Probability-Credibility Approach for a Multiobjective Linear Programming Problem

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Abstract. In real world problems, decision maker encounter uncertain environment where fuzziness and randomness coexist in the mathematical model. In this work, we consider multiobjective programming problem involving fuzzy random variables coefficients. We assume that the coefficients in the constraints and in the objective functions are represented by fuzzy random variables but the decision variables are crisp. First, based on fuzzy random theory we formulate the crisp equivalent model of the chance-constrained multiobjective programming problem under some assumptions. Then, we study the convexity of the set of feasible solutions. In order to obtain the decision maker's satisfactory solution, we consider an interactive procedure based on an interactive fuzzy satisfying method. Finally, an example is provided for illustrating purpose.

Keywords: Credibility measure, Fuzzy random variable, Multiobjective linear programming.

AMS Classification: 90C05; 90C29; 90C70; 90C15.

1. Introduction

In real life optimization problem, the coefficients used in mathematical models are obviously unclear and vague. Various approaches have been used to model uncertainty that arises from imprecise or lack of information on mathematical programming problems, such as, stochastic programming model [3] and fuzzy programming model ([3], [27]).

But in many cases, we may encounter situations where fuzziness and randomness coexist. When the coefficients are random variables and their realizations are fuzzy variables, we have the concept of fuzzy random variable, introduced by Kwakernaak [8]. Thereafter, this concept have received great attention and many studies have been conducted in this area of research. To mention but a few, see for example Kruse and Meyer [7], Puri and Ralescu [17], Wang et al. [23], Qiao et al. [18], Lopez-Diaz and Ralescu [15], Couso and Dubois [1], Couso and Sanchez [2], Hao and Liu [5].

Recently, multi-criteria linear programming problem with fuzzy random variables have received an increasing attention. Without being exhaustive, we mention some references [6, 9, 16, 21, 22, 25, 24]. An overview on fuzzy stochastic multiobjective programming field is given in the book of Sakawa et al. [20].

This problem remains an important direction of research and new results in this field continue to be discovered. In this paper, we consider fuzzy stochastic multiobjective linear programming problem with coefficients in the constraints and in the objective functions are represented by fuzzy random variables but the decision variables are crisp. We assume that the realizations of the random variables are LR-fuzzy variables in a credibility space. First, under some assumptions we formulate the deterministic equivalent of the chance-constrained programming problem, then we study the convexity of the set of feasible solutions. Finally, we propose an interactive procedure based on the interactive fuzzy satisfying method of Sakawa [19] and we conclude with an illustrative example.

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311.

The rest of this paper is organized as follows. In section 2, we recall some definitions and results on fuzzy theory, we also recall the definition of fuzzy random variables. In section 3, we consider fuzzy stochastic multiobjective linear programming problem with coefficients in the constraints and in the objective functions are represented by fuzzy random variables but the decision variables are crisp. A chance constrained multiobjective programming model for this problem is formulated using a probability-credibility approach.Then, we give it's crisp equivalent model, we study the convexity of the feasible sets and we develop an interactive procedure which gives the satisfying solution for the decision maker. An illustrative example is presented in section 4 which clarifies the developed interactive procedure.

2. Preliminaries

In this section, the basic definitions [26] involving fuzzy sets, fuzzy numbers and arithmetic on fuzzy numbers [4] are reviewed.

2.1. Basic definitions

Definition 1. *Let X denote a universal set. A fuzzy subset $\widetilde{A}$ of X is defined by its membership function $\mu_{\widetilde{A}} : X \mapsto [0,1]$, which assigns a real number $\mu_{\widetilde{A}}(x)$ in the interval $[0,1]$, for each element $x \in X$, where the value of $\mu_{\widetilde{A}}(x)$ at x shows the degree of membership of x in $\widetilde{A}$. Therefore, a fuzzy set $\widetilde{A}$ is completely characterized by the set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))/x \in X\}$.*

Definition 2. *A fuzzy set $\widetilde{A}$ in X is convex if*
$\mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\widetilde{A}}(x),\ \mu_{\widetilde{A}}(y)\}, \forall x, y \in X$ *and* $\forall \lambda \in [0,1]$.

Definition 3. *A fuzzy set $\widetilde{A}$ in X is said to be normal if there exist at least $x_0 \in X$ such that*

$$\mu_{\widetilde{A}}(x_0) = 1.$$

Definition 4. *A fuzzy number is a convex normalized fuzzy set of the real line $\mathbb{R}$ whose membership function is piecewise continuous.*

Definition 5. *A fuzzy number $\widetilde{A}$, denoted as $(\overline{m}, \overline{n}, \overline{\alpha}, \overline{\beta})_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}$ is given by*

$$\mu_{\widetilde{A}}(t) = \begin{cases} L(\frac{\overline{m}-t}{\overline{\alpha}}) & if \quad \overline{m}-\overline{\alpha} \leq t \leq \overline{m},\ \overline{\alpha} > 0 \\ 1 & if \quad \overline{m} \leq t \leq \overline{n} \\ R(\frac{t-\overline{n}}{\overline{\beta}}) & if \quad \overline{n} \leq t \leq \overline{n}+\overline{\beta},\ \overline{\beta} > 0 \\ 0 & otherwise \end{cases}$$

where $L,\ R : [0,\ 1] \longrightarrow [0,\ 1]$ are two continuous non increasing shape functions such that $R(0) = L(0) = 1$ and $L(1) = R(1) = 0$. $[\overline{m}, \overline{n}]$ is the core of $\widetilde{A}$; $\mu_{\widetilde{A}}(t) = 1\ \ \forall t \in [\overline{m}, \overline{n}]$; $\overline{m}, \overline{n}$ are the lower and upper modal values of $\widetilde{A}$ and $\overline{\alpha} > 0,\ \overline{\beta} > 0$ are the left hand and the right hand spreads.
The support of $\widetilde{A}$ is $[\overline{m}-\overline{\alpha},\ \overline{n}+\overline{\beta}]$.

Remark 1. *Among the various type of LR flat fuzzy numbers, we have trapezoidal fuzzy numbers, obtained from Definition 5, by taking $L(t) = R(t) = max(0, 1-|t|)$. The membership function of the trapezoidal fuzzy number $\widetilde{A} = (\overline{m}, \overline{n}, \overline{\alpha}, \overline{\beta})_{LR}$ is given as follows :*

$$\mu_{\widetilde{A}}(t) = \begin{cases} 1-(\frac{\overline{m}-t}{\overline{\alpha}}) & if\, \overline{m}-\overline{\alpha} \leq t \leq \overline{m},\ \overline{\alpha} > 0 \\ 1 & if\, \overline{m} \leq t \leq \overline{n} \\ 1-(\frac{t-\overline{n}}{\overline{\beta}}) & if\, \overline{n} \leq t \leq \overline{n}+\overline{\beta},\ \overline{\beta} > 0 \\ 0 & otherwise \end{cases}$$

2.2. Arithmetic on fuzzy numbers

Definition 6. *Let $\widetilde{A} = (m_1, n_1, \alpha_1, \beta_1) \in F(\mathbb{R})$ and $\widetilde{B} = (m_2, n_2, \alpha_2, \beta_2) \in F(\mathbb{R})$ where $F(\mathbb{R})$ denotes the set of all real LR flat fuzzy numbers.Then, arithmetic on fuzzy numbers are defined as:*
The addition: $\widetilde{A} + \widetilde{B} = (m_1+m_2,\ n_1+n_2,\ \alpha_1+\alpha_2,\ \beta_1+\beta_2)$.
The image of $\widetilde{A}$: $-\widetilde{A} = (-n_1,\ -m_1,\ \beta_1,\ \alpha_1)$.

The subtraction : $\widetilde{A} - \widetilde{B} = (m_1 - n_2,\ n_1 - m_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2)$.
The scalar multiplication :

$$\begin{cases} k.\widetilde{A} = (km_1, kn_1, k\alpha_1, k\beta_1) & \text{if } k \geq 0,\ k \in \mathbb{R} \\ k.\widetilde{A} = (kn_1, km_1, -k\beta_1, -k\alpha_1) & \text{if } k < 0,\ k \in \mathbb{R} \end{cases}$$

2.3. Credibility measure

The credibility theory developed by Liu [12, 13], is an appropriate tool for treating fuzziness. This theory has been successfully used in many areas of research such as fuzzy optimization, management science, engineering technology, etc [10].

Definition 7. *Given a Universe X, we denote $P(X)$ the power set of X. A set function Cr on $P(X)$ is called a credibility measure if*
$Cr(X) = 1$;
$Cr(A) \leq Cr(B)$ *whenever* $A \subset B$;
$Cr(A) + Cr(A^C) = 1$ *for each* $A \in P(X)$, *where* A^C *is the complement set of* A;
$Cr(\cup_i A_i) = \sup_i Cr(A_i)$ *for any collection* (A_i) *in* $P(X)$ *with* $\sup_i Cr(A_i) < 0.5$.
$Cr(A \cup B) \leq Cr(A) + Cr(B)$ *for any* $A,\ B \in P(X)$.

Definition 8. [14] *A fuzzy variable is a function from a credibility space $(X, P(X), Cr)$ to the set of real numbers.*

Definition 9. *Let ξ be a fuzzy variable defined on the credibility space $(X, P(X), Cr)$. Then, its membership function is derived from the credibility measure by*

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1.$$

Definition 10. *A set function Pos on $P(X)$ is called a possibility measure if*
$Pos(X) = 1$;
$Pos(\emptyset) = 0$; *and*
$Pos(\cup_i A_i) = \sup_i Pos(A_i)$ *for any collection* (A_i) *in* $P(X)$.

- *The necessity measure for $A \in P(X)$ is defined as $Nec(A) = 1 - Pos(A^c)$.*

Remark 2. *For each $A \in P(X)$,*
1) $Pos(A)$ is the measure of best case of event A, and it is the maximal chance of the event A holds. It means that the decision maker is optimistic.
2) $Nec(A)$ gives the measure of worst case of event A, and it is the minimal chance of A holds. It means that the decision maker is pessimistic.
3) The credibility measure of a fuzzy event A, $Cr(A)$, satisfy

$$Cr(A) = \frac{1}{2}(Pos(A) + Nec(A))$$

which means that the decision maker takes compromise attitude.

Theorem 1. *(Credibility Inversion Theorem)* [14]
Let ξ be a fuzzy variable with a membership function μ_ξ. Then for any set B of real numbers, we have

$$Cr\{B\} = \frac{1}{2}[\sup_{t \in B} \mu_\xi(t) + 1 - \sup_{t \in B^C} \mu_\xi(t)]. \tag{1}$$

2.4. Fuzzy random variables

Focusing on the credibility measure, Liu [11] defined a fuzzy random variable as follows.

Definition 11. [11]
A fuzzy random variable is a function ξ from a probability space (Ω, F, P) to the set of fuzzy variables such that $Cr\{\xi(\omega) \in B\}$ is a measurable function of ω for any Borel set B of $\mathbb{R}$.

3. Chance constrained multiobjective programming model with fuzzy stochastic coefficients

3.1. Problem description

In the classical multi-objective linear programming problems, the coefficients of objective functions or constraints are assumed to be completely known. However, in many real life problems, decision makers are faced with the situations where both fuzziness and randomness exist. Fuzzy random variable, introduced by Kwakernaak [8], is an efficient tool to describe the phenomena in which randomness and fuzziness appear simultaneously. Therefore, in this section we focus on fuzzy stochastic multi-criteria linear programming model($\widetilde{P}_M$), where coefficients are fuzzy random variables.

$$(\widetilde{P}_M)\begin{cases} maximize\ (\widetilde{C}_1x, \widetilde{C}_2x, ..., \widetilde{C}_kx) \\ \text{subject to} \\ \sum_{j=1}^{n} \widetilde{a}_{ij}x_j \leq \widetilde{b}_i, \quad i = 1, 2, ..., m \\ x_j \geq 0, \quad j = 1, 2, ..., n \end{cases}$$

where $x = (x_1, x_2, ..., x_n)^T$ is a decision vector; $\widetilde{C}_r = (\widetilde{C}_{r1},\ \widetilde{C}_{r2}, ...,\ \widetilde{C}_{rn})$, $r = 1, 2..., k$ are coefficient vectors of objective function $\widetilde{C}_r$.
$\widetilde{C}_{ri}$, $\widetilde{a}_{ij}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $\widetilde{b}_i$, $i = 1, 2, ..., n$ are fuzzy random variables on probability space (Ω, F, P).
A chance constrained multiobjective programming model for problem $(\widetilde{P}_M)$ is formulated as follows [10]

$$(P_M)\begin{cases} maximize\ (f_1, f_2, ..., f_k) \\ \text{subject to} \\ P\left\{\omega : Cr(\sum_{j=1}^{n} \widetilde{C}_{rj}x_j \geq f_r) \geq \delta_r\right\} \geq p_r^0, \quad r = 1, 2..., k \\ P\left\{\omega : Cr(\sum_{j=1}^{n} \widetilde{a}_{ij}x_j \leq \widetilde{b}_i) \geq \gamma_i\right\} \geq p_i, \quad i = 1, 2, ..., m \\ x_j \geq 0, \quad j = 1, 2, ..., n \end{cases}$$

where P and Cr denote, respectively probability and credibility measures. p_i, p_r^0, δ_r, γ_i, $i = 1, 2, ..., m$; $r = 1, 2, ..., k$ are predetermined confidence levels defined by a decision maker.

The set of Pro-Cr feasible solutions of problem (P_M), is denoted by X_p^{Cr}.
$X_p^{Cr} = \bigcap_{i=1}^{i=m} X_p^{Cr}(p_i; \gamma_i)$, where for $i = 1, 2, ..., m$,

$$X_p^{Cr}(p_i; \gamma_i) = \left\{ x \geq 0, : P\left\{\omega : Cr\left(\sum_{j=1}^{n} \widetilde{a}_{ij}x_j \leq \widetilde{b}_i\right) \geq \gamma_i\right\} \geq p_i\right\}.$$

In Definition 12, we introduce the notion of (p_r^0-Pro δ_r-Cr)-efficient solution of problem $(\widetilde{P}_M)$.

Definition 12. *For a confidence levels p_r^0, $\delta_r \in [0, 1]$, $r = 1, 2, ..., k$, x^* is said to be a (p_r^0-Pro,δ_r-Cr)-Pareto efficient solution to problem $(\widetilde{P}_M)$ if and only if x^* is a Pareto optimal solution of problem (P_M).*

Remark 3. *A Pareto optimal solution of problem (P_M) is considered as a satisfactory solution for problem $(\widetilde{P}_M)$.*

3.2. Crisp equivalent model

One way of solving model $(\widetilde{P}_M)$ is to convert the chance constraints of the model into their respective crisp equivalents. This process is generally a difficult work and only successful for some special cases. Consequently, we need to make the following assumptions.

Assumption 1. :
Assume that $\widetilde{C}_{rj},\ \widetilde{a}_{ij},\ \widetilde{b}_i,\ r=1,2,...,k,\ i=1,2,...,m,\ j=1,2,...,n$ are L-R fuzzy random variables on probability space (Ω,F,P) such that $\widetilde{C}_{rj}=(m_{rj},n_{rj},\ \alpha_{rj},\ \beta_{rj})_{LR}$, $\widetilde{a}_{ij}=(m^{ij},\ n^{ij},\ \alpha^{ij},\ \beta^{ij})_{LR}$ and $\widetilde{b}_i=(m^i,\ n^i,\ \alpha^i,\ \beta^i)_{LR}$ which are characterized by the following membership functions

$$\mu_{\widetilde{C}_{rj}(\omega)}(x)=\begin{cases} L(\frac{m_{rj}(\omega)-t}{\alpha_{rj}}) & \text{if } m_{rj}(\omega)-\alpha_{rj}\le t\le m_{rj}(\omega), \quad \omega\in\Omega\\ 1 & \text{if } m_{rj}(\omega)\leqslant t\le n_{rj}(\omega) \qquad j=1,2,...,n\\ R(\frac{t-n_{rj}(\omega)}{\beta_{rj}}) & \text{if } n_{rj}(\omega)\le t\le n_{rj}(\omega)+\beta_{rj}, \quad r=1,2,....,k\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{\widetilde{a}_{ij}(\omega)}(x)=\begin{cases} L(\frac{m^{ij}(\omega)-t}{\alpha^{ij}}) & \text{if } m^{ij}(\omega)-\alpha^{ij}\le t\le m^{ij}(\omega), \quad \omega\in\Omega\\ 1 & \text{if } m^{ij}(\omega)\leqslant t\le n^{ij}(\omega), \qquad j=1,2,...,n\\ R(\frac{t-n^{ij}(\omega)}{\beta^{ij}}) & \text{if } n^{ij}(\omega)\le t\le n^{ij}(\omega)+\beta^{ij}, \quad i=1,2,....,m\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{\widetilde{b}_i(\omega)}(x)=\begin{cases} L(\frac{m^{i}(\omega)-t}{\alpha^{i}}) & \text{if } m^{i}(\omega)-\alpha^{i}\le t\le m^{i}(\omega), \quad \omega\in\Omega\\ 1 & \text{if } m^{i}(\omega)\le t\le n^{i}(\omega)\\ R(\frac{t-n^{i}(\omega)}{\beta^{i}}) & \text{if } n^{i}(\omega)\le t\le n^{i}(\omega)+\beta^{i} \quad i=1,2,....,m\\ 0 & \text{otherwise}\end{cases}$$

Let m_{rj}, n_{rj}, m^{ij},n^{ij}, m^i and n^i denoted by $m_{rj}\sim N(\mu_{m_{rj}},\sigma^2_{m_{rj}}),\ n_{rj}\sim N(\mu_{n_{rj}},\sigma^2_{n_{rj}})$, $m^{ij}\sim N(\mu_{m^{ij}},\sigma^2_{m^{ij}}),\ n^{ij}\sim N(\mu_{n^{ij}},\sigma^2_{n^{ij}})$ be normally distributed and independent random variables on (Ω,F,P) such that

$$\alpha_{rj}>0,\ \beta_{rj}>0,\ \alpha^{ij}>0,\ \beta^{ij}>0,\ \alpha^i>0,\ \beta^i>0.$$

In the following, we convert the constraints of problem (P_M) into their respective crisp equivalents. This conversion is based on the following theorem.

Theorem 2. *Let $\widetilde{A}=(\overline{m},\overline{n},\overline{\alpha},\overline{\beta})_{LR}$ be an L-R flat fuzzy number with continuous membership function. For a given confidence level $\gamma\in[0,1]$,*
1- When $\gamma\le 0.5$, $Cr\{\widetilde{A}\ge\theta\}\ge\gamma$ if and only if $\theta\le\overline{n}+\overline{\beta}R^{-1}(2\gamma)$, and
2- When $\gamma>0.5$, $Cr\{\widetilde{A}\ge\theta\}\ge\gamma$ if and only if $\theta\le\overline{m}-\overline{\alpha}L^{-1}(2(1-\gamma))$.

Proof. Assume that $\widetilde{A}=(\overline{m},\overline{n},\overline{\alpha},\overline{\beta})_{LR}$ be an LR flat fuzzy number with continuous membership functions. Using equation (1), we prove that the credibility of the fuzzy event $Cr\{\widetilde{A}\ge\theta\},\ \theta\in\mathbb{R}$ is given as follows :

$$Cr\{\xi\ge\theta\}=\frac{1}{2}[\sup_{t\ge\theta}\mu_\xi(t)+1-\sup_{t<\theta}\mu_\xi(t)]$$
$$=\begin{cases} 1 & \text{if} \quad \theta\le\overline{m}-\overline{\alpha}, \quad \overline{\alpha}>0\\ 1-\frac{1}{2}L(\frac{\overline{m}-\theta}{\overline{\alpha}}) & \text{if} \quad \overline{m}-\overline{\alpha}\le\theta\le\overline{m}\\ \frac{1}{2} & \text{if} \quad \overline{m}\le\theta\le\overline{n}\\ \frac{1}{2}R(\frac{\theta-\overline{n}}{\overline{\beta}}) & \text{if} \quad \overline{n}\le\theta\le\overline{n}+\overline{\beta}, \quad \overline{\beta}>0\\ 0 & \text{if} \quad \overline{n}+\overline{\beta}<\theta\end{cases}$$

- If $\gamma\le 0.5$, we have
$Cr\{\widetilde{A}\ge\theta\}\ge\gamma\Longleftrightarrow\frac{1}{2}R(\frac{\theta-\overline{n}}{\overline{\beta}})\ge\gamma\Longleftrightarrow\theta\le\overline{n}+\overline{\beta}R^{-1}(2\gamma)$.
- If $0.5<\gamma\le 1$, we have
$Cr\{\widetilde{A}\ge\theta\}\ge\gamma\Longleftrightarrow 1-\frac{1}{2}L(\frac{\overline{m}-\theta}{\overline{\alpha}})\ge\gamma\Longleftrightarrow\theta\le\overline{m}-\overline{\alpha}L^{-1}(2(1-\gamma))$. □

Based on Theorem 2 and probability properties of normal random variables, we give the crisp equivalents of the constraints $P\left\{\omega: Cr(\sum_{j=1}^n\widetilde{C}_{rj}x_j\ge f_r)\ge\delta_r\right\}\ge p_r^0,\quad r=1,2...,k$ and $P\left\{\omega: Cr(\sum_{j=1}^n\widetilde{a}_{ij}x_j\le\widetilde{b}_i)\ge\gamma_i\right\}\ge p_i,\quad i=1,2,...,m$ in Theorem 3 and Theorem 4 respectively.

Theorem 3. *Under Assumption 1, for $i = 1, 2, .., m$, we have the following equivalences*
• *if $\gamma_i \leq \frac{1}{2}$, for $x \geq 0$,*

$$P\left\{\omega : Cr\left(\sum_{j=1}^{n}\widetilde{a}_{ij}(\omega)x_j \leq \widetilde{b}_i(\omega)\right) \geq \gamma_i\right\} \geq p_i \Longleftrightarrow$$

$$-\beta^i R^{-1}(2\gamma_i) - \mu_{n^i} - \sum_{j=1}^{j=n}\left(\alpha^{ij}R^{-1}(2\gamma_i) - \mu_{m^{ij}}\right)x_j + \Phi^{-1}(p_i)\sqrt{(\sigma_{n^i})^2 + x^T M^i x} \leq 0.$$

• *if $0.5 < \gamma_i \leq 1$, for $x \geq 0$,*

$$P\left\{\omega : Cr\left(\sum_{j=1}^{n}\widetilde{a}_{ij}(\omega)x_j \leq \widetilde{b}_i(\omega)\right) \geq \gamma_i\right\} \geq p_i \Longleftrightarrow$$

$\alpha^i L^{-1}(2(1-\gamma_i)) - \mu_{m^i} + \sum_{j=1}^{j=n}\left(\beta^{ij}L^{-1}(2(1-\gamma_i) + \mu_{n^{ij}}\right)x_j + \phi^{-1}(p_i)\sqrt{(\sigma_{m^i})^2 + x^T N^i x} \leq 0,$ *where* $N((\mu_{n^{ij}})_{n\times 1},\ N^i)$ *and* $(m^{ij}(w))_{n\times 1} \sim N((\mu_{m^{ij}})_{n\times 1},\ M^i)$

Proof. Let $\widetilde{A_i} = \widetilde{b}_i - \sum_{j=1}^{j=n}\widetilde{a}_{ij}x_j$. Using arithmetic on LR fuzzy numbers, we obtain

$$\widetilde{A_i} = (m^i, n^i, \alpha^i, \beta^i) - \left(\sum_{j=1}^{j=n}m^{ij}x_j, \sum_{j=1}^{j=n}n^{ij}x_j, \sum_{j=1}^{j=n}\alpha^{ij}x_j, \sum_{j=1}^{j=n}\beta^{ij}x_j\right)$$

$$= \left(m^i - \sum_{j=1}^{j=n}n^{ij}x_j,\quad n^i - \sum_{j=1}^{j=n}m^{ij}x_j,\quad \alpha^i + \sum_{j=1}^{j=n}\beta^{ij}x_j,\quad \beta^i + \sum_{j=1}^{j=n}\alpha^{ij}x_j\right).$$

Using Theorem 2 with $\theta = 0$, we have

- If $\gamma_i \leq 0.5$,

$$Cr\{\widetilde{A_i} \geq 0\} \geq \gamma_i \Longleftrightarrow -\left(n_i - \sum_{j=1}^{j=n}m^{ij}x_j\right) \leq \left(\beta^i + \sum_{j=1}^{j=n}\alpha^{ij}x_j\right)R^{-1}(2\gamma_i)$$

$$\Longleftrightarrow \sum_{j=1}^{j=n}(m^{ij} - \alpha^{ij}R^{-1}(2\gamma_i))x_j \leq n^i + \beta^i R^{-1}(2\gamma_i).$$

Consequently, $P\left\{\omega : Cr\left(\sum_{j=1}^{n}\widetilde{a}_{ij}(\omega)x_j \leq \widetilde{b}_i(\omega)\right) \geq \gamma_i\right\} \geq p_i$ is equivalent to

$$P\left\{\omega : \sum_{j=1}^{j=n}(m^{ij}(\omega) - \alpha^{ij}R^{-1}(2\gamma_i))x_j \leq n^i(\omega) + \beta^i R^{-1}(2\gamma_i)\right\} \geq p_i. \tag{2}$$

This is equivalent to

$$P\left\{\omega : (-n^i(\omega) + \sum_{j=1}^{j=n}m^{ij}(\omega)x_j \leq \beta^i R^{-1}(2\gamma_i) + \sum_{j=1}^{j=n}\alpha^{ij}R^{-1}(2\gamma_i)x_j\right\} \geq p_i.$$

Set $T^i = -n^i(\omega) + \sum_{j=1}^{j=n}m^{ij}(\omega)x_j \sim N(-\mu_{n^i} + \sum_{j=1}^{j=n}\mu_{m^{ij}}x_j, V(T^i))$,
where
$V(T^i) = +(\sigma_{n^i})^2 + x^T M^i x$ and
$(m^{ij}(w))_{n\times 1} \sim N((\mu_{m^{ij}})_{n\times 1},\ M^i)$, then

$$(2) \Longleftrightarrow P\left\{\omega : \frac{(T^i-E(T^i))}{\sqrt{V(T^i)}} \leq \frac{\beta^i R^{-1}(2\gamma_i) + \sum_{j=1}^{j=n}(\alpha^{ij}R^{-1}(2\gamma_i))x_j - E(T^i)}{\sqrt{(\sigma_{n^i})^2 + x^T M^i x}}\right\} \geq p_i.$$

$\frac{(T^i-E(T^i))}{\sqrt{V(T^i)}} \sim N(0,1)$, consequently if we note by Φ the standardized normal distribution, we obtain

$$(2) \Longleftrightarrow \Phi\left(\frac{\beta^i R^{-1}(2\gamma_i) + \mu_{n^i} + \sum_{j=1}^{j=n}(\alpha^{ij}R^{-1}(2\gamma_i) - \mu_{m^{ij}})x_j}{\sqrt{(\sigma_{n^i})^2 + x^T M^i x}}\right) \geq p_i.$$

$$(2) \Longleftrightarrow \frac{\beta^i R^{-1}(2\gamma_i) + \mu_{n^i} + \sum_{j=1}^{j=n}(\alpha^{ij}R^{-1}(2\gamma_i) - \mu_{m^{ij}})x_j}{\sqrt{(\sigma_{n^i})^2 + x^T M^i x}} \geq \Phi^{-1}(p_i)$$

$$(2) \Longleftrightarrow -\beta^i R^{-1}(2\gamma_i) - \mu_{n^i} - \sum_{j=1}^{j=n}\left(\alpha^{ij}R^{-1}(2\gamma_i) - \mu_{m^{ij}}\right)x_j$$
$$+\Phi^{-1}(p_i)\sqrt{(\sigma_{n^i})^2 + x^T M^i x} \leq 0.$$

- If $0.5 < \gamma_i \leq 1$,

$$Cr\{\widetilde{A_i} \geq 0\} \geq \gamma_i \Longleftrightarrow \left(m^i - \sum_{j=1}^{j=n} n^{ij}x_j\right) \geq \left(\alpha^i + \sum_{j=1}^{j=n}\beta^{ij}x_j\right) L^{-1}(2(1-\gamma_i))$$
$$\Longleftrightarrow \sum_{j=1}^{j=n}(n^{ij} + \beta^{ij}L^{-1}(2(1-\gamma_i)))x_j \leq m^i - \alpha^i L^{-1}\left(2(1-\gamma_i)\right).$$

We prove in the same way as in part 1) that

$$P\left\{\omega : Cr(\sum_{j=1}^{n}\widetilde{a}_{ij}x_j \leq \widetilde{b}_i) \geq \gamma_i\right\} \geq p_i$$
$$\Longleftrightarrow \alpha^i L^{-1}(2(1-\gamma_i)) - \mu_{m^i} + \sum_{j=1}^{j=n}(\beta^{ij}L^{-1}(2(1-\gamma_i) + \mu_{n^{ij}})x_j$$
$$+\phi^{-1}(p_i)\sqrt{(\sigma_{m^i})^2 + x^T N^i x} \leq 0.$$

Where $K^i = -m^i + \sum_{j=1}^{j=n} n^{ij}x_j \sim N(-\mu_{m^i} + \sum_{j=1}^{j=n}\mu_{n^{ij}}x_j,\ V(K^i))$;
$V(K^i) = +(\sigma_{m^i})^2 + x^T N^i x$ and $(n^{ij}(\omega))_{n\times 1} \sim N((\mu_{n^{ij}})_{n\times 1},\ \ N^i)$.
We conclude the proof as in part 1). □

Theorem 4. *Under Assumption 1, for $r = 1, 2, ..., k$, we have*

- *if* $\delta_r \leq 0.5$, *then* $P\left\{\omega : Cr\left(\sum_{j=1}^{n}\widetilde{C}_{rj}x_j \geq f_r\right) \geq \delta_r\right\} \geq p_r^0$

$$\Longleftrightarrow f_r \leq +\sum_{j=1}^{n}\mu_{n_{rj}} + R^{-1}(2\delta_r)\sum_{j=1}^{n}\beta_{rj} - \Phi^{-1}(p_r^0)\sqrt{x^T V_r x}.$$

- *if* $0.5 < \delta_r \leq 1$, *then* $P\left\{\omega : Cr(\sum_{j=1}^{n}\widetilde{C}_{rj}x_j \geq f_r) \geq \delta_r\right\} \geq p_r^0$

$$\Longleftrightarrow f_r \leq +\sum_{j=1}^{n}\mu_{m_{rj}} + R^{-1}(2\delta_r)\sum_{j=1}^{n}\alpha_{rj} - \Phi^{-1}(p_r^0)\sqrt{x^T W_r x},$$

where V_r is the covariance matrix of the random vector $(n_{rj}(\omega))_{n\times 1}$; W_r is the covariance matrix of the vector $(m_{rj}(\omega))_{n\times 1}$ and Φ is the standardized normal distribution.

Proof. We have $\sum_{j=1}^{n} \widetilde{C}_{rj}x_j = \left(\sum_{j=1}^{n} m_{rj}(\omega)x_j, \sum_{j=1}^{n} n_{rj}(\omega)x_j, \sum_{j=1}^{n} \alpha_{rj}x_j, \sum_{j=1}^{n} \beta_{rj}x_j\right)_{LR}$

- If $\delta_r \le 0.5$, using Theorem 2, we have

$$Cr\left(\sum_{j=1}^{n} \widetilde{C}_{rj}x_j \ge f_r\right) \ge \delta_r \iff f_r \le \sum_{j=1}^{n} n_{rj}(\omega)x_j + R^{-1}(2\delta_r)\sum_{j=1}^{n} \beta_{rj}x_j$$

$$\iff -\sum_{j=1}^{n} n_{rj}(\omega)x_j \le -f_r + \sum_{j=1}^{n} R^{-1}(2\delta_r)\beta_{rj}x_j.$$

Then,

$P\left\{\omega : Cr\left(\sum_{j=1}^{n} \widetilde{C}_{rj}x_j \ge f_r\right) \ge \delta_r\right\} \ge p_r^0$

$$\iff P\left\{\omega : -\sum_{j=1}^{n} n^{rj(\omega)}x_j \le -f_r + \sum_{j=1}^{n} R^{-1}(2\delta_r)\beta_{rj}x_j\right\} \ge p_r^0.$$

Set $Y_r = -\sum_{j=1}^{n} n_{rj}(\omega)x_j$, then $E(Y_r) = -\sum_{j=1}^{n} \mu_{n_{rj}}x_j$

and $Var(Y_r) = Var\left(\sum_{j=1}^{n} n_{rj}(\omega)x_j\right) = x^T V_r x$, where V_r is the covariance matrix of the random vector $(n_{rj})_{j=1,n}$.

Consequently

$P\left\{\omega : Cr\left(\sum_{j=1}^{n} \widetilde{C}_{rj}x_j \ge f_r\right) \ge \delta_r\right\} \ge p_r^0 \iff$

$$P\left\{\omega : \frac{-\sum_{j=1}^{n} n_{rj}(\omega)x_j - E(Y_r)}{\sqrt{x^T V_r x}} \le \frac{-f_r + \sum_{j=1}^{n} R^{-1}(2\delta_r)\beta_{rj}x_j - E(Y_r)}{\sqrt{x^T V_r x}}\right\} \ge p_r^0.$$

$$\iff \Phi\left(\frac{-f_r + \sum_{j=1}^{n} R^{-1}(2\delta_r)\beta_{rj}x_j - E(Y_r)}{\sqrt{x^T V_r x}}\right) \ge p_r^0$$

$$\iff \frac{-f_r + \sum_{j=1}^{n} R^{-1}(2\delta_r)\beta_{rj}x_j - E(Y_r)}{\sqrt{x^T V_r x}} \ge \Phi^{-1}(p_r^0)$$

$$\iff f_r \le +\sum_{j=1}^{n} \mu_{n_{rj}}x_j + R^{-1}(2\delta_r)\sum_{j=1}^{n} \beta_{rj}x_j - \Phi^{-1}(p_r^0)\sqrt{x^T V_r x}.$$

- If $0.5 < \delta_r \leq 1$, using Theorem 2, we have

$$Cr\left(\sum_{j=1}^{n}\widetilde{C}_{rj}x_j \geq f_r\right) \geq \delta_r \Longleftrightarrow f_r \leq \sum_{j=1}^{n} m_{rj}x_j - L^{-1}(2(1-\delta_r))\sum_{j=1}^{n}\alpha_{rj}x_j$$

$$\Longleftrightarrow f_r \leq \sum_{j=1}^{n}\left(m_{rj} - L^{-1}(2(1-\delta_r))\alpha_{rj}\right)x_j.$$

In the same way as in the precedent case, we prove that
$P\left\{\omega : Cr\left(\sum_{j=1}^{n}\widetilde{C}_{rj}x_j \geq f_r\right) \geq \delta_r\right\} \geq p_r^0$

$$\Longleftrightarrow f_r \leq +\sum_{j=1}^{n}\mu_{m_{rj}}x_j - L^{-1}(2(1-\delta_r))\sum_{j=1}^{n}\alpha_{rj}x_j - \Phi^{-1}(p_r^0)\sqrt{x^T W_r x},$$

with $Z_r = -\sum_{j=1}^{n} m_{rj}(\omega)x_j$, $E(Z_r) = -\sum_{j=1}^{n}\mu_{m_{rj}}x_j$ and

$Var(Z_r) = Var\left(\sum_{j=1}^{n} m_{rj}(\omega)x_j\right) = x^T W_r x$, W_r covariance matrix of the random vector $(m_{rj}(\omega))_{n\times 1}$.

□

The crisp equivalent model of problem (P_M) is given in Theorem 5.

Theorem 5. *Under Assumption 1, problem (P_M) is equivalent to the following one*

$$(P_{M1})\begin{cases} maximize\ (f_1, f_2, ..., f_k) \\ subject\ to \\ F_r(x) \geq f_r, \quad r = 1,2,...,k \\ q_i(x) \geq 0, \quad i = 1,2,...,m \\ x \geq 0 \end{cases}$$

where, for $r = 1,2,...,k$, $F_r(x) =$

$$\begin{cases} \sum_{j=1}^{n}\mu_{n_{rj}}x_j + R^{-1}(2\delta_r)\sum_{j=1}^{n}\beta_{rj}x_j - \Phi^{-1}(p_r^0)\sqrt{x^T V_r x}, & if \quad 0 \leq \delta_r \leq 0.5 \\ \sum_{j=1}^{n}\mu_{m_{rj}}x_j - L^{-1}(2(1-\delta_r))\sum_{j=1}^{n}\alpha_{rj}x_j - \Phi^{-1}(p_r^0)\sqrt{x^T W_r x}, & if \quad 0.5 < \delta_r \leq 1 \end{cases}$$

-If $0 \leq \delta_r \leq 0.5$,

$q_i(x) = \beta^i R^{-1}(2\gamma_i) - \mu_{n^i} + \sum_{j=1}^{j=n}(\alpha^{ij}R^{-1}(2\gamma_i) + \mu_{m^{ij}})x_j - \Phi^{-1}(p_i)\sqrt{(\sigma_{n^i})^2 + x^T M^i x}.$

- If $0.5 < \delta_r \leq 1$,

$q_i(x) = -\alpha^i L^{-1}(2(1-\gamma_i)) + \mu_{m^i} + \sum_{j=1}^{j=n}(\beta^{ij}L^{-1}(2(1-\gamma_i) - \mu_{n^{ij}})x_j - \phi^{-1}(p_i)\sqrt{(\sigma_{m^i})^2 + x^T N^i x}$

Proof. The objective vector functions are the same and from Theorem 3 and Theorem 4, we deduce that the constraint regions of the two problems coincide. □

Remark 4. *Under Assumption 1, from the reference* [9]*, we have the equivalence :*

$$(P_{M1}) \Longleftrightarrow (P_{M2})\begin{cases} maximize\ (F_1(x), F_2(x), ..., F_k(x)) \\ subject\ to \\ q_i(x) \geq 0, \quad i = 1,2,...,m \\ x \geq 0 \end{cases}$$

3.3. Convexity of the set of Pro-Cr feasible solutions

The convexity plays an important role in optimization problems and the convexity of the set of Pro-Cr feasible solutions to problem (P_M) is needed for the resolution of our problem in section 3.4 below. Therefore, we give here sufficient conditions to the convexity of the sets $X_p^{Cr}(p_i;\gamma_i)$, $i=1,2,...,m$.

Theorem 6. *Under Assumption 1, for $p_i \geq \frac{1}{2}$, the set $X_p^{Cr}(p_i;\gamma_i)$ is a convex set, $i=1,2,...,m$.*

Proof. 1) If $\gamma_i \leq \frac{1}{2}$, then from Theorem 2 we have, for $x \geq 0$,

$$P\left\{\omega : Cr(\sum_{j=1}^{n}\widetilde{a}_{ij}x_j \leq \widetilde{b}_i) \geq \gamma_i\right\} \geq p_i$$

$$\Longleftrightarrow q_i(x) = +\beta^i R^{-1}(2\gamma_i) + \mu_{n^i} + \sum_{j=1}^{j=n}(\alpha^{ij}R^{-1}(2\gamma_i) - \mu_{m^{ij}})x_j$$

$$-\Phi^{-1}(p_i)\sqrt{(\sigma_{n^i})^2 + x^T M^i x} \geq 0.$$

If $p_i \geq \frac{1}{2}$, then $\phi^{-1}(p_i) \geq 0$, consequently the function $q_i(x)$ is a concave function. Then $X_p^{Cr}(p_i;\gamma_i) = \{x \geq 0 \ \ / \ \ q_i(x) \geq 0\}$ is a convex set, $i=1,2,...,m$.

2) If $\gamma_i \in]\frac{1}{2},1]$, from Theorem 2, we have for $x \geq 0$

$$P\left\{\omega : Cr(\sum_{j=1}^{n}\widetilde{a}_{ij}x_j \leq \widetilde{b}_i) \geq \gamma_i\right\} \geq p_i$$

$$\Longleftrightarrow q_i(x) = -\alpha^i L^{-1}(2(1-\gamma_i)) + \mu_{m^i} - \sum_{j=1}^{j=n}(\beta^{ij}L^{-1}(2(1-\gamma_i) + \mu_{n^{ij}})x_j$$

$$-\phi^{-1}(p_i)\sqrt{(\sigma_{m^i})^2 + x^T N^i x} \geq 0.$$

We conclude the proof as in part 1). □

3.4. Interactive procedure to obtain a satisfactory solution of (P_{M2})

The interactive fuzzy satisfying method proposed by Sakawa [19] is considered here to solve problem (P_{M2}). We proceed as follows. For each of the objective functions $F_r(x)$, $\ r=1,2,...,k$ assume that the decision maker have fuzzy goals such as "$F_r(x)$ should be substantially larger than or equal to some specific value". This kind of statement can be quantified by using a corresponding membership function. Then, problem (P_{M2}) is transformed into the following one:

$$(P_{M3})\begin{cases} \max\left(\mu_1(F_1(x)), \mu_2(F_2(x)), ..., \mu_k(F_k(x))\right) \\ \textit{subject to} \\ x \in X_p^{Cr}(p_i;\gamma_i), \quad i=1,2,...,m \end{cases}$$

Fuzzy goals for the objectives are characterized by the following membership function

$$\mu_r(F_r(x)) = \begin{cases} 0 & F_r(x) \leq F_r^0 \\ \frac{F_r(x)-F_r^0}{F_r^1-F_r^0} & F_r^0 \leq F_r(x) \leq F_r^1, \qquad r=1,2,...,k \\ 1 & F_r^1 < F_r(x) \end{cases}$$

F_r^1, F_r^0 are given by solving the optimization problems respectively :

$$F_r^1 = \max_{x\in X_p^{Cr}(p_i;\gamma_i), i=1,...,m}(F_r(x)) \quad \textit{and} \quad F_r^0 = \min_{x\in X_p^{Cr}(p_i;\gamma_i), i=1,...,m}(F_r(x)) \tag{3}$$

which are convex for $p_i > \frac{1}{2}$ (see Theorem 6).
In order to solve problem (P_{M3}), the decision maker consider for each $\mu_r(F_r(x))$, a reference membership function value $\overline{\mu}$ which leads to the following problem

$$(P_{M4})\begin{cases} \min_{r=1,...,k}\max\left(\overline{\mu_r} - \mu_r(F_r(x))\right) \\ \textit{subject to} \\ x \in X_p^{Cr}(p_i;\gamma_i), \quad i=1,2,...,m \end{cases} \tag{4}$$

Problem (P_{M4}) is equivalent to

$$(P_{M5})\begin{cases} \min \lambda \\ F_r(x) \geq (\overline{\mu_r} - \lambda)(F_r^1 - F_r^0) + F_r^0, \qquad r = 1, 2, ..., k \\ 0 \leq \lambda \leq 1 \\ x \in X_p^{Cr}(p_i; \gamma_i), \quad i = 1, 2, ..., m \end{cases} \tag{5}$$

λ is an auxiliary variable.

(P_{M5}) is a nonlinear programming problem which is convex problem for $p_i > \frac{1}{2}, i = 1, 2, ..., m.$

Remark 5. *The relationship between the optimal solution of problem (P_{M5}) and the Pareto optimal solution of problem (P_{M2}) is illustrated as follows (see Sakawa* [19]*).*

(1) If $x^* \in X_i^{Cr}(p_i, \gamma_i)$, $i = 1, 2, ..., m$ is a unique optimal solution for problem (P_{M5}) for some $\mu_r, r = 1, 2, ..., k$, then x^* is a Pareto optimal solution for problem (P_{M2}).

(2) If x^* is a Pareto optimal solution for problem (P_{M2}) with $0 < \mu_r(F_r(x^*)) < 1$ holding for all $r = 1, ..., k$, then there exists $\mu_r, r = 1, 2, ..., k$ such that x^* is an optimal solution for problem (P_{M5}).

Following the above discussions, we propose an interactive satisfying method to obtain a satisfactory solution.

Interactive procedure.

Step 1 : The decision maker chooses references membership values $\overline{\mu}_i$, $i = 1, 2, ..., m.$

Step 2 : Solve the problem mono-criteria optimization problem (P_{M5}). The optimal solution of problem (P_{M5}) denoted by x^* is a satisfactory solution of problem (P_{M3}).

Step 3 : If the decision maker is satisfied by the obtained $\mu_r(F_r(x^*))$, $r = 1, 2, ..., k$, then stop, x^* is a satisfactory solution for problem (P_{M3}). Otherwise, the decision maker goes to Step 1.

Remark 6. *We note that obtaining the optimal solution in Step 2, is an easy task due to the properties of problem (P_{M5}). Problem (P_{M5}) is solved optimally by conventional computational methods, the obtained solution is a Pareto optimal solution of problem (P_{M3}) (or (P_M)) which is a (p_r^0-Pro δ_r-Cr)-Pareto optimal solution of problem $(\widetilde{P}_M)$ (see Definition 12).*
This interactive procedure begins with the information provided by the decision maker. It stops when the decision maker is satisfied by the solution obtained and it is not possible to restrict the number of iteration since the convergence depends on the behavior of the decision maker. However, it should be noted to the decision maker that any improvement of one objective function value can be obtained only at the detriment of at least one of the other objective function values.

4. Numerical example

In this section, we provide a numerical example to illustrate the feasibility of the proposed approach. Consider the fuzzy stochastic linear program :

$$(\widetilde{P}_M)\begin{cases} \textit{maximize } (\widetilde{C}_{11}(\omega)x_1 + \widetilde{C}_{12}(\omega)x_2, \widetilde{C}_{21}(\omega)x_1 + \widetilde{C}_{22}(\omega)x_2) \\ \text{subject to} \\ \widetilde{a}_{11}(\omega)x_1 + \widetilde{a}_{12}(\omega)x_2 \leq \widetilde{b}_1(\omega) \\ \widetilde{a}_{21}(\omega)x_1 + \widetilde{a}_{22}(\omega)x_2 \leq \widetilde{b}_2(\omega) \\ x_j \geq 0, \quad j = 1, 2 \end{cases}$$

where

$\widetilde{C}_{rj}(\omega)$, $\widetilde{a}_{ij}(\omega)$, $\widetilde{b}_i(\omega)$, $i, r, j = 1, 2$ are trapezoidal fuzzy random variables.

$\widetilde{C}_{11} = (N(42, 2), N(40, 3), 1, 2)$; $\widetilde{C}_{21} = (N(36, 1), N(25, 2), 1, 2)$;

$\widetilde{C}_{12} = (N(32, 3), N(50, 1), 2, 1)$; $\widetilde{C}_{22} = (N(72, 2), N(71, 3), 2, 2)$;

$\widetilde{a}_{11} = (N(15, 1), N(-18.2, 2), 1, 2)$; $\widetilde{a}_{21} = (N(110, 2), N(-33, 3), 1, 2)$;

$\widetilde{a}_{12} = (N(12, 3), N(-71.4, 3), 3, 4)$; $\widetilde{a}_{22} = (N(112, 1), N(-43, 3), 1, 2)$;

$\widetilde{b}_1 = (N(-129, 2), N(120, 3), 2, 4)$; $\widetilde{b}_2 = (N(-100, 1), N(50, 3), 1, 3)$;

According to the model (P_M), the Pro-Cr constrained multi-objective programming problem is given as follows:

Table 1. The interactive processes

Iteration	1	2	3	4
$\overline{\mu}_1$	0.9	1	1	1
$\overline{\mu}_2$	0.9	0.9	0.8 5	0.8
$x^* = (x_1^*, x_2^*)$	(1.9419, 0.8301)	(2.2925, 0.5513)	(2.4674, 0.4112)	(2.642, 0.2708)
λ	0.0363	0.0884	0.0653	0.0426
F_1	101.2674	106.8820	109.5940	112.2539
F_2	123.8828	116.4083	112.5544	108.6369
$\mu_1(F_1)$	0.8637	0.9116	0.9347	0.9574
$\mu_1(F_2)$	0.8637	0.8116	0.7847	0.7574

$$\begin{cases} maximize\ (f_1, f_2) \\ subject\ to \\ P\left\{\omega : Cr\left(\sum_{j=1}^{2} \widetilde{C}_{1j}x_j \geq f_1\right) \geq \delta_1\right\} \geq p_1^0, \\ P\left\{\omega : Cr\left(\sum_{j=1}^{2} \widetilde{C}_{2j}x_j \geq f_2\right) \geq \delta_2\right\} \geq p_2^0, \\ P\left\{\omega : Cr\left(\sum_{j=1}^{2} \widetilde{a}_{1j}x_j \leq \widetilde{b}_1\right) \geq \gamma_1\right\} \geq p_1, \\ P\left\{\omega : Cr\left(\sum_{j=1}^{2} \widetilde{a}_{2j}x_j \leq \widetilde{b}_2\right) \geq \gamma_2\right\} \geq p_2, \\ x_j \geq 0, \quad j = 1, 2. \end{cases} \tag{6}$$

Assume that the decision makers set confidence levels $\delta_1 = \delta_2 = \gamma_1 = \gamma_2 = 0.9$ and $p_1^0 = p_2^0 = p_1 = p_2 = 0.9$, then $L^{-1}(2(1-\gamma_1)) = L^{-1}(2(1-\gamma_2)) = L^{-1}(2(1-\delta_1) = L^{-1}(2(1-\delta_2)) = L^{-1}(0.2) = 0.8$ and $\Phi^{-1}(0.9) = 1.2816$. Problem (6) is formulated as the following problem according to model (P_{M3}):

$$\begin{cases} maximize\ (F_1(x) = 42x_1 + 32x_2 - 0.8x_1 - 1,6x_2 - 1.2816\sqrt{2x_1^2 + 3x_2^2}) \\ maximize\ (\ F_2(x) = 36x_1 + 72x_2 - 0.8x_1 - 1.6x_2 - 1.2816\sqrt{1x_1^2 + 2x_2^2}) \\ \text{subject to} \\ -130.6 + 16.6x_1 + 68,2x_2 - 1.2816\sqrt{2 + 2x_1^2 + 3x_2^2} \geq 0, \\ -100.2 + 31.4x_1 + 41.4x_2 - 1.2816\sqrt{1 + 3x_1^2 + 3x_2^2} \geq 0, \\ x_1 \geq 0, \quad x_2 \geq 0. \end{cases} \tag{7}$$

which is equivalent to (according to model (P_{M5}))

$$\begin{cases} \min \lambda \\ subject\ to \\ F_1(x) = 41.2x_1 + 30.4x_2 - 1.2816\sqrt{2x_1^2 + 3x_2^2} \geq (\overline{\mu_1} - \lambda)(F_1^1 - F_1^0) + F_1^0, \\ F_2(x) = 35.2x_1 + 70.4x_2 - 1.2816\sqrt{1x_1^2 + 2x_2^2} \geq (\overline{\mu_2} - \lambda)(F_2^1 - F_2^0) + F_2^0, \\ 0 \leq \lambda \leq 1 \\ -130.6 + 16.6x_1 + 68,2x_2 - 1.2816\sqrt{2 + 2x_1^2 + 3x_2^2} \geq 0 \\ -100.2 + 31.4x_1 + 41.4x_2 - 1.2816\sqrt{1 + 3x_1^2 + 3x_2^2} \geq 0 \\ x_1 \geq 0, \quad x_2 \geq 0, \end{cases}$$

where

$$\begin{cases} F_r^1 = maxF_r(x) \\ -130.6 + 16.6x_1 + 68,2x_2 - 1.2816\sqrt{2 + 2x_1^2 + 3x_2^2} \geq 0 \\ -100.2 + 31.4x_1 + 41.4x_2 - 1.2816\sqrt{1 + 3x_1^2 + 3x_2^2} \geq 0 \\ x_1 \geq 0, \quad x_2 \geq 0, \end{cases}$$

$$\begin{cases} F_r^0 = minF_r(x) \\ -130.6 + 16.6x_1 + 68,2x_2 - 1.2816\sqrt{2 + 2x_1^2 + 3x_2^2} \geq 0 \\ -100.2 + 31.4x_1 + 41.4x_2 - 1.2816\sqrt{1 + 3x_1^2 + 3x_2^2} \geq 0 \\ x_1 \geq 0, \quad x_2 \geq 0. \end{cases}$$

The interactive processes, corresponding to the confidence levels $\delta_1 = \delta_2 = \gamma_1 = \gamma_2 = 0.9$ and $p_1^0 = p_2^0 = p_1 = p_2 = 0.9$, are summarized in Table 1.

At iteration 1, the obtained optimal solution is $x^* = (1.9419,\ 0.8301)$ and $\lambda = 0.0363$. The result is shown in the second column of Table 1. Since decision makers prefer to enlarge the value of F_1 instead of reducing the value of F_2, the decision maker updates the reference values of $(\overline{\mu}_1, \overline{\mu}_2) = (1,\ 0.85)$. In the third column, we have iteration 2, the optimal solution is $x^* = (2.4674,\ 0.4112)$ and the optimal value of $\lambda = 0.0653$ for the updated reference value. We assume that the decision maker is not satisfied with the optimal value and prefer to increase the value of F_1 at the expense of F_2. Iteration 3, the decision maker updates the reference values again $(\overline{\mu}_1, \overline{\mu}_2) = (1,\ 0.8)$ and obtain an optimal solution $x^* = (2.642,\ 0.2708)$ and the optimal value $\lambda = 0.0426$. Since the decision maker is satisfied with the optimal value, the algorithm terminates.

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