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BOUNDARY CONTROL OF STOCHASTIC ELLIPTIC SYSTEMS INVOLVING LAPLACE OPERATOR

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Abstract. The purpose of the present paper is to study the boundary control for Neumann or Dirichlet stochastic elliptic systems are introduced. The existence of the unique state process for these systems is derived, then the set of equations and inequalities that characterizes the boundary control is obtained.

1. Introduction

A stochastic partial differential equation (SPDE) is a partial differential equation in which one or more of the terms is a stochastic process, and resulting in a solution which is itself a stochastic process. SPDE consists of a partial differential equation containing a deterministic part and an additional random white noise term. Optimal control represents a study has many biological and physical and mechanical applications and can easily be linked with many sciences because of its extreme importance. It determine control and state trajectories for a dynamic system. Control either be added to the region (Distributed control) or on the boundary (Boundary control).

The model of the system represented by partial differential operators in different of works started by Lions [10]. So many problems concerning the distributed (or boundary) control of systems governed by partial differential operators with Dirichlet or Neumann conditions appeared for one or two equations in [14].

The necessary and sufficient conditions of optimality for systems governed by elliptic operators have been studied by Lions in [10].

In the present work, we focus on the boundary control for Neumann and Dirichlet stochastic elliptic systems in scalar case.

This paper is organized as follows: In section 1; some definitions and notations are mentioned. In section 2, the existence and uniqueness of the state process for Neumann stochastic elliptic systems is stated; then the set of equations and inequalities that characterizes the boundary control for these systems is found. In section 3, Dirichlet elliptic problems conditions are considered.

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1. Notations

In this section, we first wish to collect some basic definitions, lemmas that will be important to us in the sequel. These and other related results and their proofs can be considered.

we shall consider some definitions introduced in [3-9], [12]concerning the Stochastic Sobolev space , the embedding , which are necessary to introduce our work. Let$G$be an open set in $\mathbb{R}^n, (\Omega, \mathcal{F}, P)$be a probability space, where Ω is a sample space, $\mathcal{F}$ is an σ-algebra and P is a probability measure.

We introduce

$$V = L^2(\Omega, \mathcal{F}, P; G) = \left\{ v : G \times \Omega \to \mathbb{R} |\ v \text{ is measurable and } \int_\Omega \|v\|^2 dp < \infty \right\},$$

with inner product $(.,.)_V : V \times V \to \mathbb{R}$, defined as

$$(u, v)_V = \left(\int_\Omega \left(\int_G \nabla u(x)\ \nabla v(x)\ dx \right) dp \right)$$

For instance,$H^1(G)$is a Hilbert space with a norm$\|...\|_{H^1(G)}; H_0^1(G)$ is the subspace of $H^1(G)$whose function value is zero on the boundary of$G$, and its norm is $\|u|^2_{H_0^1(G)} = \int_G (\nabla u)^2 dx$.

With these standard Sobolev spaces, we define stochastic Sobolev spaces as follows:

$$V = L^2(\Omega; H^1(G)) = \{ v : G \times \Omega \longrightarrow \mathbb{R} |\ \|v\|^2_{L^2(\Omega; H^1(G))} < \infty \},$$

where

$$\|v\|^2_{L^2(\Omega; H^1(G))} = \int_G (\|v\|^2_{H^1(G)}) dp = \mathbb{E}(\|v\|^2_{H^1(G)}).$$

Note that, we can write Stochastic Sobolev spaces by $L^2(\Omega, \mathcal{F}, P; G)orL^2(\Omega; H^1(G))$

For the weak formulation of our stochastic elliptic partial differential equation, we introduce the following notations:

$$a(u, v) = \mathbb{E}(\int_G \nabla u\ \nabla v dx)$$

and

$$(u, v) = \mathbb{E}(\int_G u\ v dx),$$

where$\mathbb{E}$is the expected value.

we introduce the Riesz isomorphism

$$A : H^1(\Omega, \mathcal{F}, P; G) \to H^1(\Omega, \mathcal{F}, P; G)$$

associated with the standard scalar product of$H^1(\Omega, \mathcal{F}, P; G)$,that is,

$$< Au, v >_{H^1(\Omega, \mathcal{F}, P; G)} = \mathbb{E} \left(\int_G ((\nabla u\ \nabla v + u\ v) dx \right), \forall u, v \in H^1(\Omega, \mathcal{F}, P; G)$$

2. Existence and Uniqueness for Stochastic Elliptic Systems and Dirichlet Conditions

In this section, we discuss the boundary optimal control of the stochastic elliptic systems involving Laplace operator and prove the existence of an optimal solution based on the Lions theory.

Let us consider the following stochastic elliptic equations:

$$\begin{cases} -\Delta u(x) = W(x) & \text{in } G \\ u(x) = 0 & \text{on } \partial G \end{cases} \tag{2.1}$$

where$G$is a bounded, continous and strictly domain in$\mathbb{R}^n$with boundary∂G While$u(x) \in H_0^1(\Omega, \mathcal{F}, P; G)$is a state process and$W(x)$is a white noise. We prove the existence and uniqueness of the state process for system (2.1) in the following subsection.

2.2 Formulation of the Optimal Control Problem. In this subsection we wish to formulate mixed initial boundary value Dirichlet problem for stochastic elliptic systems and we investigate necessary conditions for an optimal control policy.

The space $[L^2(\Omega, \mathcal{F}, P; \partial G)]$ being the space of controls.

For a control $y \in [L^2(\partial, \mathcal{F}, P; \partial G)]$, the state process of the system u is given by the solution of the following system:

$$\begin{cases} -\Delta u(y) = W & \text{in } G \\ u(y) = y & \text{on } \partial G. \end{cases} \tag{2.2}$$

The observation equation is given by $\chi(y) \equiv u(y)$, the cost functional is given by:

$$C(y) = \mathbb{E}\biggl(\int_{\partial G} (((u(y) - u(0) + (u(0) - \chi_d))^2) dx\biggr) + \int_\Omega \biggl(\int_{\partial G} M(z^2)\ dx\biggr)\ dp, \tag{2.3}$$

whereχ_d in $[L^2(\Omega, \mathcal{F}, P; \partial G)]$.

Then, the control problem is defined by:

$$\begin{cases} y \in Y_{ad} & \text{such that} \\ C(z) = \inf C(y) & \forall z \in Y_{ad}, \end{cases}$$

where$Y_{ad}$is a closed convex subset from$[L^2(\Omega, \mathcal{F}, P; \partial G)]$.

Since the cost functional (2.3) can be written as:

$$C(y) \quad = \quad \mathbb{E}\biggl(\int_{\partial G} (((u(y) - u(0) + (u(0) - \chi_d))^2) dx\biggr) + \int_\Omega \biggl(\int_{\partial G} M(z^2)\ dx\biggr)\ dp,$$

where

$$\Pi(y, z) = \mathbb{E}\biggl(\int_{\partial G} \{(u(y) - u(0))^2 + (u(z) - u(0))^2\} dx\biggr) + \int_\Omega \int_{\partial G} \biggl(M(z^2)\biggr)\ dx\ dp, \tag{2.4}$$

$M > 0$ is a positive constant, then

$$L(z) = \mathbb{E}\biggl(\int_{\partial G} (-u(0) + \chi_d)(u(i) - u(0)\ dx\biggr), \tag{2.5}$$

and$\Pi(y,y)$is a stochatic coercive on$[L^2(\Omega,\mathcal{F},P;\partial G)]$.Since$L(z)$is continuous on$[L^2(\Omega,\mathcal{F},P;\partial G)]$, then there exists a unique optimal control from the general theory in [10].

Moreover, we have the following theorem which gives the characterization of the optimal control.

Theorem 2.2.

If the state $u(y)$ is given by (2.1) and if the cost functional is given by (2.3), then there exists a unique optimal control $y \in Y_{ad}$ such that $C(y) \leq C(z)$ $\forall z \in Y_{ad}$; Moreover, it is characterized by:

$$\begin{cases} -\Delta h(y) = u(y) - \chi_d & \text{in } G \\ \\ h(y) = 0 & \text{on } \partial G, \end{cases}$$

where$h(y)$is the adjoint state process.

Proof.

Since $C(y)$ is differentiable and Y_{ad} is bounded, then the optimal control$z$is characterized (see e.g [8,9]). Using equations (2.4), (2.5), we get

$$\Pi(y, z-y) \geq L(z-y), \tag{2.6}$$

and

$$\begin{aligned} \Pi(y,z-y) \quad &- \quad L(z-y) \\ &= \mathbb{E}\Big(\int_{\partial G}((u(y)-u(0))((u(z-y)-u(0))))dx\Big) \\ &- \mathbb{E}\Big(\int_{\partial G}(((u(0)-\chi_d)((u(z-y)-u(0)))))dx\Big) \\ &+ \int_\Omega\Big(\int_{\partial G} My(z-y)\ dx\Big)dp \\ &= \int_\Omega\Big(\int_{\partial G} My(z-y))\ dx\Big)dp \\ &+ \mathbb{E}\Big(\int_{\partial G}((u(y)-\chi_d)(u(z)-u(y))dx\Big) \geq 0, \end{aligned}$$

with$(B^*h(y),u(y)) = (h(y),Bu(y))$,and B is defined by:

$$B\ \Phi = B\ \{u(y)\} = (-\Delta u(y)).$$

Applying the derivative in the sense of distribution, we get

$$B^*h(y) = u(y) - \chi_d,$$

where$B = -\Delta$ and . So,

$$\begin{aligned} \Pi(y,z-y) \quad &- \quad L(z-y) \\ &= \int_\Omega\Big(\int_{\partial G}(My, z-y)dx\Big)dp + \mathbb{E}\Big(\int_{\partial G}(h(-\Delta u(z)))dx\Big) \geq 0 \end{aligned}$$

Hence, from (2.6) we obtain$\mathbb{E}\left(\int_{\partial G}((h+My)(z-y)dx\right) \geq 0$ □

Remark 2.1

If constraints are absent, i.e. when$Y_{ad} = Y$,then$h(z) + My = 0, z_j \neq y_jory = -\dfrac{h(z)}{M}$the differential problem of finding the vector-function satisfies the the following relations.

For the state process equations

$$\begin{cases} Bu + \dfrac{h(z)}{M} = W & \text{in } G \\ \\ u = 0 & \text{on } \partial G. \end{cases}$$

For the adjoint state process equations

$$\begin{cases} Bh(y) - u(y) = -\chi_d & \text{in } G \\ \\ h(y) = 0 & \text{on } \partial G. \end{cases}$$

3 Neumann Stochastic Elliptic Systems

In this section, we study the optimal control problem for stochastic elliptic system with Neumann conditions.

$$\begin{cases} -\Delta u(x) = W(x) & \text{in } G_1 \\ \\ \dfrac{\partial u(x)}{\partial V_A} = g & \text{on } \partial G, \end{cases} \tag{3.1}$$

where$g \in H^{\frac{1}{2}}(\Omega, \mathcal{F}, P; \partial G)$.

3.1 Existence and Uniqueness of Solution. In this subsection, we study the existence and uniqueness of solutions for stochastic systems governed by Neumann problems. Since

$$[H_0^1(\Omega, \mathcal{F}, P; \partial G)]^2 \subseteq [H^1(\Omega, \mathcal{F}, P; \partial G)]^2,$$

then

$$\|u\|^2_{[H_0^1(\Omega,\mathcal{F},P;\partial G)]^2} \subseteq \|u\|^2_{[H^1(\Omega,\mathcal{F},P;\partial G)]^2},$$

which proves the coerciveness of bilinear form$a(u,u)$on$[H^1(\Omega, \mathcal{F}, P; \partial G)]^2$

$$b(u,u) \geq c\|u\|^2_{[H^1(\Omega,\mathcal{F},P;\partial G)]^2} \text{ (Stochastic coerciveness)} \tag{3.2}$$

Theorem 3.1.

Assume that (3.2) holds, and then there exists a unique solution$u$of system (3.1).

Proof.

Since the bilinear form$b(u,\Psi)$is continuous and stochastic coercive on$[H^1(\Omega,\mathcal{F},P;\partial G)]^2$, then by Lax Milgram lemma there exist a unique solution of:

$$b(u,\Psi)=L(\Psi),\forall u\in[H^1(\Omega,F,\mathbf{P};\partial G)]^2, \tag{3.3}$$

where$L(\Psi)$is continuous linear form defined on $[H^1(\Omega,\mathcal{F},P;\partial G)]^2$by using Green's formula, we obtain (3.1):

$$L(\Psi)=\mathbb{E}\Big(\int_{\partial G}(W\ \Psi)\ dx+\int_{\partial G}(g\ \Psi)\ d\partial G\Big),$$

then (3.3) is equivalent to

$$\begin{aligned} b(u,\Psi) &= \mathbb{E}\Big(\int_{\partial G}(\nabla u\ \nabla\Psi)\ dx\Big)\\ &+ \mathbb{E}\Big(\int_{\partial G}\frac{\partial u(x)}{\partial V_A}\Psi\Big)\\ &= \mathbb{E}\Big(\int_{\partial G}(W\ \Psi)\ dx+\int_{\partial G}(g\ \Psi)\ d\partial G\Big) \end{aligned}$$

Hence (3.3) is equivalent to (3.1) and there exists a unique solution of (3.1).

3.2 Formulation of the Optimal Control Problem with Neumann Conditions. Here, we formulate the problem and establish necessary and sufficient conditions for the optimal control of distributed type. The space$[L^2(\Omega,\mathcal{F},P;\partial G)]^2$is the space of controls. For a control$y\in[L^2(\Omega,\mathcal{F},P;\partial G)]^2$,the state$u(y)$of the system is given by the solution of

$$\begin{cases} -\Delta u(y)=W(y) & \text{in } G_1\\ \dfrac{\partial u(y)}{\partial V_A}=g+y & \text{on } \partial G. \end{cases} \tag{3.4}$$

The observation is given by$\chi(y)=u(y)$,the cost functional is given again by (3.4). The optimal control is characterize by the following theorem:

Theorem 3.2.

Assume that (3.2) holds, if the cost functional is given by (2.7), then there exists an optimal control$y=(y_1,y_2)\in[L^2(\Omega,\mathcal{F},P;\partial G)]^2$.Moreover, it is characterized by the following equations and inequalities:

$$\begin{cases} -\Delta h(y)=0 & \text{in } G\\ \dfrac{\partial h(y)}{\partial V_A^*}=M\dfrac{\partial u(y)}{\partial V_A}-\chi_d. \end{cases}$$

Together with (3.4), where$p(u)$is the adjoint state

$$\mathbb{E}\left(\int_{\partial G}((\frac{h(y)}{\partial V_A}+My)(z-y)dx\right)\geq 0\ \square$$

Remark 3.1

If constraints are absent, i.e. when$Y_{ad}=Y$,then$h(y)+Ny=0$or$y=-\dfrac{h(y)}{N}$the differential problem of finding the vector-function satisfies the following relations:

For the state process equations

$$\begin{cases} AU=W & \text{in } G \\ \\ \dfrac{\partial U(y)}{\partial V_A}+\dfrac{h(u)}{N}=g & \text{on } \partial G. \end{cases}$$

For the adjoint state process equations

$$\begin{cases} Ah(y)=0 & \text{in } G \\ \\ \dfrac{\partial h(y)}{\partial V_A^*}=-M\dfrac{\partial h(y)}{\partial V_A^*}=-\chi_d, & \text{on } \partial G. \end{cases}$$

4 Dirichlet and Neumann Elliptic Systems

In this section, we study the distributed control problem for elliptic systems involving Laplace operator. We consider the following elliptic equations:

$$\begin{cases} -\Delta u(x)=f(x) & \text{in } G \\ \\ u(x)=0 & \text{on } \partial G \end{cases} \tag{4.1}$$

where$G$is a bounded, continous and strictly domain in$\mathbb{R}^n$with boundary∂G While$u(x)\in H_0^1(\partial G)$, $f\in L^2(\partial G)$is a state process and$W(x)$is a wiener process. We derive the existence and uniqueness of state of the system (4.1) in the following subsection.

The space $[L^2(\partial G)]$ being the space of controls. For a control $y\in[L^2(\partial G)]$, the state process of the system u is given by the solution of the following system:

$$\begin{cases} -\Delta u(y)=W & \text{in } G \\ \\ u(y)=y & \text{on } \partial G, \end{cases} \tag{4.2}$$

. The observation equation is given by$\chi(y)\equiv u(y)$,the cost functional is given by:

$$C(y)=\left(\int_{\partial G}(((u(y)-u(0)+(u(0)-\chi_d))^2)dx\right)+\left(\int_{\partial G}M(z^2)\ dx\right), \tag{4.3}$$

whereχ_d in $[L^2(\partial G)]$.

Then, the control problem is defined by:

$$\begin{cases} y \in Y_{ad} & \text{such that} \\ \\ C(z) = \inf C(y) & \forall z \in Y_{ad}, \end{cases}$$

where$Y_{ad}$is a closed convex subset from$[L^2(G)]$.
Since the cost functional (4.3) can be written as:

$$C(y) \quad = \quad \left(\int_{\partial G}(((u(y) - u(0) + (u(0) - \chi_d))^2)dx\right) + \left(\int_{\partial G} M(z^2)\ dx\right),$$

where

$$\Pi(y, z) = \left(\int_{\partial G}\{(u(y) - u(0))^2 + (u(z) - u(0))^2\}dx\right) + \int_{\partial G}\left(M(z^2)\right)\ dx, \quad (4.4)$$

$M > 0$ is a positive constant, then

$$L(z) = \mathbb{E}\left(\int_{\partial G}(-u(0) + \chi_d)(u(i) - u(0)\ dx\right), \quad (4.5)$$

and$\Pi(y, y)$is a coercive on$[L^2(\partial G)]$.Since$L(z)$is continuous on$[L^2(\partial G)]$, then there exists a unique optimal control from the general theory in [10].

Since$C(y)$is differentiable and$Y_{ad}$is bounded, then the optimal control$z$is characterized (see e.g [10,11]). Using equations (4.4), (4.5), we get

$$\Pi(y, z - y) \geq L(z - y), \quad (4.6)$$

$$\left(\int_{\partial G}((h + My)(z - y)dx\right) \geq 0$$

If constraints are absent, i.e. when$Y_{ad} = Y$,then$h(z) + My = 0, z_j \neq y_jory = -\dfrac{h(z)}{M}$the differential problem of finding the vector-function satisfies the the following relations.

For the state equation

$$\begin{cases} Bu + \dfrac{h(z)}{M} = f & \text{in } G \\ \\ u = 0 & \text{on } \partial G. \end{cases}$$

For the adjoint state equation

$$\begin{cases} Bh(y) - u(y) = -\chi_d & \text{in } G \\ \\ h(y) = 0 & \text{on } \partial G. \end{cases}$$

There is no change in Neumann, where the difference are also in bilinear form, linear form and the cost functional.

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