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ON THE DYNAMICS OF AN SIRI EPIDEMIC MODEL WITH A GENERALIZED INCIDENCE FUNCTION

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Abstract. The duration of the latent period has the influence on the spread of infectious diseases in the infected population. The introduction of this period into epidemiological models gave rise to different models: the SIRI epidemic model with time delay and the SEIRI epidemic model. Taking into account the infected individuals who died before the end of the latent period, we propose the modification of the existing SIRI model. The dynamics of the obtained model is analyzed in terms of the global stability. A comparison with the existing versions is presented. Numerical simulations are proposed to illustrate our results and finally a conclusion is given to close this work.
Keywords. epidemic model, global stability, Lyapunov functionals.

1 Introduction

Recently, Global dynamics of mathematical models describing the propagation of infectious diseases have been studied by several authors (see, for example, [2, 4, 6, 5, 7, 1, 9, 10]). special attention has been paid to the study of the effect of the latent period on the spread of infectious diseases. These studies have led to different models, namely the ordinary SEIR model [1, 3, 9], the SIR model with

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delay in the evolution equation of infectious individuals [6, 7, 11, 12] and SIR model with delay in the evolution equations of susceptible individuals and infected individuals [1, 4, 8, 9].

Taking into account the infected individuals who died before the end of the latent period, we propose the following SIRI epidemic model with a discrete time delay :

$$\begin{cases} \dfrac{dS}{dt} = A - \mu S - f(S, e^{-\mu\tau} I_\tau), \\ \dfrac{dI}{dt} = f(S, e^{-\mu\tau} I_\tau) - (\mu + \gamma)I + \delta R, \\ \dfrac{dR}{dt} = \gamma I - (\mu + \delta)R, \end{cases} \tag{1}$$

here $\psi_\tau = \psi(t-\tau)$ for any given function ψ, $A = \mu N$, where N is the total number of population, S is the number of susceptible individuals, I is the number of infectious individuals, μ denote birth and death rates, f is the incidence function, i.e. the number of susceptible individuals infected through their contacts with the infectious individuals, γ is the recovery rate of the infectious individuals, δ represents the rate that recovered individuals relapse and regain the infectious class and τ units of time after infection. The terme $e^{-\mu\tau}$ is the probability of surviving from time $t-\tau$ to time t.
The initial condition for the above system is

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad \theta \in [-\tau, 0] \tag{2}$$

with $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C^+ \times C^+ \times C^+$, such that $\varphi_i(\theta) \geq 0$ ($-\tau \leq \theta \leq 0$, $i = 1, 2, 3$). Here C denotes the Banach space $C([-\tau, 0], \mathbb{R})$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$, equipped with the supremum norm. The nonnegative cone of C is defined as $C^+ = C([-\tau, 0], \mathbb{R}^+)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

The first model in this optic was proposed by Cooke in ([6], 1979). In ([12], 2004) Wanblao Ma and al proposed a SIR model with a bilinear incidence rate. In ([7], 2011) Y. Enatsu and al propose a generalization of the Wanblao et al model, they considered a general incidence rate and a distributed latency period.

In this paper, it is shown that global stability of the system (1) can be attained under suitable monotonicity conditions and it is established that the basic reproduction number R_0 is a threshold parameter for the stability of this model. The rest of the paper is organized as follows. Next, in section 2, we establish the global stability of disease-free and endemic equilibria of the system (1). Finally, a comparison with existing models is proposed and some numerical simulations are provided in section 3.

2 Global stability analysis

Consider the incidence function $f(S, e^{-\mu\tau} I)$ that is a locally Lipschitz continuous function on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfaying $f(0, e^{-\mu\tau} I) = f(S, 0) = 0$ for $S \geq 0$, $I \geq 0$ and the followings hold:

(H_1) $f(S, e^{-\mu\tau} I)$ is a strictly monotone increasing function of $S \geq 0$, for any fixed $I > 0$, and $f(S, e^{-\mu\tau} I)$ is a strictly monotone increasing function of $I \geq 0$, for any fixed $S \geq 0$;

(H_2) $\phi(S, e^{-\mu\tau}I) = \frac{f(S,e^{-\mu\tau}I)}{I}$ is a bounded and monotone decreasing function of $I > 0$, for any fixed $S \geq 0$, and $K(S) = \lim_{I\to 0^+} \phi(S, e^{-\mu\tau}I)$ is a continuous and monotone increasing function on $S \geq 0$.

Let

$$R_0 := \frac{K(\frac{A}{\mu})}{\eta},$$

with $\eta = (\mu + \gamma) - \frac{\gamma\delta}{\mu+\delta}$.
System (1) always has a disease-free equilibrium $P_0 = (\frac{A}{\mu}, 0, 0)$. The following proposition gives the existence of a unique endemic equilibrium P^*.

Proposition 2.1. *Under the hypotheses* (H_1) *and* (H_2)*, if* $R_0 > 1$*, then system* (1) *also admits a unique endemic equilibrium* $P^* = (S^*, I^*, R^*)$*, where* S^**,* I^* *and* R^* *satisfying the following system:*

$$\begin{cases} A - \mu S - f(S, e^{-\mu\tau}I) = 0 \\ f(S, e^{-\mu\tau}I) - (\mu + \gamma)I + \delta R = 0 \\ \gamma I - (\mu + \delta)R = 0 \end{cases} \tag{3}$$

Proof. The proof goes in a way similar to the proof of Proposition 3 in [2]. □

In this section, we discuss the global stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of system (1). Since $\frac{d}{dt}(S+I+R) \leq A-\mu(S+I+R)$, we have $\limsup(S+I+R) \leq \frac{A}{\mu}$. Hence we discuss system (1) in the closed set

$$\Omega =: \{(\varphi_1, \varphi_2, \varphi_3) \in C^+ \times C^+ \times C^+ : \|\varphi_1 + \varphi_2 + \varphi_3\| \leq A/\mu\}.$$

It is easy to show that Ω is positively invariant with respect to system (1). Next we consider the global asymptotic stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of (1) by Lyapunov functionals, respectively.

Proposition 2.2. *If* $R_0 \leq 1$*, then the disease-free equilibrium* P_0 *is globally asymptotically stable.*

Proof. Define a Lyapunov functional

$$W_0(t) = \int_{\frac{A}{\mu}}^{S} (1 - \frac{K(\frac{A}{\mu})}{K(u)})du + I + \frac{\delta}{\mu+\delta}R + \eta\int_{t-\tau}^{t} I(u)du$$

We will show that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have

$$\begin{aligned} \frac{dW_0(t)}{dt} &= (1 - \frac{K(\frac{A}{\mu})}{K(S)})\dot{S} + f(S, e^{-\mu\tau}I_\tau) \\ &- (\mu+\gamma)I + \delta R + \eta(I - I_\tau) + \frac{\gamma\delta}{\mu+\delta}I - \delta R \\ &= (1 - \frac{K(\frac{A}{\mu})}{K(S)})(A - \mu S) \\ &+ \frac{K(\frac{A}{\mu})}{K(S)} f(S, e^{-\mu\tau}I_\tau) - \eta I_\tau \\ &= \mu(1 - \frac{K(\frac{A}{\mu})}{K(S)})(\frac{A}{\mu} - S) \\ &+ \eta I_\tau(\frac{\phi(S, e^{-\mu\tau}I_\tau)}{\eta}\frac{K(\frac{A}{\mu})}{K(S)} - 1) \end{aligned}$$

By the hypothesis (H_1) , we obtain that

$$(1-\frac{K(\frac{A}{\mu})}{K(S)})(\frac{A}{\mu}-S)\leq 0$$

Where equality holds if and only if $S=\frac{A}{\mu}$.
Furthermore, it follows from the hypothesis (H_2) that

$$\begin{aligned}\frac{K(\frac{A}{\mu})}{K(S)}\frac{\phi(S,e^{-\mu\tau}I_\tau)}{\eta} &\leq \frac{K(\frac{A}{\mu})}{K(S)}\frac{K(S)}{\eta}\\ &\leq \frac{K(\frac{A}{\mu})}{\eta}\\ &\leq R_0\end{aligned}$$

Therefore, $R_0\leq 1$ ensures that $\frac{dW_0(t)}{dt}\leq 0$ for all $t\geq 0$, where $\frac{dW_0(t)}{dt}=0$ holds if $(S,I,R)=(\frac{A}{\mu},0,0)$. Hence, it follows from system (1) that $\{P_0\}$ is the largest invariant set in $\left\{(S,I,R)|\frac{dW_0(t)}{dt}=0\right\}$. From the Lyapunov-LaSalle asymptotic stability, we obtain that P_0 is globally asymptotically stable. This completes the proof. □

The following lemma plays a key role to obtain Theorems 2.1.

Lemma 2.1. *Under the hypotheses (H_1) and (H_2) , it holds that*

$$g(y_t)-g(\widetilde{y}_{t,\tau})\geq 0$$

for all $t\geq 0$ and $0\leq\tau\leq h$, where $g(x)=x-1-\ln x\geq 0$, for $x>0$ and

$$y_t=\frac{I}{I^*}\ ,\ \widetilde{y}_{t,\tau}=\frac{f(S(t+\tau),e^{-\mu\tau}I)}{f(S(t+\tau),e^{-\mu\tau}I^*)}.$$

Proof. By the definitions of y_t and $\widetilde{y}_{t,\tau}$, we have that

$$\widetilde{y}_{t,\tau}-1=\frac{f(S(t+\tau),e^{-\mu\tau}I)-f(S(t+\tau),e^{-\mu\tau}I^*)}{f(S(t+\tau),e^{-\mu\tau}I^*)}$$

and

$$\begin{aligned}y_t-\widetilde{y}_{t,\tau} &= \frac{I}{I^*}-\frac{f(S(t+\tau),e^{-\mu\tau}I)}{f(S(t+\tau),e^{-\mu\tau}I^*)}\\ &= \frac{I}{f(S(t+\tau),e^{-\mu\tau}I^*)}\left\{\phi(S(t+\tau),e^{-\mu\tau}I^*)-\phi(S(t+\tau),e^{-\mu\tau}I)\right\}\end{aligned}$$

Then , it follows from the hypotheses (H_1) and (H_2) that

$$\begin{aligned}(y_t-\widetilde{y}_{t,\tau})(\widetilde{y}_{t,\tau}-1) &= \frac{I}{f(S(t+\tau),e^{-\mu\tau}I^*)^2}\left\{\phi(S(t+\tau),e^{-\mu\tau}I^*)-\phi(S(t+\tau),e^{-\mu\tau}I)\right\}\\ &\times\left\{f(S(t+\tau),e^{-\mu\tau}I)-f(S(t+\tau),e^{-\mu\tau}I^*)\right\}\geq 0.\end{aligned}$$

that is , either $y_t\leq\widetilde{y}_{t,\tau}\leq 1$ or $y_t\geq\widetilde{y}_{t,\tau}\geq 1$ holds for all $t\geq 0$ and $0\leq\tau\leq h$. Since $g'(x)=1-\frac{1}{x}$ for all $x>0$ and $g'(1)=0$, it follows that $g(y_t)\geq g(\widetilde{y}_{t,\tau})\geq 0$. This completes the proof. □

Theorem 2.1. *If $R_0 > 1$, then the endemic equilibrium P^* is globally asymptotically stable.*

Proof. Firstly, we prove the existence and the uniqueness of the endemic equilibrium P^*. At a fixed point (S, I, R) of system (1), the following equations hold.

$$\begin{cases} A - \mu S - f(S, e^{-\mu\tau} I) = 0, \\ f(S, e^{-\mu\tau} I) - (\mu + \gamma)I + \delta R = 0, \\ \gamma I - (\mu + \delta)R = 0, \end{cases} \tag{4}$$

Substituting the third equation into the second equation of (4), we consider the following system:

$$\begin{cases} A - \mu S - f(S, e^{-\mu\tau} I) = 0, \\ f(S, e^{-\mu\tau} I) - (\mu + \gamma - \dfrac{\gamma\delta}{\mu + \delta})I = 0, \\ R = \dfrac{\gamma I}{\mu + \delta}, \end{cases} \tag{5}$$

Using the first and the second equations in (5), we conclude that

$$S = \frac{A}{\mu} - (\mu + \gamma - \frac{\delta\gamma}{\mu + \delta})\frac{I}{\mu}. \tag{6}$$

Substituting the equation (6) into the second equation of (4), we have

$$g(I) := \frac{f(\frac{A}{\mu} - (\mu + \gamma - \frac{\delta\gamma}{\mu+\delta})\frac{I}{\mu}, e^{-\mu\tau} I)}{I} - (\mu + \gamma - \frac{\gamma\delta}{\mu + \delta}) = 0.$$

By the hypothesis (H_2), g is strictly monotone decreasing on $]0, \frac{A}{(\mu+\gamma-\frac{\gamma\delta}{\mu+\delta})}]$ satisfying:

$$\begin{aligned} \lim_{I \to 0^+} g(I) &= K(\frac{A}{\mu}) - (\mu + \gamma - \frac{\gamma\delta}{\mu + \delta}) \\ &= (\mu + \gamma - \frac{\gamma\delta}{\mu + \delta})(\frac{K(\frac{A}{\mu})}{(\mu + \gamma - \frac{\gamma\delta}{\mu+\delta})} - 1) \\ &= (\mu + \gamma - \frac{\gamma\delta}{\mu + \delta})(R_0 - 1) > 0 \end{aligned}$$

and

$$g(\frac{A}{(\mu + \gamma - \frac{\gamma\delta}{\mu+\delta})}) = -(\mu + \gamma - \frac{\gamma\delta}{\mu + \delta}) < 0.$$

Thus, there exists a unique I^* such that $g(I^*) = 0$. Hence, we conclude the existence and uniqueness of the endemic equilibrium P^*.

Finally, To prove global stability of the endemic equilibrium, we define a Lyapunov functional

$$W(t) = W_1(t) + W_2(t) + W_3(t) + W_4(t),$$

with

$$W_1(t) = S - S^* - \int_{S^*}^{S} \frac{f(S^*, e^{-\mu\tau} I^*)}{f(u, e^{-\mu\tau} I^*)} du,$$

$$W_2(t) = (I - I^* - I^* \ln \frac{I}{I^*}),$$

$$W_3(t) = \frac{\delta}{\mu+\delta}(R - R^* - R^* \ln \frac{R}{R^*}),$$

and

$$W_4(t) = f(S^*, e^{-\mu\tau} I^*) \int_{t-\tau}^{t} g(\frac{f(S(u+\tau), e^{-\mu\tau} I(u))}{f(S(u+\tau), e^{-\mu\tau} I^*)}) du$$

The time derivative of the function $W(t)$ along the positive solution of system (1) is

$$\begin{aligned} \frac{dW(t)}{dt} &= \Big(1 - \frac{f(S^*, e^{-\mu\tau} I^*)}{f(S, e^{-\mu\tau} I^*)}\Big)\Big(A - \mu S - f(S, e^{-\mu\tau} I_\tau)\Big) \\ &+ \big(1 - \frac{I^*}{I}\big)\Big(f(S, e^{-\mu\tau} I_\tau) + \delta R - (\mu+\gamma) I\Big) \\ &+ \frac{\delta}{\mu+\delta}\big(1 - \frac{R^*}{R}\big)\Big(\gamma I - (\mu+\delta) R\Big) \\ &+ f(S^*, e^{-\mu\tau} I^*)\Big(g(\frac{f(S(t+\tau), e^{-\mu\tau} I)}{f(S(t+\tau), e^{-\mu\tau} I^*)}) - g(\frac{f(S, e^{-\mu\tau} I_\tau)}{f(S, e^{-\mu\tau} I^*)})\Big). \end{aligned} \tag{7}$$

Using the relation $A = \mu S^* + f(S^*, e^{-\mu\tau} I^*)$, a simple calculations give that

$$\begin{aligned} \frac{dW(t)}{dt} &= \Big(1 - \frac{f(S^*, e^{-\mu\tau} I^*)}{f(S, e^{-\mu\tau} I^*)}\Big)\Big(-\mu(S - S^*) + f(S^*, e^{-\mu\tau} I^*)\big) \\ &+ \frac{f(S^*, e^{-\mu\tau} I^*)}{f(S, e^{-\mu\tau} I^*)} f(S, e^{-\mu\tau} I_\tau) \\ &- (\mu+\gamma) I - \frac{I^*}{I} f(S, e^{-\mu\tau} I_\tau) - \delta \frac{I^*}{I} R + (\mu+\gamma) I^* \\ &+ \frac{\delta}{\mu+\delta}\Big(\gamma I - \gamma I \frac{R^*}{R} + (\mu+\delta) R^*\Big) \\ &+ f(S^*, e^{-\mu\tau} I^*)\Big(g(\frac{f(S(t+\tau), e^{-\mu\tau} I)}{f(S(t+\tau), e^{-\mu\tau} I^*)}) - g(\frac{f(S, e^{-\mu\tau} I_\tau)}{f(S, e^{-\mu\tau} I^*)})\Big). \end{aligned} \tag{8}$$

Here by using

$$(\mu+\gamma) I^* - \frac{\delta}{\mu+\delta}\gamma I^* = f(S^*, e^{-\mu\tau} I^*),$$

$$(\mu+\delta) R^* = \gamma I^*$$

and

$$\ln \frac{f(S, e^{-\mu\tau} I_\tau)}{f(S, e^{-\mu\tau} I^*)} = \ln \frac{I}{I^*} + \ln \frac{f(S^*, e^{-\mu\tau} I^*)}{f(S, e^{-\mu\tau} I^*)} + \ln \frac{I^*}{I} \frac{f(S, e^{-\mu\tau} I_\tau)}{f(S^*, e^{-\mu\tau} I^*)}$$

a straightforward calculations give

$$\begin{aligned}\frac{dW(t)}{dt} &= -\mu\Big(1-\frac{f(S^*,e^{-\mu\tau}I^*)}{f(S,e^{-\mu\tau}I^*)}\Big)(S-S^*)\\ &\quad - f(S^*,e^{-\mu\tau}I^*)\Big(\frac{I}{I^*}-1-\ln\frac{I}{I^*}\Big)\\ &\quad - f(S^*,e^{-\mu\tau}I^*)\Big(\frac{f(S^*,e^{-\mu\tau}I^*)}{f(S,e^{-\mu\tau}I^*)}-1-\ln\frac{f(S^*,e^{-\mu\tau}I^*)}{f(S,e^{-\mu\tau}I^*)}\Big)\\ &\quad - f(S^*,e^{-\mu\tau}I^*)\Big(\frac{I^*}{I}\frac{f(S,e^{-\mu\tau}I_\tau)}{f(S^*,e^{-\mu\tau}I^*)}-1-\ln\frac{I^*}{I}\frac{f(S,e^{-\mu\tau}I_\tau)}{f(S^*,e^{-\mu\tau}I^*)}\Big)\\ &\quad + f(S^*,e^{-\mu\tau}I^*)g(\frac{f(S(t+\tau),e^{-\mu\tau}I)}{f(S(t+\tau),e^{-\mu\tau}I^*)})\\ &\quad + \frac{\delta\gamma I^*}{\mu+\delta}\Big(2-\frac{I^*R}{IR^*}-\frac{IR^*}{I^*R}\Big)\\ &= -\mu\Big(1-\frac{f(S^*,e^{-\mu\tau}I^*)}{f(S,e^{-\mu\tau}I^*)}\Big)(S-S^*)\\ &\quad + f(S^*,e^{-\mu\tau}I^*)\Big\{g(\frac{f(S(t+\tau),e^{-\mu\tau}I)}{f(S(t+\tau),e^{-\mu\tau}I^*)})-g(\frac{I}{I^*})\Big\}\\ &\quad - f(S^*,e^{-\mu\tau}I^*)\Big\{g(\frac{f(S^*,e^{-\mu\tau}I^*)}{f(S,e^{-\mu\tau}I^*)})+g(\frac{I^*}{I}\frac{f(S,e^{-\mu\tau}I_\tau)}{f(S^*,e^{-\mu\tau}I^*)})\Big\}\\ &\quad - \frac{\delta\gamma I^*}{\mu+\delta}\Big(\sqrt{\frac{I^*R}{IR^*}}-\sqrt{\frac{IR^*}{I^*R}}\Big)^2.\end{aligned} \tag{9}$$

It follows from (H_1) that

$$-\mu\Big(1-\frac{f(S^*,e^{-\mu\tau}I^*)}{f(S,e^{-\mu\tau}I^*)}\Big)\Big(S-S^*\Big)\le 0,$$

With strict equality holds if and only if $S(t)=S^*$, and using Lemma 2.1, we have

$$g(\frac{f(S(t+\tau),e^{-\mu\tau}I)}{f(S(t+\tau),e^{-\mu\tau}I^*)})-g(\frac{I}{I^*})\le 0\ ,\ \ for\ \ all\ \ 0\le\tau\le h.$$

Furthermore, since the function $g(x)=1-x+\ln x$ is always non-positive for any $x>0$, and $g(x)=0$ if and only if $x=1$, then $\frac{dW(t)}{dt}\le 0$, for all $t\ge 0$, where the equality holds only at the equilibrium point (S^*,I^*,R^*). Thus $\{P^*\}$ is the largest invariant set in $\Big\{(S,I,R)\Big|\frac{dW(t)}{dt}=0\Big\}$. Consequently, we obtain , by the Lyapunov-LaSalle asymptotic stability theorem, that P^* is globally asymptotically stable. This completes the proof. □

3 Comparison

In this section we propose a comparison between the model (1) and the following model [4]:

$$\begin{cases}\dfrac{dS}{dt}=A-\mu S-f(S,I),\\ \dfrac{dI}{dt}=e^{-\mu\tau}f(S_\tau,I_\tau)-(\mu+\gamma)I+\delta R,\\ \dfrac{dR}{dt}=\gamma I-(\mu+\delta)R,\end{cases} \tag{10}$$

The principal result of the model (10) is recalled in the following theorem.

Theorem 3.1. *[4] Assume that* (H_1) *and* (H_2) *hold.*
If $R_{01} : \frac{K(\frac{A}{\mu})e^{-\mu\tau}}{\eta} > 1$, *then the endemic equilibrium* P^* *of the system* (10) *is globally asymptotically stable.*

To facilitate the comparison, we summarize in the following table the main results of the two proposed models.

Model	(1)	(10)
Basic reproduction number, R_0	$\frac{K(\frac{A}{\mu})}{\eta}$	$\frac{K(\frac{A}{\mu})e^{-\mu\tau}}{\eta}$
Key to global stability	R_0	R_0

Table 1: The principal characteristics of the two models (1) and (10).

From Table 1, we note that the basic reproduction number R_0 is the key parameter of the global stability analysis for the two proposed cases. Consequently the models (1) and (10) generate identical global asymptotic behavior (see Figure 1 and Figure 2).

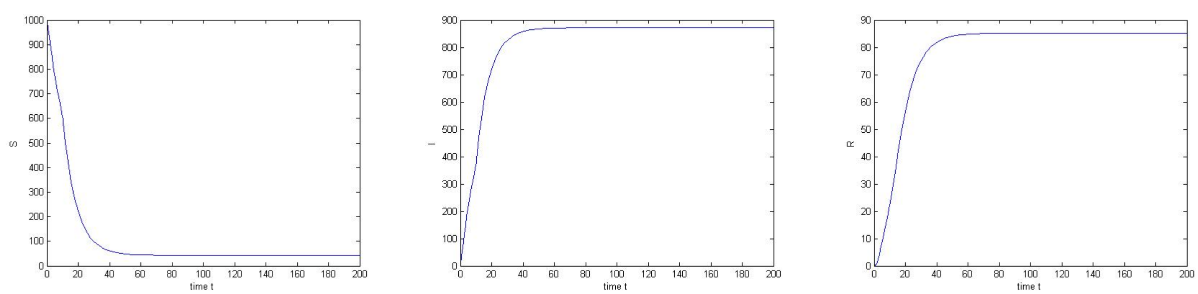

Figure 1: The dynamic behavior of the model (1) for the incidence function $f(S, I) = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ [8] and the parameters $A = 5$, $\mu = 0.005$, $\alpha_1 = 0$, $\alpha_2 = 0.9$, $\gamma = 0.02$, $\beta = 0.1$, $\delta = 0.2$, $\tau = 10$, $S(0) = 999$, $I(0) = 1$, $R(0) = 0$.

4 Conclusion

This paper investigates the effect of latent period (delay) in the stability of the following SIRI model:

$$\begin{cases} \frac{dS}{dt} = A - \mu S - f(S, e^{-\mu\tau} I_\tau), \\ \frac{dI}{dt} = f(S, e^{-\mu\tau} I_\tau) - (\mu+\gamma)I + \delta R, \\ \frac{dR}{dt} = \gamma I - (\mu+\delta)R, \end{cases} \tag{11}$$

In conclusion, the fundamental objective of the above formulation is to study the role of time delay on the existence of the endemic equilibrium and his stability. By using the reproduction

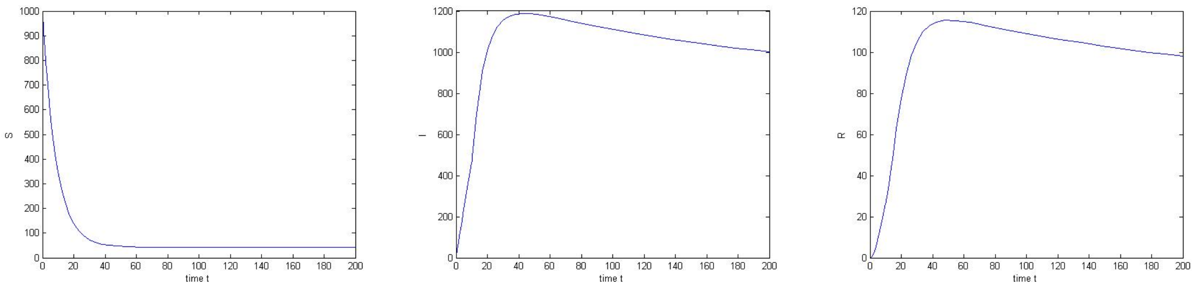

Figure 2: The dynamic behavior of the model (10) for the incidence function $f(S,I) = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ and the parameters $A = 5$, $\mu = 0.005$, $\alpha_1 = 0$, $\alpha_2 = 0.9$, $\gamma = 0.02$, $\beta = 0.1$, $\delta = 0.2$, $\tau = 10$, $S(0) = 999$, $I(0) = 1$, $R(0) = 0$.

number R_0, to establish the stability of the endemic equilibrium point, we used a Lyapunov function and the assumptions of monotony on the incidence function for a rigorous mathematical treatment.

In addition, we present a comparison of the model (11) with the following version:

$$\begin{cases} \dfrac{dS}{dt} = A - \mu S - f(S,I), \\ \dfrac{dI}{dt} = e^{-\mu\tau} f(S_\tau, I_\tau) - (\mu+\gamma)I + \delta R, \\ \dfrac{dR}{dt} = \gamma I - (\mu+\delta)R, \end{cases} \tag{12}$$

It is shown through theoretical analysis and numerical simulations, that the two models (11) and (12) have the same properties. It is also shown that, unlike other existing work, the model (11) has a basic reproduction number which depend on the time delay.

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