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2-Tuple Linguistic Induced Generalized Weighted Distance Operators and Their Application to Decision Making

Maoying Tian[1]   Min Sun[2,3*]

1. Department of Physiology, Shandong Coal Mining Health School, Taozhuang, Shandong, 277011, P. R. China
2. School of Management, Qufu Normal University, Rizhao, Shandong, 276826, P. R. China
3. School of Mathematics and Statistics, Zaozhuang University, Zaozhuang, Shandong, 277160, P. R. China

Abstract. In this paper, based on the induced ordered weighted distance(IOWD) measure, we present a wide range of 2-tuple linguistic induced generalized weighted distance operators. Firstly, we introduce the 2-tuple linguistic induced generalized ordered weighted distance(2TLIGOWD) operator, which is a generalization of the classical induced ordered weighted averaging(IOWA) operator that uses 2-tuple linguistic information, distance measures, and the generalized mean in order to provide some more general formulations. The main advantage of the operator is that it is able to deal with uncertain environment where the information is very imprecise and can be assessed with 2-tuple linguistic information. We study some desirable properties of the proposed aggregation operator and investigate its special cases. Furthermore, we generalize the 2TLIGOWD operator using the Choquet integral, and get the 2TLIGCOWD operator. The prominent characteristic of the operator is that it cannot only consider the importance of the elements or their ordered positions, but also can reflect the correlation among the elements. In addition, we further generalize the 2TLIGOWD operator and 2TLIGCOWD operator by using the hybrid averaging and the quasi-arithmetic mean getting more general aggregation operators. Finally, we present an application of the 2TLIGCOWD operator in a decision making concerning the selection of strategies with 2-tuple linguistic information.

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1 Introduction

Distance measure is one of hot topics in fuzzy set theory, which can indicate the discrepancy and deviation between difference arguments, and has gained lots of scholars' attention. While during the last decades, many distance measures continue to be introduced and published, which have been applied in a variety of scientific fields, such as medical diagnosis, pattern recognition, (group) decision making, data mining, etc. However, almost all of them only take the importance of the given individual distances into consideration, and cannot emphasize the importance of the ordered position of the given individual distances. Recently, by the use of the ordered weighted averaging(OWA) operator defined by Yager[1], Xu and Chen[2] first introduced the ordered weighted distance(OWD) operator, whose prominent characteristic is that it can alleviate (or intensify) the influence of unduly larger (or smaller) deviations on the aggregation results by assigning them low (or high) weights. Over the past few years, the OWD operator has been studied by many scholars. In [3], Zeng and Su generalized the OWD operator to uncertain situation and developed the intuitionistic fuzzy ordered weighted distance(IFOWD) operator. Zeng and Su[4] further presented a linguistic induced generalized ordered weighted averaging distance(LIGOWAD) operator, which includes a wide range of linguistic aggregation distance operators as special cases. Quite recently, motivated by the induced variable, Su et al.[5] introduced a fuzzy-induced Euclidean ordered weighted averaging distance(FIEOWAD) operator that extends the OWA operator by the Euclidean distance and the uncertain information represented by triangular fuzzy numbers. Xu and Xia[6] developed a number of hesitant ordered weighted distance(HOWD) operators and hesitant ordered weighted similarity measures. For further research on the use of the OWD operator, refer to [7,8].

In the real life, the decision makers cannot provide exact numbers to express their opinions because of time pressure, lack of knowledge and people's limited expertise related with problem domain. In fact, human judgements may be stated in linguistic terms. Therefore, decision making problem under linguistic environments is an interesting research topic, and has received much attention from researchers during the last several years. In this paper, we focus on the decision making problem based on the 2-tuple linguistic representation model, which is proposed by Herrera and Martinez in 2000[9]. In recent years, 2-tuple linguistic model has been intensively studied by many researchers[10-15] and many 2-tuple linguistic weighted aggregation operators been proposed. However, in the existing literatures, there is little research about the aggregation distance operator to aggregate the 2-tuple linguistic information. Therefore, much improved is needed in regard to the 2-tuple linguistic information aggregation distance operators. This is our main motivation. In this paper, we first develop the 2-tuple linguistic induced generalized ordered weighted distance(2TLIGOWD) operator. The main advantage of the operator is that it is able to deal with uncertain environment where the information is very imprecise and can be assessed with 2-tuple linguistic information. Then, noting that the 2TLIGOWD operator only considers the situation where all the elements in the fuzzy numbers are independent, however, in real life, there is always some inter-dependent characteristics among fuzzy numbers. Therefore, based on the Choquet integral, which is a useful tool to depict the correlations between the aggregated arguments, we introduce the 2-tuple linguistic induced generalized Choquet ordered weighted distance(2TLIGCOWD) operator. The prominent characteristic of the operator is that it cannot only consider the importance of the elements or their ordered positions, but also can reflect the correlation among the elements. Another advantage is that it is able to deal with complex attitudinal characters(or complex degrees of orness) in the decision process by using order-inducing variables. In addition, we further generalize the 2TLIGOWD operator and 2TLIGCOWD operator by using the hybrid averaging and the quasi-arithmetic mean. The result is the 2TLIGHWD operator, the Q2TLIGOWD operator, the 2TLIGCHWD operator and the Q2TLIGCOWD operator.

To accomplish this, the rest of the paper is organized as follows. Section 2 introduces some basic concepts related to 2-tuple linguistic information. In Section 3, we present the 2TLIGOWD operator, and discusses its desirable properties and special cases. The 2TLIGHWD operator and the Q2TLIGOWD operator are also presented in this section. In Section 4, we generalize the 2TLIGOWD operator, the 2TLIGHWD operator and the Q2TLIGOWD operator using the Choquet integral getting the 2TLIGCOWD operator, the 2TLIGCHWD operator and the Q2TLIGCOWD operator, respectively. A practical example is illustrated to show the advantages of the presented operator in Section 5. Finally, Section 6 draws the main conclusions of the paper.

2 Preliminaries

In this section, we give some relevant concepts of the 2-tuple linguistic information.

Let $S = \{s_i | i = 0, 1, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value of the linguistic variable, and it should satisfies the following characteristics[10]:

(1). The set is ordered: $s_i > s_j$, if $i > j$;

(2). There is a negation operator: neg(s_i)=s_j, such that $j = t - i$;

(3). Maximum operator: $\max\{s_i, s_j\} = s_i$, if $s_i \geq s_j$;

(4). Minimum operator: $\min\{s_i, s_j\} = s_i$, if $s_i \leq s_j$.

For example, a linguistic set S of nine terms can be defined as follows:

$$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, s_4 = \text{fair}, \\ s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\}. \tag{1}$$

Definition 1.[9] Let $S = \{s_0, s_1, \ldots, s_t\}$ be a linguistic term set and $\beta \in [0, t]$ be a value representing the result of a symbolic aggregation, then the 2-tuple that expresses the equivalent information to β is obtained with the function $\Delta : [0, t] \to S \times [-0.5, 0.5)$ defined by

$$\Delta : [0, t] \to S \times [-0.5, 0.5) \tag{2}$$

$$\Delta(\beta) = (s_i, \alpha_i), \text{with} \begin{cases} s_i, & i = \text{round}(\beta), \\ \alpha_i = \beta - i, \alpha_i \in [-0.5, 0.5), \end{cases}$$

where round$(\cdot)$ is the usual round operation, s_i has the closest index label to β and α_i is the value of the symbolic translation.

From Definition 1, it is obvious that the conversion of a linguistic term into a linguistic term into a linguistic 2-tuple consists of adding a value zero as symbolic translation:

$$s_i \in S \Rightarrow (s_i, 0). \tag{3}$$

Definition 2.[9] Let $S = \{s_0, s_1, \ldots, s_t\}$ be a linguistic term set and (s_i, α_i) be a linguistic 2-tuple. There is always a function Δ^{-1}, such that, from a 2-tuple it returns its equivalent numerical value $\beta \in [0, t] \subset R$, which is

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, t], \tag{4}$$

$$\Delta^{-1}(s_i, \alpha_i) = i + \alpha_i = \beta.$$

In order to measure the deviation between 2-tuple linguistic variables, Liu[15] defined a linguistic distance as follows.

Definition 3.[15] Let (s_k, α_k) and (s_l, α_l) be two 2-tuple linguistic variables, then

$$d((s_k, \alpha_k), (s_l, \alpha_l)) = \frac{|(k + \alpha_k) - (l + \alpha_l))|}{t} \tag{5}$$

is called a distance between 2-tuple linguistic variables (s_k, α_k) and (s_l, α_l).

3 2-tuple linguistic induced generalized aggregation distance operator

The IOWA operator proposed by Yager and Filev[16] is an important extension of the OWA operator. Their main difference is that the reordering step is not carried out with the values of the arguments a_i, but is developed with order-inducing variables that reflect a more complex reordering process.

Definition 4. An IOWA operator of dimension n is a mapping IOWA: $R^n \times R^n \to R$ that has an associated weighting vector $w = (w_1, w_2, \ldots, w_n)^\top$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$ such that

$$\mathrm{IOWA}(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \tag{6}$$

where b_j is the a_i value of IOWA pair $\langle u_i, a_i\rangle$ having the jth largest of u_i; u_i is the order-inducing variable and a_i is the argument variable.

The IOWD operator is a distance measure that uses the reordering ideas of the IOWA operator. For two sets $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the IOWD operator is defined as follows:

Definition 5. An IOWD operator of dimension n is a mapping IOWD: $R^n \times R^n \times R^n \to R$ that has an associated weighting vector $w = (w_1, w_2, \ldots, w_n)^\top$ such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$ such that

$$\mathrm{IOWD}(\langle u_1, a_1, b_1\rangle, \langle u_2, a_2, b_2\rangle, \ldots, \langle u_n, a_n, b_n\rangle) = \sum_{j=1}^{n} w_j d_j, \tag{7}$$

where d_j is the $|a_i - b_i|(i = 1, 2, \ldots, n)$ value of the IOWD triplet $\langle u_i, a_i, b_i\rangle$ having the jth largest u_i; u_i is the order inducing variable; $|a_i - b_i|$ is the argument variable represented in the form of individual distance.

The IOWD operator is commutative, monotone, bounded and idempotent. It is very useful in many scientific fields, including group decision making, medical diagnosis, data mining, and pattern recognition, etc. However, the IOWD operator is mainly used to aggregate the data taking the form of exact numerical, in what follow, we shall extend the IOWD operator to 2-tuple linguistic environment and develop the 2-tuple linguistic induced generalized ordered weighted distance(2TLIGOWD) operator.

Definition 6. Let $X = \{(x_j | x_j = (r_{x_j}, \alpha_{x_j})(j = 1, 2, \ldots, n)\}$ and $Y = \{(y_j | y_j = (r_{y_j}, \alpha_{y_j})(j = 1, 2, \ldots, n)\}$ be two collections of linguistic 2-tuples. A 2TLIGOWD operator of dimension n is a mapping 2TLIGOWD: $R^n \times S^n \times S^n \to R$, that has an associated weighting vector $w = (w_1, w_2, \ldots, w_n)^\top$ such that $w_j > 0(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$\mathrm{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j d_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \tag{8}$$

where $d_{\sigma(j)}$ is one of $d(x_i, y_i)(i = 1, 2, \ldots, n)$; $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $(1, 2, \ldots, n)$, such that $u_{\sigma(j-1)} \geq u_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$, and $d(x_i, y_i)$ is the argument variable represented in the form of distance defined by Eq.(5); u_i is the order inducing variable; λ is a parameter such that $\lambda \in (-\infty, +\infty)/\{0\}$.

Example 1. Let S be a linguistic terms set defined by Eq.(1), $X = \{x_1, x_2, x_3, x_4\} = \{(s_2, 0.3), (s_4, 0.4), (s_3, -0.1), (s_6, 0.3)\}$, and $Y = \{y_1, y_2, y_3, y_4\} = \{(s_3, -0.2), (s_4, 0.1), (s_5, -0.3), (s_6, 0)\}$ be two collections of 2-tuple linguistic variables, then

$$d(x_1, y_1) = \frac{|(2+0.3)-(3-0.2)|}{8} = 0.0625.$$

Similarly, we have

$$d(x_2, y_2) = 0.0375, d(x_3, y_3) = 0.2500, d(x_4, y_4) = 0.0375.$$

Assume that the order-inducing variables $U = (5, 8, 2, 3)$ and the weighting vector $w = (0.2, 0.3, 0.2, 0.3)^\top$ and without loss of generality, let $\lambda = 2$, then we can calculate the induced distance between X and Y by using the 2TLIGOWD operator:

$$\text{2TLIGOWD}(U, X, Y) = 0.1431.$$

Note that if the weighting vector w is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the 2TLIGOWD operator can be expressed as:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\frac{1}{W}\sum_{j=1}^{n} w_j d_{\sigma(j)}^{\lambda}\right)^{1/\lambda}.$$

From a generalized perspective of the reordering step, we can distinguish between the descending 2TLIGOWD(D2TLIGOWD) operator and the ascending 2TLIGOWD(A2TLIGOWD) operator by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the D2TLIGOWD operator and w^*_{n-j+1} is the jth weight of the A2TLIGOWD operator.

The following desirable properties of the 2TLIGOWD operator can be easily proved.

Property 1. (Idempotency) If all $d(x_j, y_j) = a (j = 1, 2, \ldots, n)$ are equal, then

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = a.$$

Property 2. (Monotonicity) If $d(x_j, y_j) \geq d(x'_j, y'_j)$ for all j, then

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$$

$$\geq \text{2TLIGOWD}(\langle u_1, x'_1, y'_1\rangle, \langle u_2, x'_2, y'_2\rangle, \ldots, \langle u_n, x'_n, y'_n\rangle).$$

Property 3. (Boundary)

$$\min_j d(x_j, y_j) \leq \text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \max_j d(x_j, y_j).$$

Property 4. (Commutativity)

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$$

$$\text{2TLIGOWD}(\langle u_1, x'_1, y'_1\rangle, \langle u_2, x'_2, y'_2\rangle, \ldots, \langle u_n, x'_n, y'_n\rangle),$$

where $((x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_n, y'_n))$ is any permutation of the arguments $((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$.

If we analyze different values of the parameter λ and the ordered induced variable u_i, we obtain a group of particular cases of the 2TLIGOWD operator.

Case 1. If $\lambda = 1$, then the 2TLIGOWD operator is reduced to the following operator:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j d_{\sigma(j)},$$

which is called the 2-tuple linguistic induced ordered weighted distance(2TLIOWD) operator.

Case 2. If $\lambda \to 0$, then the 2TLIGOWD operator is reduced to the following operator:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \prod_{j=1}^{n} d_{\sigma(j)}^{w_j},$$

which is called the 2-tuple linguistic induced ordered geometric distance(2TLIOGD) operator.

Case 3. If $\lambda = -1$, then the 2TLIGOWD operator is reduced to the following operator:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} \frac{w_j}{d_{\sigma(j)}}\right)^{-1},$$

which is called 2-tuple linguistic induced ordered harmonic distance(2TLIOHD) operator.

Case 4. If $\lambda = 2$, then the 2TLIGOWD operator is reduced to the following operator:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j d_{\sigma(j)}^2\right)^{\frac{1}{2}},$$

which is called 2-tuple linguistic induced ordered quadratic distance(2TLIOQD) operator.

Case 5. If $u_j > u_{j+1}$ for all j, then the 2TLIGOWD operator is reduced to the following operator:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j d(x_j, y_j)^\lambda\right)^{\frac{1}{\lambda}}, \tag{9}$$

which is called 2-tuple linguistic generalized weighted averaging distance(2LGWAD) operator.

Case 6. If the ordered position of the u_j is the same as the ordered position $d(x_j, y_j)$ for all j, then the 2TLIGOWD operator is reduced to the following operator:

$$\text{2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j d_j^\lambda\right)^{\frac{1}{\lambda}},$$

where d_j is the jth largest of the $d(x_i, y_i)$, which is called 2-tuple linguistic generalized ordered weighted averaging distance(2LGOWAD) operator in [17].

In addition, different families of the 2TLIGOWD operator can be found by using a different manifestation in the weighting vector w such as the step-2TLIGOWD operator, the window-2TLIGOWD operator, the median-2TLIGOWD operator, the centered-2TLIGOWD operator, etc.

In [18], Merigó and Casanovas proposed the induced generalized hybrid averaging(IGHA) operator which provides a wider generalization of the generalized hybrid averaging operator because it provides a more complete attitudinal character by using inducing variables. However, when using the IGHA operator, it is assumed that the available information is clearly known and can be assessed with exact numbers. In the following, we shall generalize the IGHA operator to aggregate the 2-tuple linguistic information, and get the following operator.

Definition 7. A 2TLIGHWD operator of dimension n is a mapping 2TLIGHWD: $R^n \times S^n \times S^n \to R$, that has an associated weighting vector $w = (w_1, w_2, \ldots, w_n)^\top$ such that $w_j > 0 (j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$\text{2TLIGHWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j d_{\sigma(j)}^{\lambda} \right)^{1/\lambda}, \tag{10}$$

where $d_{\sigma(j)}$ is the $\dot{d}(x_i, y_i)$ value $(\dot{d}(x_i, y_i) = n\omega_i d(x_i, y_i), i = 1, 2, \ldots, n)$; $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $(1, 2, \ldots, n)$, such that $u_{\sigma(j-1)} \geq u_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$, and $d(x_i, y_i)$ is the argument variable represented in the form of distance defined by Eq.(5); u_i is the order inducing variable; λ is a parameter such that $\lambda \in (-\infty, +\infty)/\{0\}$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ is the weighting vector of the $d(x_i, y_i)$, with $\omega_i \in (0, 1)$ and the sum of the weights is 1.

Example 2. Use the same X, Y, U, and w, λ in Example 1, and set $\omega = (0.1, 0.3, 0.2, 0.4)^\top$. Then

$$\dot{d}(x_1, y_1) = 0.025, \dot{d}(x_2, y_2) = 0.045, \dot{d}(x_3, y_3) = 0.02, \dot{d}(x_4, y_4) = 0.06.$$

Thus we can get the weighted distance between X and Y as follows:

$$\text{2TLIGHWD}(U, X, Y) = 0.1154.$$

Theorem 1. The 2TLIGOWD operator is a special case of the 2TLIGHWD operator.

Proof. Let $\omega = (1/n, 1/n, \ldots, 1/n)^\top$, then

$$\dot{d}(x_i, y_i) = n\omega_i d(x_i, y_i) = d(x_i, y_i),$$

which completes the proof of Theorem 1.

Obviously, the 2TLIGHWD operator can not only take the importance of given individual distances $d(x_i, y_i) (i = 1, 2, \ldots, n)$ into consideration, but also can emphasize the importance of the ordered position of the given individual distances.

Going a step further, the 2TLIGOWD operator can also be generalized using quasi-arithmetic means, and get the following operator:

Definition 8. Let g be a general continuous strictly monotone function. A Q2TLIGOWD operator of dimension n is a mapping Q2TLIGOWD: $R^n \times S^n \times S^n \to R$, that has an associated weighting vector

$w = (w_1, w_2, \ldots, w_n)^\top$ such that $w_j > 0 (j = 1, 2, \ldots, n)$ and $\sum_{j=1}^n w_j = 1$. Furthermore,

$$\mathrm{Q2TLIGOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j g(d_{\sigma(j)})\right), \tag{11}$$

where $d_{\sigma(j)}$ is one of the $d(x_i, y_i)(i = 1, 2, \ldots, n)$; $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $(1, 2, \ldots, n)$, such that $u_{\sigma(j-1)} \geq u_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$, and $d(x_i, y_i)$ is the argument variable represented in the form of distance defined by Eq.(5); u_i is the order inducing variable; λ is a parameter such that $\lambda \in (-\infty, +\infty)/\{0\}$.

Example 3. Use the same X, Y, U, and w, λ in Example 1, and set $g(x) = \ln x$. Then we can get the weighted distance between X and Y as follows:

$$\mathrm{Q2TLIGOWD}(U, X, Y) = 0.0772.$$

The main advantage of the Q2TLIGOWD operator is that it provides a more general formulation because it includes the 2TLIGOWD operator as a special case. In fact, in Q2TLIGOWD operator, if $g(x) = x^\lambda$, then we can get the 2TLIGOWD operator.

4 2-tuple linguistic induced generalized Choquet aggregation distance operator

Obviously, the above aggregation distance operators with 2-tuple linguistic variables are based on the assumption that the attribute of decision makers are independent. In fact, for real decision making problems, there is always some inter-dependent characteristics among attributes. In the following, motivated by the correlation properties of the Choquet integral[19], we shall propose the 2-tuple linguistic induced generalized Choquet ordered weighted distance(2TLIGCOWD) operator.

Let $m(G_j)(j = 1, 2, \ldots, n)$ be the weight of the elements $G_j \in G = \{G_1, G_2, \ldots, G_n\}(j = 1, 2, \ldots, n)$, where m is a fuzzy measure, defined as follows:

Definition 9. A fuzzy measure m on the set G is a set function $m : \theta(G) \to [0, 1]$ satisfying the following axioms:

(1) $m(\phi) = 0, m(G) = 1$;

(2) $A \subseteq B$ implies $m(A) \leq m(B)$, for all $A, B \subseteq G$;

(3) $m(A \bigcup B) = m(A) + m(B) + \rho m(A)m(B)$, for all $A, B \subseteq G$ and $A \bigcap B = \phi$, where $\rho \in (-1, \infty)$.

Especially, if $\rho = 0$, then the condition (3) reduces to the axiom of additive measure:

$$m(A \bigcup B) = mu(A) + m(B), \text{ for all } A, B \subseteq G \text{ and } A \bigcap B = \phi.$$

If all the elements in G are independent, then we have

$$m(A) = \sum_{G_j \in A} m(\{G_j\}), \forall A \subseteq G.$$

Definition 10.[19] Let f be a positive real-valued function $f : X \to R^+$ and m be a fuzzy measure on X. The discrete Choquet integral of f with respective to m is defined by

$$C_m(f) = \sum_{j=1}^{n} f_{\sigma(j)}[m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})],$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $f_{\sigma(j-1)} \geq f_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$, $A_{\sigma(j)}) = \{x_{\sigma(j)} | j \leq k\}$, for $k \geq 1$, and $A_{\sigma(0)} = \phi$.

Based on Definition 10, in what follows, we shall introduce the correlated weighted averaging operator for 2-tuple linguistic variables.

Definition 11. Let $X = \{(x_j | x_j = (r_{x_j}, \alpha_{x_j})(j = 1, 2, \ldots, n)\}$ and $Y = \{(y_j | y_j = (r_{y_j}, \alpha_{y_j})(j = 1, 2, \ldots, n)\}$ be two collections of linguistic 2-tuples. m be a fuzzy measure on X. A 2TLIGCOWD operator of dimension n is a mapping 2TLIGCOWD: $R^n \times S^n \times S^n \to R$. Furthermore,

$$\text{2TLIGCOWD}(\langle u_1, x_1, y_1 \rangle, \langle u_2, x_2, y_2 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \left(\sum_{j=1}^{n} (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) d_{\sigma(j)}^{\lambda} \right)^{1/\lambda},$$

where $d_{\sigma(j)}$ is one of $d(x_i, y_i)(i = 1, 2, \ldots, n)$; $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $(1, 2, \ldots, n)$, such that $u_{\sigma(j-1)} \geq u_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$; $A_{\sigma(k)} = \{G_{\sigma(j)} | j \leq k\}$ for all $k \geq 1$ and $A_{\sigma(0)} = \phi$; u_i is the order inducing variable; λ is a parameter such that $\lambda \in (-\infty, +\infty)/\{0\}$.

Remark 1. If $m(x_{\sigma(j)}) = m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}), j = 1, 2, \ldots, n$, then 2TLIGCOWD operator is reduced to the 2LGWAD operator defined by Eq.(9).

Remark 2. If $m(A) = \sum_{x_j \in A} m(\{x_j\})$, for all $A \subseteq X$, then by setting $w_j = m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}), j = 1, 2, \ldots, n$, the 2TLIGCOWD operator is reduced to the 2TLIGOWD operator defined by Eq.(8). Therefore, the 2TLIGCOWD operator is a more general operator, which includes the 2TLIGOWD operator as a special case.

Similar to the 2TLIGOWD operator, the 2TLIGCOWD operator is idempotent, commutative, monotone and bounded.

Example 4. Let S be a linguistic terms set defined by Eq.(1), $G = \{G_1, G_2, G_3\}$, $X = \{x_1, x_2, x_3\} = \{(s_1, 0.2), (s_3, 0.2), (s_5, -0.1)\}$, and $Y = \{y_1, y_2, y_3\} = \{(s_5, 0.3), (s_2, 0.4), (s_6, -0.2)\}$ be two collections of 2-tuple linguistic variables, then

$$d(x_1, y_1) = \frac{|(2 + 0.3) - (3 - 0.2)|}{8} = 0.5125.$$

Similarly, we have

$$d(x_2, y_2) = 0.1, d(x_3, y_3) = 0.1125.$$

Assume that the order-inducing variables $U = (7, 2, 5)$ and without loss of generality, let $\lambda = 2$. Suppose the fuzzy measure of $G_j (j = 1, 2, 3)$ and their sets are as follows:

$$m(G_1) = 0.2, m(G_2) = 0.35, m(G_3) = 0.25, m(G_1, G_2) = 0.65,$$

$$m(G_1, G_3) = 0.7, m(G_2, G_3) = 0.55, m(G_1, G_2, G_3) = 1.$$

Then we can calculate the induced distance between X and Y by using the 2TLIGCOWD operator:

$$\text{2TLIGCOWD}(U, X, Y) = 0.2487.$$

In the following, we shall generalize the 2TLIGCOWD operator from two perspectives. First, based on the hybrid averaging, we shall present the following operator:

Definition 12. A 2TLIGCHWD operator of dimension n is a mapping 2TLIGCHWD: $R^n \times S^n \times S^n \to R$. Furthermore,

$$\text{2TLIGCHWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n}(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))d^{\lambda}_{\sigma(j)}\right)^{1/\lambda},$$

where $d_{\sigma(j)}$ is the $\dot{d}(x_i, y_i)$ value($\dot{d}(x_i, y_i) = n\omega_i d(x_i, y_i), i = 1, 2, \ldots, n$); $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $(1, 2, \ldots, n)$, such that $u_{\sigma(j-1)} \geq u_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$; $A_{\sigma(k)} = \{G_{\sigma(j)} | j \leq k\}$ for all $k \geq 1$ and $A_{\sigma(0)} = \phi$; u_i is the order inducing variable; $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ is the weighting vector of the $d(x_i, y_i)$ with $\omega_i > 0$ and the sum of the weights is 1.

Obviously, if $\omega = (1/n, 1/n, \ldots, 1/n)^\top$, then the 2TLIGCHWD operator is reduced to the 2TLIGCOWD operator.

Example 5. Use the same X, Y, U, and m, λ in Example 4, and set $\omega = (0.1, 0.3, 0.6)^\top$. Then

$$\dot{d}(x_1, y_1) = 0.1538, \dot{d}(x_2, y_2) = 0.0900, \dot{d}(x_3, y_3) = 0.2025.$$

Thus we can get the weighted distance between X and Y as follows:

$$\text{Q2TLIGCOWD}(U, X, Y) = 0.1663.$$

The 2TLIGCOWD operator can also be further extended by using the quasi-arithmetic means and get the Q2TLIGCOWD operator, which can be defined as follows:

Definition 13. Let g be a general continuous strictly monotone function. A Q2TLIGCOWD operator of dimension n is a mapping Q2TLIGCOWD: $R^n \times S^n \times S^n \to R$. Furthermore,

$$\text{Q2TLIGCOWD}(\langle u_1, x_1, y_1\rangle, \langle u_2, x_2, y_2\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$$
$$= g^{-1}\left(\sum_{j=1}^{n}(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))g(d_{\sigma(j)})\right),$$

where $d_{\sigma(j)}$ is one of $d(x_i, y_i)(i = 1, 2, \ldots, n)$; $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $(1, 2, \ldots, n)$, such that $u_{\sigma(j-1)} \geq u_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$; $A_{\sigma(k)} = \{G_{\sigma(j)} | j \leq k\}$ for all $k \geq 1$ and $A_{\sigma(0)} = \phi$; u_i is the order inducing variable; λ is a parameter such that $\lambda \in (-\infty, +\infty)/\{0\}$.

As we can see, the 2TLIGCOWD operator is a special case of the Q2TLIGCOWD operator when $g(x) = x^{\lambda}$.

Example 6. Use the same X, Y, U, and m, λ in Example 4, and set $g(x) = \ln x$. Then we can get the weighted distance between X and Y as follows:

$$\text{Q2TLIGCOWD}(U, X, Y) = 0.1471.$$

5 Decision making with the 2TLIGCOWD operator

In this section, we develop an illustrated example about the use of the 2TLIGCOWD operator in fuzzy decision-making problems. Suppose an investment company, which wants to invest a sum of money in the best option (adapted from [13]). There is a panel with five possible alternatives in which to invest the money: (1) A_1 is a car industry; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company; (5) A_5 is a TV company. The investment company must take a decision according to the following four attributes (suppose that the weighting vector of four attributes is $\omega = (0.35, 0.15, 0.20, 0.30)^{\top}$): (1) the risk analysis G_1; (2) the growth analysis G_2; (3) the social political impact analysis G_3; (4) the environment impact analysis G_4. The five possible suppliers $A_i (i = 1, 2, \ldots, 5)$ are to be evaluated using the linguistic term set:

$$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slight poor}, s_4 = \text{fair}, s_5 = \text{slight good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\}$$

by three decision makers under the above four attributes $G_i (i = 1, 2, 3, 4)$, and construct, the linguistic decision matrix $R = (r_{ij})_{5\times4} (k = 1, 2, 3)$, which are listed in Table 1.

Table 1: Linguistic decision matrix R

	G_1	G_2	G_3	G_4
A_1	$(s_4, 0.2)$	$(s_5, 0.3)$	$(s_3, 0.3)$	$(s_2, 0.2)$
A_2	$(s_8, -0.2)$	$(s_7, 0.1)$	$(s_7, 0.1)$	$(s_4, 0.4)$
A_3	$(s_4, 0.1)$	$(s_5, -0.2)$	$(s_5, -0.3)$	$(s_6, 0.2)$
A_4	$(s_8, -0.2)$	$(s_6, 0.1)$	$(s_3, 0)$	$(s_2, 0.4)$
A_5	$(s_6, 0.2)$	$(s_4, 0.2)$	$(s_7, -0.2)$	$(s_5, 0.1)$

According to the decision makers's objectives, the company establishes the following ideal candidate shown in Table 2.

Table 2: The ideal strategy

	G_1	G_2	G_3	G_4
Ideal strategy	$(s_8, 0.1)$	$(s_8, -0.1)$	$(s_7, 0.3)$	$(s_7, 0.1)$

In order to aggregate the information, the decision makers' calculate the attitudinal character of the alternatives. Due to the fact that the attitudinal character depends upon the opinion of several members of the board of directors, it is very complex. Therefore, they need to use order inducing variables in the reordering process[4]. The results are shown in Table 3.

Table 3: Order-inducing variables

	G_1	G_2	G_3	G_4
A_1	13	10	9	12
A_2	7	10	15	13
A_3	16	14	12	10
A_4	15	18	22	10
A_5	15	13	11	17

Suppose the fuzzy measure of $G_j (j = 1, 2, 3, 4)$ and their sets are as follows:

$$m(G_1) = 0.2, m(G_2) = 0.35, m(G_3) = 0.25, m(G_4) = 0.2, m(G_1, G_2) = 0.65, m(G_1, G_3) = 0.7,$$

$$m(G_1, G_4) = 0.6, m(G_2, G_3) = 0.55, m(G_2, G_4) = 0.65, m(G_3, G_4) = 0.5,$$

$$m(G_1, G_2, G_3) = 0.8, m(G_1, G_2, G_4) = 0.85, m(G_1, G_3, G_4) = 0.85,$$

$$m(G_2, G_3, G_4) = 0.7, m(G_1, G_2, G_3, G_4) = 1.$$

Calculate the distance between the ideal strategy with the ith row of $\tilde{R}$ by utilizing the 2TLIGCOWD operator, and get the aggregated results $r_i (i = 1, 2, 3, 4, 5)$. In this step, the objective is to measure the distance between the ideal strategy and the alternative $A_i (i = 1, 2, \ldots, 5)$. For different λ, the aggregated results are listed in Table 4. As we can see, depending on the parameter λ, the ordering of the alternatives may be different. However, the most desirable alternative is always A_2.

Table 4: Distances and rankings

	A_1	A_2	A_3	A_4	A_5	Rankings
$\lambda = 1$	0.4987	0.1100	0.3456	0.3287	0.2662	$A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 2$	0.5114	0.1598	0.3687	0.3960	0.2979	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
$\lambda = 3$	0.5222	0.1993	0.3840	0.4359	0.3215	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
$\lambda = 4$	0.5314	0.2262	0.3950	0.4618	0.3408	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
$\lambda = 5$	0.5392	0.2447	0.4038	0.4796	0.3569	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$

6 Conclusion

In this paper, we have presented the 2-tuple linguistic induced generalized ordered weighted distance(2TLIGOWD) operator which is a generalization of the classical ordered weighted averaging(OWA) operator that uses 2-tuple linguistic variables, distance measures, order inducing variables, and the generalized mean in order to provide some more general formulations. This operator is very useful because it is able to deal with uncertain environment where the information is very imprecise and can be assessed with 2-tuple linguistic variables. We study some desirable properties of the proposed aggregation operator and investigate its special cases.

Furthermore, we have generalized the 2TLIGOWD operator using the Choquet integral, and get the 2TLIGCOWD operator. The prominent characteristic of the operator is that it cannot only consider the importance of the elements or their ordered positions, but also can reflect the correlation among the elements. In addition, we have further generalized the above two new operators by using the hybrid averaging and the quasi-arithmetic mean, and provide some more robust formulations of 2-tuple linguistic aggregation operators. We have presented an application of the new operators to a decision making problem regarding the selection of investment strategies.

In the future, we shall continue working in the application of the developed operators to other domains, and expect to develop further generalizations by considering other types of the OWA operator.

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