

## LINEAR WEINGARTEN TYPE OF A PENCIL SURFACE IN EUCLIDEAN 3-SPACE

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**ABSTRACT.** In this paper, we study linear and non linear W-surfaces in Euclidean 3-space. Firstly, we obtain W-surfaces of a pencil surface. Then, curvatures of this surface is calculated. Finally, we give a new theorem to be linear pencil W-surface of pencil W-surfaces in  $\mathbb{E}^3$ .

### 1. INTRODUCTION

A surface  $\mathcal{M}(s, t)$  in  $\mathbb{E}^3$  is called Weingarten surface if  $\mathbf{U}(k_1, k_2) = 0$  for principal curvatures of this surface or equivalently, if there exists a non-trivial functional relation  $\phi(K, H) = 0$  with respect to its Gaussian curvature  $K$  and its mean curvature  $H$ . The existence of a non-trivial functional relation  $\phi(K, H) = 0$  on the surface  $\mathcal{M}$  parametrized by  $X(s, t)$  is equivalent to the vanishing of the corresponding Jacobian determinant, namely  $\left| \frac{\partial(K, H)}{\partial(s, t)} \right| = 0$ . Also, if the surface satisfies a linear equation with respect to  $K$  and  $H$ , that is,  $aK + bH = c$ , where  $(a, b, c) \neq (0, 0, 0)$  and  $a, b, c \in \mathbb{R}$ , [12]. Then, weingarten surface most extensively studied in research articles. For example, Karacan studied tubular W-surfaces in Euclidean 3-space in [3], Sodsiri showed ruled surfaces of Weingarten type in Minkowski space and Yoon studied polynomial translation surfaces of Weingarten type in Euclidean space, [9, 14].

To construct a surface pencil, they gave the parametric form of the surface  $\mathcal{P}(s, t) : [0, L] \times [0, T] \rightarrow \mathbb{R}^3$  as follows  $\mathcal{P}(s, t) = \alpha(s) + u(s, t)\mathbf{T}(s) + v(s, t)\mathbf{N}(s) + w(s, t)\mathbf{B}(s)$ , where  $0 \leq s \leq L$ ,  $0 \leq t \leq T$  and  $u(s, t)$ ,  $v(s, t)$  and  $w(s, t)$  are  $C^1$  functions. The values of the functions  $u(s, t)$ ,  $v(s, t)$  and  $w(s, t)$  indicate, respectively, the extension-like, flexion-like, and retortion-like effects, by the point unit through the time  $t$ , starting from  $\alpha(s)$ , in [11]. This surface has not been studied much research articles. But in recent times Li, Zhao and Wang have been working on this surface. Zhao obtained a new method for designing a developable surface pencil, Li give parametric representation of a surface pencil and Wang studied parametric representation of a surface pencil with spatial geodesic [9,11, 15].

In this paper, we study principal curvatures  $k_1$  and  $k_2$  of pencil surface in  $\mathbb{E}^3$  and we obtain Weingarten surface of pencil surface  $\mathcal{P}$  in  $\mathbb{E}^3$  for suitable  $s$  and  $t$ .

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Additionally, we give new theorem for linear Weingarten surface of pencil surface  $\mathcal{P}$  in  $\mathbb{E}^3$ .

## 2. Background on curves and surfaces

In the space  $\mathbb{E}^3$ , if  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  is denote by the moving Frenet–Serret frame along the curve  $\zeta$ , for this arbitrary curve  $\zeta$  the following Frenet–Serret formulae is given

$$(2.1) \quad \begin{aligned} \mathbf{e}_1 &= \kappa \mathbf{e}_2, \\ \mathbf{e}_2 &= -\kappa \mathbf{e}_1 + \tau \mathbf{e}_3, \\ \mathbf{e}_3 &= -\tau \mathbf{e}_2, \end{aligned}$$

where  $\kappa, \tau$  are first and second curvature of curve  $\zeta$ , respectively, and

$$\begin{aligned} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle &= \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = 1, \\ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle &= \langle \mathbf{e}_1, \mathbf{e}_3 \rangle = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = 0. \end{aligned}$$

In addition, curvatures  $\kappa$  and  $\tau$  defined by  $\kappa = \kappa(s) = \|\mathbf{e}'_1(s)\|$  and  $\tau(s) = -\langle \mathbf{e}_2, \mathbf{e}'_3 \rangle$ . Moreover we give

$$\tau = \frac{(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta})}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere  $\kappa \neq 0$  and  $\tau \neq 0$ .

We denote a surface  $\mathcal{M}$  in  $\mathbb{E}^3$  by

$$(2.2) \quad \mathcal{M}(s, t) = (m_1(s, t), m_2(s, t), m_3(s, t)).$$

Let  $U$  be the standard unit normal vector field on a surface  $\mathcal{M}$  defined by

$$(2.3) \quad \mathbf{U} = \frac{\mathcal{M}_s \wedge \mathcal{M}_t}{\|\mathcal{M}_s \wedge \mathcal{M}_t\|},$$

where  $\mathcal{M}_s = \partial \mathcal{M}(s, t) / \partial s$ ,  $\mathcal{M}_t = \partial \mathcal{M}(s, t) / \partial t$ , respectively. Then, the first fundamental form  $\mathbf{I}$  and the second fundamental form  $\mathbf{II}$  of a surface  $\mathcal{M}$  are defined by, respectively,

$$(2.4) \quad \mathbf{I} = Eds^2 + 2Fdsdt + Gdt^2,$$

$$(2.5) \quad \mathbf{II} = eds^2 + 2fdsdt + gdt^2,$$

where

$$(2.6) \quad E = \langle \mathcal{M}_s, \mathcal{M}_s \rangle, \quad F = \langle \mathcal{M}_s, \mathcal{M}_t \rangle, \quad G = \langle \mathcal{M}_t, \mathcal{M}_t \rangle,$$

$$(2.7) \quad e = \langle \mathcal{M}_{ss}, U \rangle, \quad f = \langle \mathcal{M}_{st}, U \rangle, \quad g = \langle \mathcal{M}_{tt}, U \rangle.$$

On the other hand, the Gaussian curvature  $K$  and the mean curvature  $H$  are

$$(2.8) \quad K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

and principal curvatures  $k_1$  and  $k_2$  are

$$(2.9) \quad k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K},$$

respectively, [12, 13].

### 3. Pencil W-surfaces in $\mathbb{E}^3$

Let  $\alpha : I \rightarrow E^3$  be a regular curve with parametrized by arc-length. Then, we can write pencil surface by

$$(3.1) \quad \mathcal{P}(s, t) = \alpha(s) + u(s, t)\mathbf{e}_1(s) + v(s, t)\mathbf{e}_2(s) + w(s, t)\mathbf{e}_3(s),$$

where  $0 \leq s \leq L$ ,  $0 \leq t \leq T$  and  $u(s, t)$ ,  $v(s, t)$  and  $w(s, t)$  are  $C^1$  functions, [6]. The derivative of pencil surface with respect to arc-length parameter  $s$  and the point unit through the time  $t$ , respectively.

**Lemma 3.1.** *Let  $\mathcal{P}$  be a pencil surface around a regular curve with parametrized by arc-length in  $\mathbb{E}^3$ . Then, the coefficients of the first fundamental form of pencil surface  $\mathcal{P}(s, t)$  are given by*

$$(3.2) \quad \begin{aligned} E &= 1 + u_s^2 + v_s^2 + w_s^2 + u^2\kappa^2 + v^2\kappa^2 + w^2\tau^2 - 2v\kappa \\ &\quad - 2uw\kappa\tau - 2v\kappa u_s + 2u_s + 2u\kappa v_s - 2w\tau v_s + 2v\tau w_s, \\ F &= u_t + u_s u_t + v_s v_t + w_s w_t - v\kappa u_t + u\kappa v_t - w\tau v_t + v\tau w_t, \\ G &= u_t^2 + v_t^2 + w_t^2. \end{aligned}$$

**Proof.** We take the derivative of pencil surface with respect to arc-length parameter  $s$  and the point unit through the time  $t$  in (3.1), respectively. We have

$$(3.3) \quad \mathcal{P}_s = (1 + u_s - v\kappa) \mathbf{e}_1 + (u\kappa + v_s - w\tau) \mathbf{e}_2 + (v\tau + w_s) \mathbf{e}_3$$

and

$$(3.4) \quad \mathcal{P}_t = u_t \mathbf{e}_1 + v_t \mathbf{e}_2 + w_t \mathbf{e}_3.$$

From (2.6), (3.3) and (3.4), we can calculate the coefficients of the first fundamental form at (3.2). Therefore, the proof is finished.

**Lemma 3.2.** *Let  $\mathcal{P}$  be a pencil surface around a regular curve with parametrized by arc-length in  $\mathbb{E}^3$ . The coefficients of the second fundamental form of pencil surface  $\mathcal{P}(s, t)$  are given by*

$$\begin{aligned} e &= \frac{1}{\gamma} (u\kappa u_{ss}w_t + u_{ss}v_s w_t - w\tau u_{ss}w_t - v\tau u_{ss}v_t - u_{ss}v_t w_s - 2u\kappa^2 v_s w_t \\ &\quad - 2\kappa v_s^2 w_t + 3w\kappa\tau v_s w_t + v\kappa\tau v_s v_t + 2\kappa v_s v_t w_t - u\kappa\kappa_s w_t - v\kappa_s v_s w_t \\ &\quad + v\omega\tau\kappa_s w_t - v\tau_s u_t v_s + v^2\tau\kappa_s v_t + v\kappa_s v_t w_s - u^2\kappa^3 w_t - u\kappa^2 v_s w_t \\ &\quad + 2u\omega\kappa^2 \tau w_t + u\omega\kappa^2 \tau v_t + u\kappa^2 v_t w_s - w^2\kappa^2 w_t - v\omega\kappa\tau^2 v_t - w\kappa\tau v_t w_s \\ &\quad - \kappa w_t - \kappa u_s w_t + v\kappa^2 w_t + 3v\kappa^2 u_s w_t + w^2\tau^3 u_t + v\kappa\tau u_t + \kappa u_t w_s \\ &\quad - 2\kappa u_s w_t - 2\kappa u_s^2 w_t + 2v\kappa\tau u_s u_t + 2\kappa u_s u_t w_s - v\omega\tau\tau_s u_t + v_t w_{ss} \\ &\quad + v\kappa^2 w_t - v^2\kappa^3 w_t - v^2\kappa^2 \tau u_t - v\kappa^2 u_t w_s - u\kappa_s w_t - u\kappa_s u_s w_t \\ &\quad - u\kappa\tau_s u_t + v\tau_s v_t + u\omega\kappa\kappa_s w_t + u\omega\tau\kappa_s u_t + u\kappa_s u_t w_s - v_{ss} w_t - v_{ss} u_s w_t \\ &\quad + v\kappa v_{ss} w_t + v\tau v_{ss} u_t - w\tau^2 v_t + v_{ss} u_t w_s + 2\tau w_s w_t + 2\tau u_s w_s w_t \\ &\quad - 2\tau v\kappa w_s w_t - 2v\tau^2 u_t w_s - 2\tau u_t w_s^2 - 2v\kappa\tau v_s v_t + w\tau_s u_s w_t - v\omega\kappa\tau_s w_t \end{aligned}$$

$$\begin{aligned}
& -vw\tau\tau_s u_t - w\tau_s u_t w_s + v\tau^2 w_t + v\tau^2 u_s w_t + v\tau_s u_s v_t - v^2 \tau^3 u_t - v\tau^2 u_t w_s \\
& + u\kappa\tau v_t + u\kappa\tau u_s v_t - uv\kappa^2 \tau v_t - u^2 \kappa^2 \tau u_t - u\kappa\tau u_t v_s + w\tau_s w_t - uw\kappa\tau^2 u_t \\
& + 2w\tau\tau_s \tau v_s v_t + 2\tau u_s v_s v_t - 2\tau u_t v_s^2 - v^2 \kappa\tau^2 w_t - 2u\kappa\tau u_t v_s - v^2 \kappa\tau_s v_t \\
& - 2w\tau^2 u_t v_s - w\tau^2 u_s v_t + vw\kappa\tau^2 v_t + uw\kappa\tau^2 u_t + w\tau^2 u_t v_s + u_s v_t w_{ss} \\
& - v\kappa v_t w_{ss} - u\kappa u_t w_{ss} - u_t v_s w_{ss} - w\tau u_t w_{ss}),
\end{aligned}$$

$$\begin{aligned}
f &= \frac{1}{\gamma} (u\kappa u_{st} w_t + u_{st} v_s w_t - w\tau u_{st} w_t - v\tau u_{st} v_t - u_{st} v_t w_s - u\kappa^2 v_t w_t \\
&\quad - \kappa v_s v_t w_t + w\kappa\tau v_t w_t + v\kappa\tau v_t^2 + \kappa v_t^2 w_s - \kappa u_t w_t - \kappa u_s u_t w \\
&\quad + v\kappa^2 u_t w_t + v\kappa\tau u_t^2 + \kappa u_t^2 w_s - v_{st} w_t - u_s v_{st} w_t + v\kappa v_{st} w_t + v\tau u_t v_{st} \\
&\quad + u_t v_{st} w_s + \tau w_t^2 + \tau u_s w_t^2 - v\kappa\tau w_t^2 - v\tau^2 u_t w_t - \tau u_t w_s w_t + \tau v_t^2 \\
&\quad + \tau u_s v_t^2 - v\kappa\tau v_t^2 - u\kappa\tau u_t v_t - \tau u_t v_s v_t + w\tau^2 u_t v_t + v_t w_{st} + u_s v_t w_{st} \\
&\quad - v\kappa v_t w_{st} - u\kappa u_t w_{st} - u_t v_s w_{st} + w\tau u_t w_{st}), \\
g &= \frac{1}{\gamma} (u\kappa u_{tt} w_t + u_{tt} v_s w_t - w\tau u_{tt} w_t - v\tau u_{tt} v_t - u_{tt} v_t w_s - v_{tt} w_t - u_s v_{tt} w_t \\
&\quad + v\kappa v_{tt} w_t + v\tau u_t v_{tt} + u_t v_{tt} w_s + v_t w_{tt} + u_s v_t w_{tt} - v\kappa v_t w_{tt} - u\kappa u_t w_{tt} \\
&\quad - u_t v_s w_{tt} - w\tau u_t w_{tt}).
\end{aligned}$$

**Proof.** The norm of mixed product of (3.3) and (3.4) is

$$\begin{aligned}
\|\mathcal{P}_s \wedge \mathcal{P}_t\| = \gamma &= ((v_s w_t - v_t w_s - w\tau w_t + u\kappa w_t - v\tau v_t)^2 \\
(3.5) \quad &\quad + (-w_t - u_s w_t + u_t w_s + v\tau u_t + v\kappa w_t)^2 \\
&\quad + (v_t + u_s v_t - u_t v_s + w\tau u_t - u\kappa u_t - v\kappa v_t)^2)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, unit normal vector field  $\mathbf{U}$  is obtained by

$$\begin{aligned}
\mathbf{U} &= \frac{1}{\gamma} ((v_s w_t - v_t w_s - w\tau w_t + u\kappa w_t - v\tau v_t) \mathbf{e}_1 \\
(3.6) \quad &\quad + (-w_t - u_s w_t + u_t w_s + v\tau u_t + v\kappa w_t) \mathbf{e}_2 \\
&\quad + (v_t + u_s v_t - u_t v_s + w\tau u_t - u\kappa u_t - v\kappa v_t) \mathbf{e}_3).
\end{aligned}$$

Also, we simple can calculate the coefficients of the second fundamental form from (3.5) and (3.6).

The next part of the study, we show  $\frac{\mu}{\gamma}$ ,  $\frac{\eta}{\gamma}$ ,  $\frac{\zeta}{\gamma}$  instead of the coefficients of the second fundamental form for simplicity of notation, respectively.

**Theorem 3.3.** Let  $\mathcal{P}$  be a pencil surface around a regular curve with parameterized by arc-length in  $\mathbb{E}^3$ . Principal curvatures of pencil surface  $\mathcal{P}(s, t)$  are given

by

$$\begin{aligned}
 k_1 = & \frac{1}{2\gamma\lambda} [(\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 + \zeta u^2 \kappa^2 + \zeta v^2 \kappa^2 + \zeta v^2 \tau^2 + \zeta w^2 \tau^2 \\
 & - 2\zeta v \kappa - 2\zeta u w \kappa \tau + 2\zeta u_s - 2\zeta v \kappa u_s + 2\zeta u \kappa v_s - 2\zeta w \tau v_s + 2\zeta v \tau w_s \\
 & - 2\eta u_t - 2\eta u_s u_t - 2\eta v_s v_t - 2\eta w_s w_t + 2\eta v \kappa u_t - 2\eta u \kappa v_t + 2\eta w \tau v_t \\
 (3.7) \quad & - 2\eta v \tau w_t + \mu u_t^2 + \mu v_t^2 + \mu w_t^2) + ((\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 \\
 & + \zeta u^2 \kappa^2 + \zeta v^2 \kappa^2 + \zeta v^2 \tau^2 + \zeta w^2 \tau^2 - 2\zeta v \kappa - 2\zeta u w \kappa \tau + 2\zeta u_s \\
 & - 2\zeta v \kappa u_s + 2\zeta u \kappa v_s - 2\zeta w \tau v_s + 2\zeta v \tau w_s - 2\eta u_t - 2\eta u_s u_t - 2\eta v_s v_t \\
 & - 2\eta w_s w_t + 2\eta v \kappa u_t - 2\eta u \kappa v_t + 2\eta w \tau v_t - 2\eta v \tau w_t + \mu u_t^2 + \mu v_t^2 \\
 & + \mu w_t^2)^2 - 4\lambda (\mu \zeta - \eta^2))^{\frac{1}{2}}],
 \end{aligned}$$

and

$$\begin{aligned}
 k_2 = & \frac{1}{2\gamma\lambda} [(\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 + \zeta u^2 \kappa^2 + \zeta v^2 \kappa^2 + \zeta v^2 \tau^2 + \zeta w^2 \tau^2 \\
 & - 2\zeta v \kappa - 2\zeta u w \kappa \tau + 2\zeta u_s - 2\zeta v \kappa u_s + 2\zeta u \kappa v_s - 2\zeta w \tau v_s + 2\zeta v \tau w_s \\
 & - 2\eta u_t - 2\eta u_s u_t - 2\eta v_s v_t - 2\eta w_s w_t + 2\eta v \kappa u_t - 2\eta u \kappa v_t + 2\eta w \tau v_t \\
 (3.8) \quad & - 2\eta v \tau w_t + \mu u_t^2 + \mu v_t^2 + \mu w_t^2) - ((\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 \\
 & + \zeta u^2 \kappa^2 + \zeta v^2 \kappa^2 + \zeta v^2 \tau^2 + \zeta w^2 \tau^2 - 2\zeta v \kappa - 2\zeta u w \kappa \tau + 2\zeta u_s \\
 & - 2\zeta v \kappa u_s + 2\zeta u \kappa v_s - 2\zeta w \tau v_s + 2\zeta v \tau w_s - 2\eta u_t - 2\eta u_s u_t - 2\eta v_s v_t \\
 & - 2\eta w_s w_t + 2\eta v \kappa u_t - 2\eta u \kappa v_t + 2\eta w \tau v_t - 2\eta v \tau w_t + \mu u_t^2 + \mu v_t^2 \\
 & + \mu w_t^2)^2 - 4\lambda (\mu \zeta - \eta^2))^{\frac{1}{2}}],
 \end{aligned}$$

where

$$\begin{aligned}
 \lambda = & u_t^2 + v_t^2 + w_t^2 + u_s^2 u_t^2 + u_t^2 v_s^2 + u_t^2 w_s^2 + u_s^2 v_t^2 + v_s^2 v_t^2 + v_t^2 w_s^2 - 2u w \kappa \tau u_t^2 \\
 & + 2u_s u_t^2 - 2v \kappa u_s u_t^2 + 2u \kappa u_t^2 v_s - 2w \tau u_t^2 v_s + 2v \tau u_t^2 w_s - 2u w \kappa \tau v_t^2 + 2u_s v_t^2 \\
 & - 2v \kappa u_s v_t^2 + 2u \kappa v_s v_t^2 - 2w \tau v_s v_t^2 + 2v \tau v_t^2 w_s - 2u w \kappa \tau w_t^2 - 2v \kappa u_s w_t^2 \\
 & + 2u \kappa v_s w_t^2 - 2w \tau v_s w_t^2 + 2v \tau w_s w_t^2 + 2u_s w_t^2 + u_s^2 w_t^2 + v_s^2 w_t^2 + w_s^2 w_t^2 \\
 & + u^2 \kappa^2 u_t^2 + v^2 \kappa^2 u_t^2 + v^2 \tau^2 u_t^2 + w^2 \tau^2 u_t^2 + u^2 \kappa^2 v_t^2 + v^2 \kappa^2 v_t^2 + v^2 \tau^2 v_t^2 \\
 & + w^2 \tau^2 v_t^2 + u^2 \kappa^2 w_t^2 + v^2 \kappa^2 w_t^2 + v^2 \tau^2 w_t^2 + w^2 \tau^2 w_t^2 - 2v \kappa u_t^2 - 2v \kappa v_t^2 \\
 & - 2v \kappa w_t^2 - (u_t + u_s u_t + v_s v_t + w_s w_t - v \kappa u_t + u \kappa v_t - w \tau v_t + v \tau w_t)^2.
 \end{aligned}$$

**Proof.** Firstly, we must find Gaussian curvature  $K$  and the mean curvature  $H$ . From equation (2.8), Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
 (3.9) \quad K = & \frac{\mu \zeta - \eta^2}{\gamma^2 \lambda}, \\
 H = & \frac{1}{2\gamma\lambda} (\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 + \zeta u^2 \kappa^2 + \zeta v^2 \kappa^2 + \zeta v^2 \tau^2 \\
 (3.10) \quad & + \zeta w^2 \tau^2 - 2\zeta v \kappa - 2\zeta u w \kappa \tau + 2\zeta u_s - 2\zeta v \kappa u_s + 2\zeta u \kappa v_s \\
 & - 2\zeta w \tau v_s - 2\eta u_t + 2\zeta v \tau w_s - 2\eta u_s u_t - 2\eta v_s v_t - 2\eta w_s w_t \\
 & + 2\eta v \kappa u_t - 2\eta u \kappa v_t + 2\eta w \tau v_t - 2\eta v \tau w_t + \mu u_t^2 + \mu v_t^2 + \mu w_t^2).
 \end{aligned}$$

From (2.9), (3.9) and (3.10), we get the proof of Theorem 3.3.

**Theorem 3.4.** *Let  $\mathcal{P}$  be a pencil surface around a regular curve with parameterized by arc-length in  $\mathbb{E}^3$ . If  $\mathcal{P}$  is Weingarten pencil surface, then we have*

$$\frac{\partial k_1}{\partial s} \frac{\partial k_2}{\partial t} - \frac{\partial k_1}{\partial t} \frac{\partial k_2}{\partial s} = 0.$$

**Theorem 3.5.** *Let  $\mathcal{P}$  be a Weingarten pencil surface around a regular curve with parameterized by arc-length in  $\mathbb{E}^3$ . If  $\mathcal{P}$  is a linear Weingarten surface in  $\mathbb{E}^3$ , then we write*

$$\begin{aligned} & \frac{a+b}{2\gamma\lambda}(\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 + \zeta u^2\kappa^2 + \zeta v^2\kappa^2 + \zeta w^2\tau^2 - 2\zeta v\kappa - 2\zeta uw\kappa\tau \\ & + 2\zeta u_s - 2\zeta v\kappa u_s + 2\zeta u\kappa v_s - 2\zeta w\tau v_s + 2\zeta v\tau w_s - 2\eta u_t - 2\eta u_s u_t - 2\eta v_s v_t - 2\eta w_s w_t \\ & + 2\eta v\kappa u_t - 2\eta u\kappa v_t + 2\eta w\tau v_t - 2\eta v\tau w_t + \mu u_t^2 + \mu v_t^2 + \mu w_t^2) + \frac{a-b}{2\gamma\lambda}((\zeta + \zeta u_s^2 + \zeta v_s^2 \\ & + \zeta w_s^2 + \zeta u^2\kappa^2 + \zeta v^2\kappa^2 + \zeta w^2\tau^2 - 2\zeta v\kappa - 2\zeta uw\kappa\tau + 2\zeta u_s - 2\zeta v\kappa u_s + \mu u_t^2 \\ & + 2\zeta u\kappa v_s - 2\zeta w\tau v_s + 2\zeta v\tau w_s - 2\eta u_t - 2\eta u_s u_t - 2\eta v_s v_t - 2\eta w_s w_t + 2\eta v\kappa u_t - 2\eta u\kappa v_t \\ & + 2\eta w\tau v_t - 2\eta v\tau w_t + \mu v_t^2 + \mu w_t^2)^2 - 4\lambda(\mu\zeta - \eta^2))^{\frac{1}{2}} - c = 0. \end{aligned}$$

**Proof.** If  $\mathcal{P}$  Weingarten pencil surface is linear Weingarten surface, it ensures that  $c = ak_1 + bk_2$ , which  $k_1$  and  $k_2$  are principal curvatures. From 3.7 and 3.8 equations, it is easily reached proof of the theorem.

**Corollary 3.6.** *Let  $\mathcal{P}$  be a pencil surface in  $\mathbb{E}^3$ . If Pencil surface  $\mathcal{P}$  is minimal surface, then we write*

$$\begin{aligned} & \frac{1}{2\gamma\lambda}(\zeta + \zeta u_s^2 + \zeta v_s^2 + \zeta w_s^2 + \zeta u^2\kappa^2 + \zeta v^2\kappa^2 + \zeta w^2\tau^2 \\ & + \zeta w^2\tau^2 - 2\zeta v\kappa - 2\zeta uw\kappa\tau + 2\zeta u_s - 2\zeta v\kappa u_s + 2\zeta u\kappa v_s \\ & - 2\zeta w\tau v_s - 2\eta u_t + 2\zeta v\tau w_s - 2\eta u_s u_t - 2\eta v_s v_t - 2\eta w_s w_t \\ & + 2\eta v\kappa u_t - 2\eta u\kappa v_t + 2\eta w\tau v_t - 2\eta v\tau w_t + \mu u_t^2 + \mu v_t^2 + \mu w_t^2) = 0. \end{aligned}$$

**Example 3.7.** Let us consider an unit speed curve in  $\mathbb{E}^3$  by

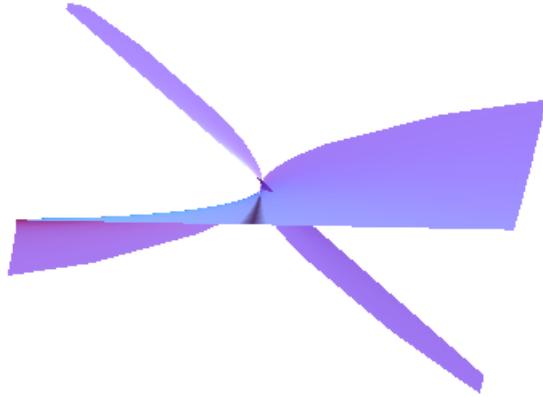
$$\zeta(s) = \frac{1}{\sqrt{5}}(\sqrt{1+s^2}, 2s, \ln(s+\sqrt{1+s^2})).$$

One can calculate its Frenet apparatus as the following

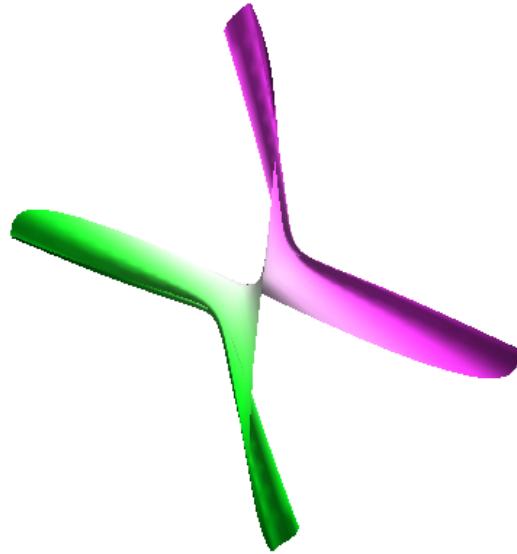
$$\begin{aligned} \mathbf{t}(s) &= \frac{1}{\sqrt{5(1+s^2)}}(s, 2\sqrt{1+s^2}, 1), \\ \mathbf{n}(s) &= \frac{1}{\sqrt{1+s^2}}(1, 0, -s), \\ \mathbf{b}(s) &= \frac{1}{\sqrt{5(1+s^2)}}(2s, -\sqrt{1+s^2}, 2). \end{aligned}$$

Then, we can write pencil W-surface by

$$\begin{aligned}\mathcal{P}(s, t) = & \frac{1}{\sqrt{5}}(\sqrt{1+s^2}, 2s, \ln(s+\sqrt{1+s^2})) + \frac{1}{\sqrt{5(1+s^2)}}u(s, t)(s, 2\sqrt{1+s^2}, 1) \\ & + \frac{1}{\sqrt{1+s^2}}v(s, t)(1, 0, -s) + \frac{1}{\sqrt{5(1+s^2)}}w(s, t)(2s, -\sqrt{1+s^2}, 2).\end{aligned}$$



$$\begin{aligned}u(s, t) &= t, \quad v(s, t) = st^3, \\ w(s, t) &= 1, \quad -2 \leq s \leq 5, \quad -5 \leq t \leq 5.\end{aligned}$$



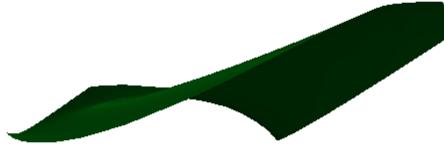
$$\begin{aligned}u(s, t) &= \sin t^3, \quad v(s, t) = \cos t \sin t, \\ w(s, t) &= \sin t, \quad -2 \leq s \leq 2, \quad -2 \leq t \leq 2.\end{aligned}$$

**Example 3.8.** Let us consider a curve in  $\mathbb{E}^3$  by

$$\mu(u) = (3u - u^3, 3u^2, 3u + u^3).$$

Then, we can write pencil W-surface by

$$\begin{aligned}\mathcal{P}(u, v) = & (3u - u^3, 3u^2, 3u + u^3) + \frac{1}{\sqrt{2}(1+u^2)}a(u, v)(1-u^2, 2u, 1+u^2) \\ & + \frac{1}{\sqrt{1+u^2}}b(u, v)(-2u, 1-u^2, 0) + \frac{1}{\sqrt{2}(1+u^2)}c(u, v)(u^2-1, -2u, 1+u^2).\end{aligned}$$



$$\begin{aligned}a(u, v) = & u^v, \quad b(u, v) = -uv, \quad c(u, v) = v^u \sin u^2 \\ 0 \leq u \leq 2, \quad -1 \leq v \leq 3.\end{aligned}$$

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