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Point-Curve Bisector in Minkowski Plane

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Abstract. The set of points equidistant from the two objects is called the bisector. For instance the bisector of a point and a line is a parabola in Euclidean plane. The aim of this paper is to compare the bisector construction of the point-curve between in Euclidean and Minkowski planes.

Keyword: Minkowski plane, bisector, medial surface.

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1 Introduction

Debasish Dutta and Christoph M. Hoffmann [4] described an algorithm for computing the skeleton (medial-axis surface) of an object defined using constructive solid geometry (CSG). Gershon Elber and Myung-Soo Kim introduced a simple and robust method for computing the bisector of two planar rational curves and related studies followed in [5-7]. Some more results about rational bisectors of point-surface and sphere-surface pairs have been given in [8]. Bisectors of plane curves and space curves are obtained in Minkowski space[11,12]. In this study, making use of method in [7], we will extend our point of view to bisector of point-curve in Minkowski plane.

Let $\mathbb{R}^2_1$ be a Minkowski plane with Lorentzian metric

$$ds^2 = dx^2 - dy^2 \tag{1}$$

If $\langle X, Y \rangle = 0$ for all X and Y, the vectors X and Y are called perpendicular in the sense of Lorentz, where $\langle , \rangle$ is the induced inner product in $\mathbb{R}^2_1$.

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The norm of $X \in \mathbb{R}^2_1$ is denoted by $\|X\|$ and defined as

$$\|X\| = \sqrt{|\langle X, X\rangle|} \tag{2}$$

We say that a lorentzian vector X is spacelike, lightlike or timelike If $\langle X, X\rangle > 0$, $\langle X, X\rangle = 0$, $\langle X, X\rangle < 0$, respectively. A smooth regular curve is said to be a timelike, spacelike or lightlike curve if the tangent vector is a timelike, spacelike, or lightlike vector, respectively [1-3].

As an elementary example of the bisector of two points in Minkowski plane. Let us consider two points $A = (2, 0)$ and $M = (0, 4)$, the set of points $B(x, y)$ equidistant from A and M are obtained by

$$\left|(x-2)^2 - y^2\right| = \left|x^2 - (y-4)^2\right| \tag{3}$$

Case1: If $(x-2)^2 - y^2 = x^2 - (y-4)^2$, then we have an equation of line given by

$$x + 2y - 5 = 0 \tag{4}$$

Case2: If $(x-2)^2 - y^2 = -x^2 + (y-4)^2$, then we have an equation of hyperbola given by

$$x^2 - 2x - y^2 + 4y - 6 = 0 \tag{5}$$

Fig. 1a and Fig. 1b show that the bisector curves of two points in $\mathbb{R}^2$ and $\mathbb{R}^2_1$, respectively.

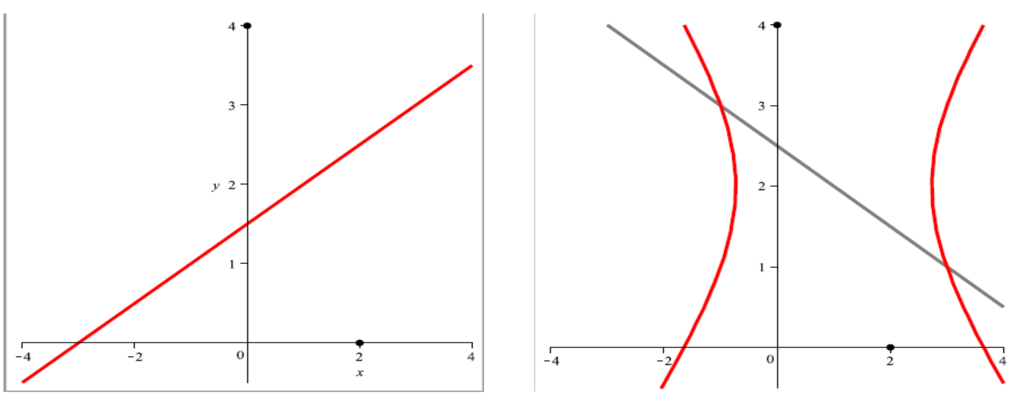

Figure 1a: The bisector of two points in Minkowski plane.

Observe that the bisector of two points is even not a trivial case in Minkowski plane.

2 Bisector Construction of Point-Curve

Let us consider the unit speed curve given by

$$C(s) = (x(s), y(s)) \tag{6}$$

The tangent vector of $C(s)$ is given by

$$T(s) = (x', y') \tag{7}$$

Now let us consider a fixed point $Q(q_1, q_2)$ in Minkowski plane. If a point B is on the bisector of the curve $C(s)$ and the point Q, then B is contained in the normal line $L(s)$ of $C(s)$. Thus, we have

$$< B(s) - C(s), T(s) >= 0 \tag{8}$$

In addition, the point B is also at an equal distance from $C(s)$ and Q. Thus, the bisector curve $B(s) = (B_x(s), B_y(s))$ satisfies the following equation

$$\|B(s) - Q\| = \|C(s) - Q\| \tag{9}$$

From (2) and (9) it is easy to see that

$$\left|(B(s) - C(s))^2\right| = \left|(B(s) - Q)^2\right| \tag{10}$$

Thus, It follows that there are two cases to consider. Now we distinguish the following two cases.

Case1: If $((B(s) - C(s))^2 = (B(s) - Q)^2$, then by rearranging this equations, we have

$$< B(s), C(s) - Q >= \frac{C(s)^2 - Q^2}{2} \tag{11}$$

By using (6), (7), (8) and (11), we may express results in the matrix form as

$$\begin{bmatrix} x'(s) & -y'(s) \\ x_{12}(s) & -y_{12}(s) \end{bmatrix} \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \begin{bmatrix} d(s) \\ m(s) \end{bmatrix} \tag{12}$$

where

$$m(s) = \frac{C(s)^2 - Q^2}{2}, d(s) =< C(s), T(s) > \tag{13}$$

(14) $$x_{12}(s) = x(s) - q_1, \qquad y_{12}(s) = y(s) - q_2$$

Combining (6), (7) and (13), we have

(15) $$d(s) = x'(s)x(s) - y(s)y'(s)$$

By Cramer's rule, the equation (12) can be solved as follows:

(16) $$B_x(s) = \frac{\begin{vmatrix} d(s) & -y'(s) \\ m(s) & -y_{12}(s) \end{vmatrix}}{\begin{vmatrix} x'(s) & -y'(s) \\ x_{12}(s) & -y_{12}(s) \end{vmatrix}}, B_y(s) = \frac{\begin{vmatrix} x'(s) & d(s) \\ x_{12}(s) & m(s) \end{vmatrix}}{\begin{vmatrix} x'(s) & -y'(s) \\ x_{12}(s) & -y_{12}(s) \end{vmatrix}}$$

The bisector curve $B(s)$ has a simple representation as long as the common denominator of B_x and B_y in equation (16) does not vanish.

Case2: If $(B(s) - Q)^2 = -(B(s) - C(s))^2$, then from (9) we obtain

(17) $$\left.\begin{array}{rcl} < B(s), T(s) > & = & < C(s), T(s) > \\ < B(s), B(s) - (C(s) + Q) > & = & -(C(s)^2 + Q^2)/2 \end{array}\right\}$$

Combining (6), (7) and (17), we have the system of equation given by

(18) $$\left.\begin{array}{c} B_x^2 - B_x(x + q_1) - B_y^2 + B_y(y + q_2) = (y^2 - x^2 + q_2^2 - q_1^2)/2 \\ B_x x' - B_y y' = x'x - yy' \end{array}\right\}$$

Thus, $B(s) = (B_x(s), B_y(s))$ can be easily obtained depend on s by using the available mathematical software.

Example1: Fig. 2b shows a elementary example of bisector curve of a line and a point in Minkowski plane.

In this example, let us consider the fixed point $Q(2, 0)$. Assume that the base curve $C(s)$ is given by parametrization

(19) $$C(s) = (1, s)$$

Case1: From (13), (14) and (15), we have

(20) $$d(s) = -s, m(s) = -\frac{3 + s^2}{2}$$

(21) $$(x_{12}(s), y_{12}(s)) = (-1, s)$$

Substituting (20) and (21) into (16), we have the bisector curve given by parametrization

$$B(s) = (-\frac{1}{2}s^2 + \frac{3}{2}, s) \tag{22}$$

Case2: From (19) and (18), we get

$$\left.\begin{array}{c} 2B_x^2(s) - 6B_x(s) + 5 - 2B_y^2(s) + 2B_y(s) = s^2 \\ B_y(s) = s \end{array}\right\} \tag{23}$$

The solutions of (23) can be obtained as follows:

$$B_x(s) = \frac{3}{2} - \frac{1}{2}\sqrt{2s^2 - 1}, B_y(s) = s \tag{24}$$

and

$$B_x(s) = \frac{3}{2} + \frac{1}{2}\sqrt{2s^2 - 1}, B_y(s) = s \tag{25}$$

Thus, the parametrization of $B(s)$ becomes

$$B(s) = (\frac{3}{2} - \frac{1}{2}\sqrt{2s^2 - 1}, s) \tag{26}$$

and

$$B(s) = (\frac{3}{2} + \frac{1}{2}\sqrt{2s^2 - 1}, s) \tag{27}$$

Example2: In this example, we obtained the bisector curves of point $Q(2,0)$ and pseudo-circle $C(s)$ in both Euclidean and Minkowski planes.

Suppose that $C(s)$ is parameterized by

$$C(s) = (\cosh(s), \sinh(s)) \tag{28}$$

Case1: By using (13), (14) and (15) implies that

$$d(s) = 0, m(s) = -\frac{3}{2} \tag{29}$$

$$(x_{12}(s), y_{12}(s)) = (\cosh(s) - 2, \sinh(s)) \tag{30}$$

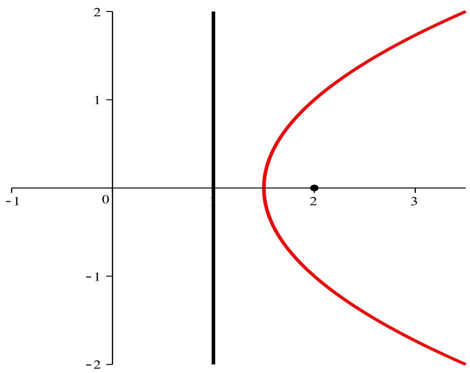

Figure 2a: The bisector of a point a line in Euclidean plane.

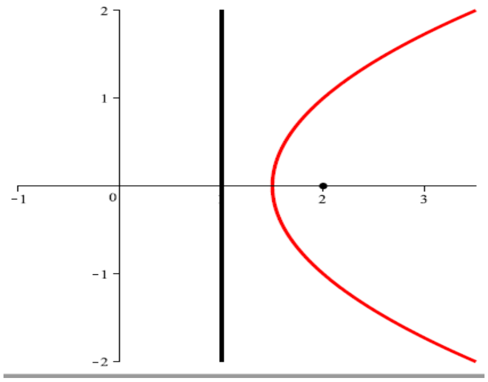

Figure 2b: The bisector of a point a line in Minkowski plane.

Substituting (29) and (30) into (16) gives the bisector curve given by parametrization

$$B(s) = \left(\frac{3\cosh(s)}{4\cosh(s)-2}, \frac{3\sinh(s)}{4\cosh(s)-2}\right) \tag{31}$$

Case2: From (28) and (18) we have

$$\left.\begin{array}{c} 2B_x^2(s) + B_x(s)(2\cosh(s)-4) - 2B_y^2(s) + 2B_y(s)\sinh(s) + 1 = 0 \\ B_x(s)\sinh(s) - B_y(s)\cosh(s) = 0 \end{array}\right\} \tag{32}$$

The solutions of the above system of equation obtained as follows:

(33)

$$B_x(s) = \tfrac{\coth(s)}{2}(\sinh 2(s) + \sinh(s) - \sinh(s)\sqrt{2\cosh 2(s) + 4\cosh(s) - 7})$$

$$B_y(s) = \tfrac{1}{2}(\sinh 2(s) + \sinh(s) - \sinh(s)\sqrt{2\cosh 2(s) + 4\cosh(s) - 7})$$

and

(34)

$$B_x(s) = \tfrac{\coth(s)}{2}(\sinh 2(s) + \sinh(s) + \sinh(s)\sqrt{2\cosh 2(s) + 4\cosh(s) - 7})$$

$$B_y(s) = \tfrac{1}{2}(\sinh 2(s) + \sinh(s) + \sinh(s)\sqrt{2\cosh 2(s) + 4\cosh(s) - 7})$$

Consequently, the bisector curves are illustrated in Fig. 3b.

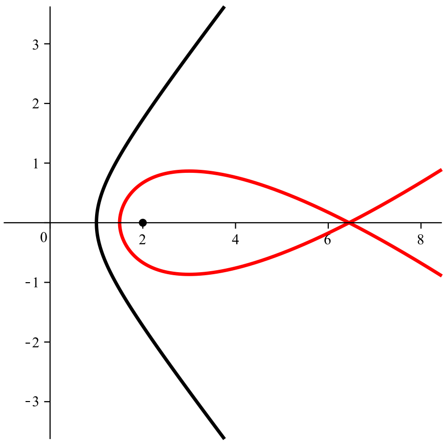

Figure 3a: The bisector of a pseudo-circle and a point in Euclidean plane.

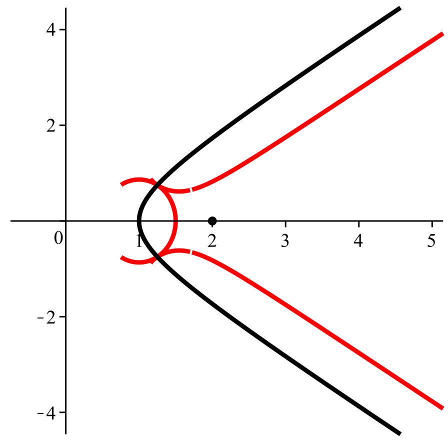

Figure 3b: The bisector of a pseudo-circle and a point in Minkowski plane.

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