A Tool for NCPs

Christian Bidard*

Abstract

The non-existence of a solution to a nonlinear complementarity problem implies the existence of solutions for a related set of nonlinear complementarity problems.

AMS Classification. 65K05, 90C33

In economics, the Gale-Nikaido-Debreu lemma ([3],[4],[5]) provides the key to solve the general equilibrium problem, which is a nonlinear complementarity problem.

GND Lemma. *Let s be a continuous function defined on a convex compact set $K \subset R^n$, with values in R^n, and which satisfies the Walras identity $\mathbf{x}^T s(\mathbf{x}) = 0$. There exists $\mathbf{x}^*$ in K such that $\mathbf{x}^T s(\mathbf{x}^*) \geq 0$ for any $\mathbf{x} \in K$.*

Proof. Let H be a compact convex subset of R^n containing the image $s(K)$, and φ be the upper-semi continuous correspondence from H to K defined by $\varphi(\mathbf{y}) = \left\{ \mathbf{x}; \mathbf{x} \in K, \mathbf{x}^T\mathbf{y} = \min_{\mathbf{z}\in K} \mathbf{z}^T\mathbf{y} \right\}$. The product correspondence $s \times \varphi$ from the convex compact set $K \times H$ into itself is upper-semicontinuous. By the Kakutani theorem, it admits a fixed point $(\mathbf{x}^*, \mathbf{y}^*)$, for which $\min_{\mathbf{x}\in K} \mathbf{x}^T s(\mathbf{x}^*) = \min_{\mathbf{x}\in K} \mathbf{x}^T \mathbf{y}^* = \varphi(\mathbf{y}^*)^T \mathbf{y}^* = \mathbf{x}^{*T}\mathbf{y}^* = \mathbf{x}^{*T} s(\mathbf{x}^*) = 0$. ■

Let $\|\mathbf{x}\| = \sum_i |x_i|$. When function s is defined on the unit simplex $S = \{\mathbf{x} \in R^n; \mathbf{x} \geq 0, \|\mathbf{x}\| = 1\}$ and satisfies the Walras identity, the lemma shows the existence of a solution to the nonlinear complementarity problem NCP(s): $\mathbf{x} \geq 0, s(\mathbf{x}) \geq 0, \mathbf{x}^T s(\mathbf{x}) = 0$. When the Walras identity is not met, we may force it by introducing one more dimension ([1]) and considering the extension $\overline{s}(\mathbf{x}, t) = (s(t^{-1}\mathbf{x}), -t^{-1}\mathbf{x}^T s(t^{-1}\mathbf{x}))$, which is defined on the set $S_\varepsilon = \{(\mathbf{x}, t) \in R^{n+1}; \mathbf{x} \geq 0, t \geq \varepsilon, \|\mathbf{x}\| + t = 1\}$.

*University Paris Nanterre, EconomiX, 200 avenue de la République, 92001 Nanterre (France). E-mail: christian.bidard@u-paris10.fr

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

Theorem 1 *Let f: $R^n_+ \to R^n$ be a continuous function such that NCP(f) has no solution. For any positive vector $\mathbf{u}$, there exist infinitely many semipositive vectors $\mathbf{z}_k$ with $\|\mathbf{z}_k\| \to \infty$ and positive scalars λ_k such that*

$$f(\mathbf{z}_k) + \lambda_k \mathbf{u} \geq \mathbf{0} \quad [\mathbf{z}_k] \tag{1}$$

Proof. If the result holds for some positive vector $\mathbf{u}$, it holds for any positive vector $\mathbf{v}$ by applying it to function g defined by $g_i(\mathbf{z}) = v_i^{-1} u_i f_i(\mathbf{z})$. We may therefore assume $\mathbf{u}^T = \mathbf{u}_n^T = (1, ..., 1)$. By the GND lemma applied to the extension of f, there exists $(\mathbf{x}_\varepsilon, t_\varepsilon)$ in S_ε such that

$$\forall (\mathbf{x}, t) \in S_\varepsilon \quad \mathbf{x}^T f(t_\varepsilon^{-1}\mathbf{x}_\varepsilon) - t t_\varepsilon^{-1} \mathbf{x}_\varepsilon^T f(t_\varepsilon^{-1}\mathbf{x}_\varepsilon) \geq 0 \tag{2}$$

We have $(\mathbf{x}_\varepsilon, t_\varepsilon) \neq (\mathbf{0}, 1)$, otherwise $\mathbf{z} = \mathbf{0}$ would be a solution of NCP(f). For a given $\varepsilon > 0$ and the choice $t = 1$, $\mathbf{x} = \mathbf{0}$, inequality (2) shows that the scalar $\lambda_\varepsilon = -(1 - t_\varepsilon)^{-1} \mathbf{x}_\varepsilon^T f(t_\varepsilon^{-1}\mathbf{x}_\varepsilon)$ is nonnegative. For the choice $\mathbf{x} = (1 - t_\varepsilon)\mathbf{e}_i, t = t_\varepsilon$, it shows that $f_i(t_\varepsilon^{-1}\mathbf{x}_\varepsilon) + \lambda_\varepsilon \geq 0$, or

$$\min_i f_i(t_\varepsilon^{-1}\mathbf{x}_\varepsilon) \geq -\lambda_\varepsilon \tag{3}$$

By definition of $-\lambda_\varepsilon$, the right-hand side term is a positive barycentre of the coordinates $f_i(t_\varepsilon^{-1}\mathbf{x}_\varepsilon)$ associated with the positive components $x_{\varepsilon i}$ of $\mathbf{x}_\varepsilon$. Inequality (3) implies that all these coordinates are equal to $-\lambda_\varepsilon$, while the coordinates corresponding the zero components of $\mathbf{x}_\varepsilon$ are greater. Vector $\mathbf{z}_\varepsilon = t_\varepsilon^{-1}\mathbf{x}_\varepsilon$ is therefore a solution to (1) for $\lambda_k = \lambda_\varepsilon$. If, when ε tends to zero, t_ε admits a positive cluster point t_0, there also exists a cluster point $(\mathbf{x}_0, t_0)$ of $(\mathbf{x}_\varepsilon, t_\varepsilon)$. For any given $(\mathbf{x}, t) \in S_\varepsilon$, inequality (2) holds by continuity when $(\mathbf{x}_\varepsilon, t_\varepsilon)$ is replaced by its limit $(\mathbf{x}_0, t_0)$. As this holds for any $\varepsilon > 0$, we have $f(t_0^{-1}\mathbf{x}_0) \geq 0$ and $-t_0^{-1}\mathbf{x}_0^T f(t_0^{-1}\mathbf{x}_0) \geq 0$, and $\mathbf{z}_0 = t_0^{-1}\mathbf{x}_0$ is a solution to NCP(f). This being excluded, t_ε tends to zero and $\|\mathbf{z}_\varepsilon\| = t_\varepsilon^{-1}(1 - t_\varepsilon)$ tends to infinity. Eventually, λ_ε is not zero since NCP(f) has no solution, therefore λ_ε is positive. ■

The following two corollaries (corollary 2 is due to Bidard ([2])) illustrate how the Theorem can be used to prove existence results: in the first case, λ_k tends to zero, not in the second case.

Corollary 1 *Let $f(\mathbf{z}) = h(\mathbf{z}) + \mathbf{q}$, where $h : R^n_+ \to R^n$ is continuous, homogeneous of degree one and copositive ($\forall \mathbf{z} \geq 0 \;\; \mathbf{z}^T h(\mathbf{z}) \geq 0$). Under assumption ($H_1$):*

$$\left\{\mathbf{z} \geq \mathbf{0}, \mathbf{z} \neq \mathbf{0}, h(\mathbf{z}) \geq \mathbf{0}, \mathbf{z}^T h(\mathbf{z}) = 0\right\} \text{ implies } \mathbf{z}^T \mathbf{q} > 0, \tag{4}$$

NCP(f) has a solution.

Proof. Let $\mathbf{u}^T = (1, ..., 1)$. If NCP$(f)$ has no solution, there exist solutions $\mathbf{z} = \mu\mathbf{x}$ with $\mathbf{x} \geq 0, \mathbf{u}^T\mathbf{x} = 1$, μ arbitrarily great, to an infinite set of NCPs parameterized by positive scalars λ

$$h(\mu\mathbf{x}) + \mathbf{q} + \lambda\mathbf{u} \geq 0 \quad [\mathbf{x}] \tag{5}$$

From copositivity and the complementarity relationship, one gets $\mathbf{x}^T\mathbf{q} + \lambda \leq 0$, therefore the values of λ are upper bounded. There exists a subset of NCPs such that λ tends to a nonnegative scalar λ_0, $\mathbf{x}$ tends to $\mathbf{x}_0$ with $\mathbf{x}_0 \geq 0, \mathbf{u}^T\mathbf{x}_0 = 1$, and μ is arbitrarily great. Inequality (5) implies that $h(\mathbf{x}) + \mu^{-1}(\mathbf{q} + \lambda\mathbf{u}) \geq 0$, therefore $h(\mathbf{x}_0) \geq \mathbf{0}$. Similarly, the complementarity relationship in (5) implies $\mathbf{x}_0^T h(\mathbf{x}_0) = 0$. By (4), we have $\mathbf{x}_0^T\mathbf{q} > 0$ and a contradiction is obtained with inequality $\mathbf{x}_0^T\mathbf{q} + \lambda_0 \leq 0$. ■

When (H_1) is replaced by the weaker assumption (H_2):

$$\left\{\mathbf{z} \geq 0, h(\mathbf{z}) \geq 0, \mathbf{z}^T h(\mathbf{z}) = 0\right\} \text{ implies } \mathbf{z}^T\mathbf{q} \geq 0, \tag{6}$$

the above reasoning shows that $\mathbf{x}_0^T\mathbf{q} = \lambda_0 = 0$, therefore NCPs arbitrarily close to NCP(f) are solvable. For LCPs, the set of vectors $\mathbf{q}$ for which LCP$(\mathbf{q}, \mathbf{M})$ is solvable is closed, therefore LCP$(\mathbf{q}, \mathbf{M})$ itself is solvable. This may not be the case for homogeneous NCPs: consider the function $f : R_+^2 \to R^2$ defined by

$$\begin{aligned} h_1(z_1, z_2) &= z_1 \frac{z_1 + 2z_2}{z_1 + z_2}, q_1 = -1 \\ h_2(z_1, z_2) &= \frac{-z_1^2}{z_1 + z_2}, q_2 = 0 \end{aligned}$$

We have $\mathbf{z}^T h(\mathbf{z}) = z_1^2$. The set $\left\{\mathbf{z} \geq \mathbf{0}, h(\mathbf{z}) \geq \mathbf{0}, \mathbf{z}^T h(\mathbf{z}) = 0\right\}$ is the set of nonnegative vectors with a first zero component, therefore assumption (H_2) is met. However, since inequality $h_2(\mathbf{z}) + q_2 \geq 0$ implies $z_1 = 0$, we then have $h_1(\mathbf{z}) + q_1 = -1$ and NCP(f) has no solution.

Corollary 2 *Let f: $R_+^n \to R^m, g : R_+^m \to R^n$ be continuous functions, and $\mathbf{c} \in R^m$, $\mathbf{d} \in R^n$ be vectors. Under assumptions:*

(i) $\forall(\mathbf{x}, \mathbf{y}) \geq \mathbf{0} \quad x^T g(\mathbf{y}) + \mathbf{y}^T f(\mathbf{x}) \geq 0$

(ii) f is homogenous of degree one

(iii) $\{\mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, f(\mathbf{x}) \geq \mathbf{0}\} \Rightarrow \mathbf{d}^T\mathbf{x} < 0$

(iv) $\mathbf{c} << \mathbf{0}$

there exists a solution to the NCP

$$f(\mathbf{x}) \geq \mathbf{c} \quad [\mathbf{y}] \tag{7}$$

$$g(\mathbf{y}) \geq \mathbf{d} \quad [\mathbf{x}] \tag{8}$$

Proof. If the NCP has no solution, there exist solutions $(\mathbf{x}_k, \mathbf{y}_k)$ tending to infinity to infinitely many NCPs

$$f(\mathbf{x}_k) + \lambda_k \mathbf{u}_m - \mathbf{c} \quad \geq \quad \mathbf{0} \quad [\mathbf{y}_k] \tag{9}$$
$$g(\mathbf{y}_k) + \lambda_k \mathbf{u}_n - \mathbf{d} \quad \geq \quad \mathbf{0} \quad [\mathbf{x}_k] \tag{10}$$

By the complementarity relationship and condition (i), we have

$$\lambda_k(\mathbf{u}_n^T \mathbf{x}_k + \mathbf{u}_m^T \mathbf{y}_k) - \mathbf{d}^T \mathbf{x}_k - \mathbf{c}^T \mathbf{y}_k \leq 0 \tag{11}$$

If the sequence $\|\mathbf{x}_k\|$ remained bounded, $\|\mathbf{y}_k\|$ would tend to infinity and a contradiction between assumption (iv) and (11) would be obtained. We therefore assume that $\|\mathbf{x}_k\|$ tends to infinity. If the sequence $\lambda_k^{-1} \|\mathbf{x}_k\|$ were bounded from above, a contradiction would be obtained with inequality $\mathbf{u}_n^T \mathbf{x}_k - \lambda_k^{-1} \mathbf{d}^T \mathbf{x}_k \leq 0$, which follows from (11) and (iv). We therefore assume that $\lambda_k \|\mathbf{x}_k\|^{-1}$ tends to zero. Then, condition (ii) and inequality (9) imply that a cluster point $\mathbf{x}_0$ of $\|\mathbf{x}_k\|^{-1} \mathbf{x}_k$ is such that $\|\mathbf{x}_0\| = 1$ and $f(\mathbf{x}_0) \geq \mathbf{0}$, therefore $\mathbf{d}^T \mathbf{x}_0 < 0$ by condition (iii) and $\mathbf{d}^T \mathbf{x}_k < 0$ for k great enough. Again, a contradiction with inequality (11) is obtained. ■

The proof of Theorem 1 is 'almost constructive' in the following sense. Given f, consider a sequence of positive scalars ε tending to zero and, for each ε, solve the programme (P_ε): find $(\mathbf{x}_\varepsilon, t_\varepsilon) \in S_\varepsilon$ such that property (2) holds. This is a programme of the type met by the general equilibrium theory: vector $(\mathbf{x}, t)$ is transformed into the orthogonal vector $(f(t^{-1}\mathbf{x}), -t^{-1}\mathbf{x}^T f(t^{-1}\mathbf{x}))$, and the problem is to make the transformed vector nonnegative. (By contrast, in many NCP algorithms, a similar problem is solved in a dual way: starting from nonnegative vectors, the aim is to make them orthogonal.) A solution always exists, and vector $\mathbf{z}_\varepsilon = t_\varepsilon^{-1} \mathbf{x}_\varepsilon$ is a solution of (1) for some λ_ε. If $\|\mathbf{z}_\varepsilon\|$ remains bounded, a cluster point of $\mathbf{z}_\varepsilon$ is a solution of NCP(f). If $\lambda_\varepsilon = -\min_i f_i(\mathbf{z}_\varepsilon)$ tends to zero, problems close to NCP(f) admit a solution. That procedure, however, gives no hint on the solvability of NCP(f) when $\|\mathbf{z}_\varepsilon\|$ tends to infinity and λ_ε admits a lower positive bound.

References

[1] Bidard, C. (2004), *Prices, Reproduction, Scarcity* (Cambridge, Cambridge University Press)

[2] Bidard, C. (2015), An oddity property for two-person games, *Advanced Modeling and Optimization*, 17(2), 233-244

[3] Debreu, G. (1956), Market equilibrium, *Proceedings of the National Academy of Sciences of the USA*, 38, 886-93

[4] Gale, D. (1955), The law of supply and demand, *Mathematica Scandinavia*, 3, 155-69

[5] Nikaido, H. (1956), On the classical multilateral exchange problem, *Metroeconomica*, 8, 135-45