists.

Step 7:To find the best paradoxical pair, form the problem (P_1). To solve problem (P_1), formulate the related problem (RP_1) and solve it as usual. Let the paradoxical pair be (z^1, F^1) where $\sum_{i \in I'} \sum_{j \in J'} y_{ij}$ such that $F^1 > F^0$ and $z^1 < z^0$. Then the paradoxical range of flows is $[F^0, F^1]$

Step 8:Now for the specified flow $P \in [F^0, F^1]$, form the problem (P_2). In order to solve the problem (P_2) , form the related problem (RP_2) and solve it as usual. Find the optimal cost z^2 . We will observe that $z^1 < z^2 < z^0$. All objective function flow pairs in the range $[F^0, F^1]$ are paradoxical pairs.

Step 9:Repeat step 8 for all possible values of P such that $P \in [F^0, F^1]$ to find all paradoxical pairs.

9. Data taken from the account keeping books of a trading firm - D.M Chemicals, Delhi

D.M Chemicals is a trading firm which deals in the trading of soap stone across various states of India. Books of the firm provides the following information. The firm purchases soap stone (in tons) from three sellers-

- Shree Shyam Grinding Udyog, RIICO Industrial Area, Ajitgarh, Rajasthan
- Neejal Industries, 16 Duniya village, Halol 389350, District - Panchmahal, Gujarat.
- Kev Minerals, 37, Alindra Malav Road, Ta Kalol District- Panchmahal, Vadodra, Gujarat.

The firm sells its product (in tons) to three buyers-

- Jindal Mechno Bricks Pvt Ltd, VPO - Badli District, Jhajjar, Haryana.
- Poplon Chemie, Jalandhar
- Maheshwari Industries, 73, third cross behind LVK, Kalyan Mandap Kamakshi Pallya, Bangalore.

Goods (soap stone) are supplied by two types of trucks- A large truck that has a maximum capacity of supplying 50 tons of goods in one run and a small truck that has a maximum capacity of supplying 20 tons in one run. But the truck driver will not carry the goods in his truck if the quantity of goods to be supplied is less than 5 tons. D.M chemicals purchases a minimum of 20 tons of soap stone per month from each of the sellers . Moreover, each buyer has a minimum monthly demand of 20 tons of soap stone . Maximum availability of soap stone at Neejal Industries, Shree Shyam grinding Udyog and Kev Minerals is $70, 60$ and 70 tons respectively. All the buyers demanded 50 tons of soap stone monthly . Cost price per ton, Selling price per ton, profit per ton, Standard cartage per ton and actual Cartage per ton are shown in table (1). The manager of the company wishes to determine how many tons of soap stone per month, the firm should purchase from each seller and sell it to the different buyers so that the ratio of cost price to profit plus the ratio of actual cartage to standard cartage is minimum and the reserve stocks may also be kept whenever situation arises. The firm would be benefited if maximum time of transporting goods is also minimized. Data from the books of D.M Chemicals shows that the firm did the following business transactions-

Data from the books of D.M Chemicals shows that the firm did the following business transactions-

- Purchased 30 tons of soap stone from Neejal industries and sold it to Jindal mechno bricks.
- Purchased 10 tons of soap stone from Neejal industries and sold it to Poplon Chemie.
- Purchased 20 tons of soap stone from Neejal industries and sold it to Maheshwari Industries.
- Purchased 20 tons of soap stone from Shree Shyam Grinding Udyog and sold it to Jindal mechno bricks.
- Purchased 20 tons of soap stone from Shree Shyam Grinding Udyog and sold it to Poplon Chemie.
- Purchased 20 tons of soap stone from Shree Shyam Grinding Udyog and sold it to Maheshwari Industries.
- Purchased 20 tons of soap stone from Kev Minerals and sold it to Poplon Chemie.
- Purchased 10 tons of soap stone from Kev Minerals and sold it to Maheshwari Industries.

Total Purchasing cost = $Rs.194580$
Total Profit earned = $Rs.752840$
Actual Cartage paid = $Rs.307000$

TABLE 1. Cost in rupees(per ton)

Sellers↓	buyers →	Jindal mechno	Poplon Chemie	Maheshwari Ind.
Neejal Industries	$actual\ cartage \rightarrow$	2800	2500	2500
	$standard\ cartage \rightarrow$	2000	3000	3000
	$C.P \rightarrow$	1148	1148	1148
	$S.P \rightarrow$	9690	10000	8000
	$Profit \rightarrow$	8542	8852	6852
Shree Shyam	$actual\ cartage \rightarrow$	600	800	1500
	$standard\ cartage \rightarrow$	500	1000	1300
	$C.P \rightarrow$	1075	1075	1075
	$S.P \rightarrow$	1836	2000	3000
	$Profit \rightarrow$	761	925	1925
Kev Minerals	$actual\ cartage \rightarrow$	2850	3000	3000
	$standard\ cartage \rightarrow$	2000	2500	3500
	$C.P \rightarrow$	2040	2040	2040
	$S.P \rightarrow$	8333	9000	8000
	$Profit \rightarrow$	6293	6960	5960

standard Cartage = $Rs.291000$
$\frac{Costprice}{Profit} + \frac{actualCartage}{standardcartage} = \frac{194580}{752840} + \frac{307000}{291000} = 1.313444$

10. SOLUTION BY THE DEVELOPED ALGORITHM

Problem of the firm can be formulated as follows:
Let the three sellers- Neejal Industries, Shree Shyam Grinding Udyog, Kev Minerals be denoted by O_1, O_2 and O_3 respectively. Let the three buyers- Jindal mechno bricks, Poplon Chemie, Maheshwari Industries be denoted by D_1, D_2 and D_3 respectively.
Let $I = \{1, 2, ..., m\}$ be the index set of 3 sellers.
$J = \{1, 2, ..., n\}$ be the index set of 3 buyers.
x_{ij} = quantity of soap stone (in tons) purchased from the i^{th} seller and sold to the j^{th} buyer.
c_{ij} = cost price paid per ton when soap stone is purchased from the i^{th} seller and sold to the j^{th} buyer.
d_{ij} = profit per unit earned from the j^{th} buyer when the goods purchased from the i^{th} seller are supplied.
e_{ij} = actual cartage paid per ton when soap stone is purchased from the i^{th} seller and sold to the j^{th} buyer.
f_{ij} = standard cartage per ton when soap stone is purchased from the i^{th} seller

and sold to the j^{th} buyer.
l_{ij} and u_{ij} are the minimum and maximum quantity of soap stone (in tons) that can be supplied by the large and small truck from the i^{th} seller to the j^{th} buyer.
a_i = availability of soap stone at the seller i
b_j = quantity of soap stone (in tons) demanded (per month) by the buyer j.
Then the problem of the manager can be formulated mathematically as -

$$(P_0) : \min\{\frac{\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}x_{ij}} + \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}x_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{3} f_{ij}x_{ij}}\}$$

subject to

$$\sum_{j=1}^{3} x_{1j} \leq 70; \sum_{j=1}^{3} x_{2j} \leq 60; \sum_{j=1}^{3} x_{3j} \leq 70$$

$$\sum_{i=1}^{3} x_{i1} = 50; \sum_{i=1}^{3} x_{i2} = 50; \sum_{i=1}^{3} x_{i3} = 50$$

$$5 \leq x_{11} \leq 50; 5 \leq x_{12} \leq 20; 5 \leq x_{13} \leq 50$$

$$5 \leq x_{21} \leq 50; 5 \leq x_{22} \leq 20; 5 \leq x_{23} \leq 50$$

$$5 \leq x_{31} \leq 50; 5 \leq x_{32} \leq 20; 5 \leq x_{33} \leq 50$$

In order to solve the problem (P_0), we first convert it into balanced transportation problem $(P_0^{'})$ and solve it. The solution of problem $(P_0^{'})$ is shown in table (2). In

TABLE 2. Solution of problem $(P_0^{'})$

x_{ij}	D1	D2	D3	D4	u_i^1	u_i^2	u_i^3	u_i4
$O1$	**10**	$\overline{20}$	**40**	$\underline{0}$	0	0	0	0
$O2$	**35**	$\overline{20}$	$\underline{5}$	**0**	-73	-7781	-2200	-1500
$O3$	$\underline{5}$	**10**	$\underline{5}$	**50**	0	0	0	0
v_j^1	1148	2040	1148	0				
v_j^2	8542	6960	6852	0				
v_j^3	2800	3000	2500	0				
v_j^4	2000	2500	3000	0				

Notes.Entries of the form $\underline{a}$ and $\overline{b}$ represent non-basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

table (2), $C^0 = 185660; D^0 = 722165; E^0 = 281750; F^0 = 296500; a_1^{'} = 70; a_2^{'} = 60; a_3^{'} = 20$ Since in table (3), $\delta_{ij}^1 \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall (i,j) \in N_2$, therefore, the solution in table (2) is an optimal solution of problem $(P_0^{'})$ and hence yields an optimal solution of (P_0) with $z^0 = \frac{185660}{722165} + \frac{281750}{296500} = 1.20734$ and

TABLE 3. Computation of δ_{ij}^1 and δ_{ij}^2

NB	$O1D2$	$O2D2$	$O2D3$	$O2D4$	$O3D1$	$O3D3$
θ_{ij}	10	5	30	0	5	35
$c_{ij} - z_{ij}^1$	-892	-892	0	73	892	892
$d_{ij} - z_{ij}^2$	1892	1746	2854	7781	-2249	-892
$e_{ij} - z_{ij}^3$	-500	0	1200	2200	50	500
$f_{ij} - z_{ij}^4$	500	0	-200	1500	0	500
δ_{ij}^1 and δ_{ij}^2	0.05305	0.00939	0.1163	0	0.01118	0.05957

flow $F^0 = 150$.We also verified this optimal solution by using a computing software Excel Solver and obtained the following report.
Result: Solver found a solution. All Constraints and optimality conditions are satisfied. Solver Engine Engine: GRG Nonlinear; Solution Time: 0.016 Seconds;Iterations: 0 Subproblems: 0 Solver Options: Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative
On comparing this result with the existing data that we get from the books of the firm, we will see that the difference in the ratio is $1.313444 - 1.207341 = 0.106103$. That is, the savings of the firm would increase by 8.07% if the transactions would be done according to the developed algorithm (shown in table (2).

Existence of paradox:

Now,there exists a cell O_2D_3 with $[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)] = 722165(1075) - 185660(-929) = 948805515 > 0$ and $[F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)] = 296500(300) - 281750(1500) = -333675000 < 0$.
For $\lambda = 1$, $[D^0(u_p^1 + v_q^1) - C^0(u_p^2 + v_q^2)]F^0(F^0 + (u_p^4 + v_q^4)) + [F^0(u_p^3 + v_q^3) - E^0(u_p^4 + v_q^4)]D^0(D^0 + (u_p^2 + v_q^2))$
$= [948805515](296500)(296500 + 1500) - 333675000[722165](722165 - 929) = -8.996148061 \times 10^{19} < 0$
Therefore, a paradox exists in this case. Best paradoxical pair is found by solving the problem (P_1). Form the related problem (RP_1) with an additional supply point and an additional destination. Table (4) below shows an optimal solution of the related problem (RP_1). For (RP_1),
$0 \le x_{14} \le 50; 0 \le x_{24} \le 60; 0 \le x_{34} \le 100; 0 \le x_{41} \le 100; 0 \le x_{42} \le 10; 0 \le x_{43} \le 100; x_{44} \ge 0;$ $A_1' = \sum_{j=1}^{3} u_{1j} = 120; A_2' = \sum_{j=1}^{3} u_{2j} = 120; A_3' = \sum_{j=1}^{3} u_{3j} = 120; A_4' = \sum_{i=1}^{3}\sum_{j=1}^{3} u_{ij} = 360; B_1' = \sum_{i=1}^{3} u_{i1} = 150; B_2' = \sum_{i=1}^{3} u_{i2} = 60; B_3' = \sum_{i=1}^{3} u_{i3} = 150$ Using excel solver, we verify that the solution given in table (4) is an optimal solution of the problem (RP_1) and consequently of the problem (P_1) with

TABLE 4. Best Paradoxical Pair

x_{ij}	D_1	D_2	D_3	D_4	$A_i^{'}$
O_1	**10**	$\overline{20}$	$\overline{50}$	**40**	120
O_2	**35**	$\overline{20}$	$\underline{5}$	$\overline{60}$	120
O_3	$\underline{5}$	**10**	$\overline{50}$	**55**	120
O_4	$\overline{100}$	$\overline{10}$	**45**	**205**	360
$B_j^{'}$	150	60	150	360	

$z^0 = \frac{288940}{1058885} + \frac{441750}{484000} = 1.185579$ and flow $F^0 = 205$.Therefore, the best paradoxical pair is $(1.185579, 205)$. The paradoxical range of flow is $[150, 205]$.The firm would benefit more if the transactions would be done according to the best paradoxical pair as shown in table (4). The savings of the firm would increase by $\frac{1.313444-1.185579}{1.313444} \times 100 = 9.74\%$

Paradoxical Solution for specified flow P $= 160$ Now, consider the paradoxical pair with in this range for $P = 160$. For this, form the related problem $(RP_2), b_4^{''} = a_4^{''} = 360 - 160 = 200; x_{44} = 0$ Using excel solver, we verify that

TABLE 5. Paradoxical solution for $P = 160$

x_{ij}	D_1	D_2	D_3	D_4	$A_i^{'}$
O_1	**8**	$\overline{20}$	$\overline{50}$	**42**	120
O_2	**37**	$\overline{20}$	$\underline{5}$	**58**	120
O_3	$\underline{5}$	**10**	$\underline{5}$	$\overline{100}$	120
O_4	$\overline{100}$	$\overline{10}$	**90**		200
$B_j^{'}$	150	60	150	200	

the solution given in table (5) is an optimal solution of the problem (RP_2) and consequently of the problem (P_2) with $z^0 = 1.188767$ and $F^0 = 160$.Therefore, the paradoxical pair is $(1.188767, 160)$.

11. CONCLUSION

This paper gives a sufficient condition for the existence of paradox in a fractional plus fractional capacitated transportation problem. Moreover, an algorithm is devised which provides the best paradoxical pair and paradoxical solution for any specified flow within the paradoxical range of flows. The developed algorithm is used to solve the problem of the manager of a trading firm namely, D.M Chemicals , Delhi. It is found that the firm would be benefited by 8.07% if the business transactions would be done according to the developed algorithm. Further, it is shown that the firm would benefit by 9.74% if the transactions would be done according to the best paradoxical pair.

As future work, it is intended to apply the proposed algorithm to a sum of n fractional functions when the decision variables are bounded. Moreover, the developed algorithm can also be applied in a solid fixed charge capacitated transportation problem, indefinite quadratic transportation problem. Developed algorithm can be applied on the problems of the real world.

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