Point-Curve Bisector in Minkowski Plane

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Abstract. The set of points equidistant from the two objects is called the bisector. For instance the bisector of a point and a line is a parabola in Euclidean plane. The aim of this paper is to compare the bisector construction of the point-curve between in Euclidean and Minkowski planes.

Keyword: Minkowski plane, bisector, medial surface.

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1 Introduction

Debasish Dutta and Christoph M. Hoffmann [4] described an algorithm for computing the skeleton (medial-axis surface) of an object defined using constructive solid geometry (CSG). Gershon Elber and Myung-Soo Kim introduced a simple and robust method for computing the bisector of two planar rational curves and related studies followed in [5-7]. Some more results about rational bisectors of point-surface and sphere-surface pairs have been given in [8]. Bisectors of plane curves and space curves are obtained in Minkowski space[11,12]. In this study, making use of method in [7], we will extend our point of view to bisector of point-curve in Minkowski plane.

Let $\mathbb{R}^2_1$ be a Minkowski plane with Lorentzian metric

\begin{equation}
    ds^2 = dx^2 - dy^2
\end{equation}

If $\langle X, Y \rangle = 0$ for all $X$ and $Y$, the vectors $X$ and $Y$ are called perpendicular in the sense of Lorentz, where $\langle , \rangle$ is the induced inner product in $\mathbb{R}^2_1$. 

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The norm of \( X \in \mathbb{R}^2 \) is denoted by \( \| X \| \) and defined as
\[
\| X \| = \sqrt{\langle X, X \rangle}
\]

We say that a lorentzian vector \( X \) is spacelike, lightlike or timelike if \( \langle X, X \rangle > 0 \), \( \langle X, X \rangle = 0 \), \( \langle X, X \rangle < 0 \), respectively. A smooth regular curve is said to be a timelike, spacelike or lightlike curve if the tangent vector is a timelike, spacelike, or lightlike vector, respectively [1-3].

As an elementary example of the bisector of two points in Minkowski plane. Let us consider two points \( A = (2, 0) \) and \( M = (0, 4) \), the set of points \( B(x, y) \) equidistant from \( A \) and \( M \) are obtained by
\[
\left| (x - 2)^2 - y^2 \right| = \left| x^2 - (y - 4)^2 \right|
\]

**Case1:** If \( (x - 2)^2 - y^2 = x^2 - (y - 4)^2 \), then we have an equation of line given by
\[
x + 2y - 5 = 0
\]

**Case2:** If \( (x - 2)^2 - y^2 = -x^2 + (y - 4)^2 \), then we have an equation of hyperbola given by
\[
x^2 - 2x - y^2 + 4y - 6 = 0
\]

Fig. 1a and Fig. 1b show that the bisector curves of two points in \( \mathbb{R}^2 \) and \( \mathbb{R}^2_1 \), respectively.

![Figure 1a: The bisector of two points in Minkowski plane.](image)

Observe that the bisector of two points is even not a trivial case in Minkowski plane.
2 Bisector Construction of Point-Curve

Let us consider the unit speed curve given by

$$ C(s) = (x(s), y(s)) $$

The tangent vector of $C(s)$ is given by

$$ T(s) = (x', y') $$

Now let us consider a fixed point $Q(q_1, q_2)$ in Minkowski plane. If a point $B$ is on the bisector of the curve $C(s)$ and the point $Q$, then $B$ is contained in the normal line $L(s)$ of $C(s)$. Thus, we have

$$ < B(s) - C(s), T(s) > = 0 $$

In addition, the point $B$ is also at an equal distance from $C(s)$ and $Q$. Thus, the bisector curve $B(s) = (B_x(s), B_y(s))$ satisfies the following equation

$$ \|B(s) - Q\| = \|C(s) - Q\| $$

From (2) and (9) it is easy to see that

$$ |(B(s) - C(s))^2| = |(B(s) - Q)^2| $$

Thus, It follows that there are two cases to consider. Now we distinguish the following two cases.

**Case1:** If $((B(s) - C(s))^2 = (B(s) - Q)^2$, then by rearranging this equations, we have

$$ < B(s), C(s) - Q > = \frac{C(s)^2 - Q^2}{2} $$

By using (6), (7), (8) and (11), we may express results in the matrix form as

$$ \begin{bmatrix} x'(s) & -y'(s) \\ x_{12}(s) & -y_{12}(s) \end{bmatrix} \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \begin{bmatrix} d(s) \\ m(s) \end{bmatrix} $$

where

$$ m(s) = \frac{C(s)^2 - Q^2}{2}, d(s) = < C(s), T(s) > $$
Combining (6), (7) and (13), we have

\[ d(s) = x'(s)x(s) - y(s)y'(s) \]  

By Cramer’s rule, the equation (12) can be solved as follows:

\[
\begin{align*}
B_x(s) &= \left| \begin{array}{cc}
 d(s) & -y'(s) \\
 m(s) & -y_{12}(s)
\end{array} \right|, \\
B_y(s) &= \left| \begin{array}{cc}
 x'(s) & d(s) \\
 x_{12}(s) & m(s)
\end{array} \right| \\
B_x(s) &= \left| \begin{array}{cc}
 x'(s) & -y'(s) \\
 x_{12}(s) & -y_{12}(s)
\end{array} \right|. \\
\end{align*}
\]

The bisector curve \( B(s) \) has a simple representation as long as the common denominator of \( B_x \) and \( B_y \) in equation (16) does not vanish.

**Case 2:** If \((B(s) - Q)^2 = -(B(s) - C(s))^2\), then from (9) we obtain

\[
\begin{align*}
< B(s), T(s) > &= < C(s), T(s) > \\
< B(s), B(s) - (C(s) + Q) > &= -(C(s)^2 + Q^2)/2
\end{align*}
\]

Combining (6), (7) and (17), we have the system of equation given by

\[
\begin{align*}
 B_x^2 - B_x(x + q_1) - B_y^2 + B_y(y + q_2) &= (y^2 - x^2 + q_2^2 - q_1^2)/2 \\
 B_x &- B_yy' = x'x - yy'
\end{align*}
\]

Thus, \( B(s) = (B_x(s), B_y(s)) \) can be easily obtained depend on \( s \) by using the available mathematical software.

**Example 1:** Fig. 2b shows a elementary example of bisector curve of a line and a point in Minkowski plane.

In this example, let us consider the fixed point \( Q(2,0) \). Assume that the base curve \( C(s) \) is given by parametrization

\[ C(s) = (1, s) \]

**Case 1:** From (13), (14) and (15), we have

\[
\begin{align*}
 d(s) &= -s, m(s) = -3 + s^2/2 \\
(x_{12}(s), y_{12}(s)) &= (-1, s)
\end{align*}
\]
Substituting (20) and (21) into (16), we have the bisector curve given by parametrization

\[ B(s) = \left( -\frac{1}{2} s^2 + \frac{3}{2}, s \right) \]  

**Case2:** From (19) and (18), we get

\[
\begin{aligned}
2B_x^2(s) - 6B_x(s) + 5 - 2B_y^2(s) + 2B_y(s) &= s^2 \\
B_y(s) &= s
\end{aligned}
\]

The solutions of (23) can be obtained as follows:

\[ B_x(s) = \frac{3}{2} - \frac{1}{2} \sqrt{2s^2 - 1}, \quad B_y(s) = s \]

and

\[ B_x(s) = \frac{3}{2} + \frac{1}{2} \sqrt{2s^2 - 1}, \quad B_y(s) = s \]

Thus, the parametrization of \( B(s) \) becomes

\[ B(s) = \left( \frac{3}{2} - \frac{1}{2} \sqrt{2s^2 - 1}, s \right) \]

and

\[ B(s) = \left( \frac{3}{2} + \frac{1}{2} \sqrt{2s^2 - 1}, s \right) \]

**Example2:** In this example, we obtained the bisector curves of point \( Q(2, 0) \) and pseudo-circle \( C(s) \) in both Euclidean and Minkowski planes.

Suppose that \( C(s) \) is parameterized by

\[ C(s) = (\cosh(s), \sinh(s)) \]

**Case1:** By using (13), (14) and (15) implies that

\[ d(s) = 0, \quad m(s) = -\frac{3}{2} \]

\[ (x_{12}(s), y_{12}(s)) = (\cosh(s) - 2, \sinh(s)) \]
Substituting (29) and (30) into (16) gives the bisector curve given by the parametrization

\[ B(s) = \left( \frac{3 \cosh(s)}{4 \cosh(s) - 2}, \frac{3 \sinh(s)}{4 \cosh(s) - 2} \right) \]  

**Case 2:** From (28) and (18) we have

\[
\begin{align*}
2B_x^2(s) + B_x(s)(2 \cosh(s) - 4) - 2B_y^2(s) + 2B_y(s) \sinh(s) + 1 &= 0 \\
B_x(s) \sinh(s) - B_y(s) \cosh(s) &= 0
\end{align*}
\]

The solutions of the above system of equation obtained as follows:

\[ B_x(s) = \frac{\coth(s)}{2} \left( \sinh(2(s) + \sinh(s) - \sinh(s) \sqrt{2 \cosh(2(s) + 4 \cosh(s) - 7)} \right) \]

\[ B_y(s) = \frac{1}{2} \left( \sinh(2(s) + \sinh(s) - \sinh(s) \sqrt{2 \cosh(2(s) + 4 \cosh(s) - 7)} \right) \]

and

\[ B_x(s) = \frac{\coth(s)}{2} \left( \sinh(2(s) + \sinh(s) + \sinh(s) \sqrt{2 \cosh(2(s) + 4 \cosh(s) - 7)} \right) \]

\[ B_y(s) = \frac{1}{2} \left( \sinh(2(s) + \sinh(s) + \sinh(s) \sqrt{2 \cosh(2(s) + 4 \cosh(s) - 7)} \right) \]

Consequently, the bisector curves are illustrated in Fig. 3b.
Figure 3a: The bisector of a pseudo-circle and a point in Euclidean plane.  
Figure 3b: The bisector of a pseudo-circle and a point in Minkowski plane.

References


