AMO - Advanced Modeling and Optimization, Volume 18, Number 2, 2016 .

$L(3,1)$-Labeling of Some Simple Graphs

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Abstract. $L(3,1)$-labeling is a particular model for frequency assignment problem of $L(h,k)$-labeling. An $L(3,1)$-labeling of a graph G is a function f from the set of vertex $V(G)$ to the set of positive integers for any two vertices u, v where label difference $|f(u) - f(v)| \geq 3$ for distance $d(u,v) = 1$ and label difference $|f(u) - f(v)| \geq 1$ for distance $d(u,v) = 2$. In $L(3,1)$ labeling λ is the smallest positive integer which denotes the maximum label used. In this article, we consider some simple graphs like cycle, path, complete graph, complete bipartite graph, star graph, bi-star graph and tree etc. To find the bounds of λ for $L(3,1)$-labeling. In this arti1cle, we obtained the boundary conditions for λ on the basis of maximum degree of G.

Keywords: $L(3,1)$-labeling, graph labeling

1. Introduction

One of the most important area in graph theory is graph labeling, which has various applications in different areas like coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network, addressing, data base management, radio frequency assignment etc. In recent era assignment of radio frequency become a very critical problem because request of frequency increases day by day due to installation of more stations. Interference is a basic problem to assign the frequency in different stations, two neighbour stations having same frequency perform a direct collision whereas station nearby the neighbour station performs hidden collision.

There exist two special type of collisions in the graph labeling problems namely direct collisions and hidden collisions. In direct collisions a station and its neighbour must have different frequencies whereas when a station received same frequency from its neighbour known as hidden collision. Bertossi et al. [1,2] studied the case of avoiding hidden collision in the multi hop radio networks. To avoid collisions from its neighbour and next to neighbour station we require distinct labeling.

A communication network is composed of station or node, each of which has computing power and can transmit and receive messages over communication links, wireless or cabled. The basic network topologies include fully connected, mesh, star, ring, tree, bus. A single network may consist of several interconnected substation of different topologies. It might be useful to assign each user terminal a node label subject to the constraint that all connecting edges (communication links) receive distinct labels.

A graph theory model of frequency assignment problem introduced by Hale[4] in 1980 as vertex coloring problem. Vertices of the graph denoted as station and

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[**]**AMO - Advanced Modeling and Optimization**

ISSN:1841- 4311

edges denoted as proximity of the station

Robert in 1991 introduced a direction in frequency assignment problem in which station are consider very close or close. Very close station are those having adjacency between them whereas close indicates those which are at distance two apart. Griggs and Yeh [3] defined $L(2,1)$ labeling of a graph $G = (V, E)$ where f is a function which assigns label to every $u, v \in V$ from the set of positive integer such that $|f(u)-f(v)| \geq 2$ if $d(u,v) = 1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v) = 2$. Now a days $L(2,1)$ labeling is applied on intersection graphs and cartesian product of cycles,paths etc[6-9,12,13].

Another labeling technique also present to solve various type of problem is $L(0,1)$-labeling. In $L(0,1)$-labeling of a graph $G = (V, E)$ where f is a function which assigns label to every $u, v \in V$ from the set of positive integers such that $|f(u) - f(v)| \geq 0$ if $d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$. Still so many applications are there in $L(0,1)$-labeling [5,10,11] applied on interval graphs, cactus graphs, permutation graphs, etc.

In a particular case when interference become high between two adjacent stations then we need to assign frequency difference more than two and for next to neighbour cases difference should be more than or equal to one. In $L(3,1)$-labeling of a graph $G = (V, E)$ where f is a function which assigns label to every $u, v \in V$ from the set of positive integer such that $|f(u)-f(v)| \geq 3$ if $d(u,v) = 1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v) = 2$. The $L(3,1)$ labeling number, $\lambda(G)$ of G is the smallest number λ such that G has an $L(3,1)$ labeling with λ as the maximum label.

In this paper, we apply $L(3,1)$-labeling technique to label the graphs, cycles, paths, complete graph, complete bipartite graph, star graph, bi-star graph, tree etc.

2. Preliminaries

Definition 1. *Let G be a graph with set of vertices V and set of edges E. Let f be a function $f : V \to N$, where f is an $L(3,1)$-labeling of G if, for all $u, v \in V$, $|f(u)-f(v)| \geq 3$ if $d(u,v) = 1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v) = 2$.*

Definition 2. *The difference between maximum and minimum values of f for all possible f is called span of the labeling, and it is denoted by $\lambda_{3,1}(G)$ or simply $\lambda(G)$ or λ. positive integer λ to be used to label a graph G by $L(3,1)$ labeling.*

Definition 3. *A complete graph is simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph with n vertices is denoted by K_n. Here all vertices $u, v \in V$, $(u, v) \in E$.*

Definition 4. *A graph G is called a complete bipartite graph if it vertices can be partitioned into two subsets V_1 and V_2 such that no edges has both end points in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A star S_n is a complete bipartite graph $K_{1,n}$.*

Definition 5. *A path is a trail in which all vertices (except possibly the first and last) are distinct. A trail is a walk in which all edges are distinct. A walk of length k in a graph is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, ..., v_{k-1}, e_{k-1}, v_k$ which begins and ends with vertices. If the graph is directed, then e_i is an arc from v_i to v_{i+1}.*

Definition 6. *A simple graph with n vertices $(n \geq 3)$ and n edges is called a cycle graph if all its edges form a cycle of length n. If the degree of each vertex in the graph is two, then it is called a cycle graph, denoted by C_n*

Definition 7. *Let G be a connected acyclic graph then G is known as tree. G is called an $n - ary\ tree$ if G is a rooted tree such that the root has degree n and all the other vertices have degree $n + 1$.*

3. Labeling of Some Special Classes of Graphs

3.1. Paths

First, we consider P_2. We start label one vertex by 0, so the other vertex must be at least 3. So $\lambda(P_2) = 3$ (see figure 1).

Proposition 1. $\lambda(P_3) = 4$.

Proof. : For P_3, we can label the leftmost vertex 0, the middle vertex 4, and the right most vertex 1, according to figure 1. So, $\lambda(P_3) \leq 4$. To label P_3, we need

Fig. 1. Path with 2 vertices

$0,1,2,3,4$ numbers to be use to label the path with 3 vertices. Labeling can't be possible by the use of just $0,1,2,3$. Label 1 or 2 couldn't be use anywhere otherwise it violets the adjacency rules for $L(3,1)$ labeling. So we have only 0 and 3 to label three vertices. By pigeon hole principle, two of these vertices must receive the same label, which necessarily violets the condition. □

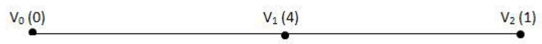

Fig. 2. Path with 3 vertices

Lemma 1. *If H is a subgraph of G, then $\lambda(H) \leq \lambda(G)$.*

Proof. : Let $\lambda(G) = m$ with corresponding labeling $f : V(G) \to 0,1,...,m$. Then $g : V(H) \to 0,1,...,m$, defined by $g(v) = f(v)$ for all $v \in V(H)$, is a labeling of H that uses no label greater than m. Thus $\lambda(H) \leq m = \lambda(G)$. The idea is we can use the same labels we use on G to label the corresponding vertices of H. □

Proposition 2. $\lambda(P_4) = 4$.

Proof. : Since P_3 is a subgraph of P_4, from our previous result we know $\lambda(P_4) \geq \lambda(P_3) = 3$. Figure 2 shows we can label P_4 with no label greater than 5. Thus $\lambda(P_4) = 4$ and the result follows. □

Proposition 3. $\lambda(P_n) = 5$ *for* $n \geq 5$.

Proof. : We have already shown $\lambda(P_4) = 4$. Since P_4 is a subgraph of P_5, from our previous result we know $\lambda(P_5) \geq \lambda(P_4) = 4$. Figure 3 shows we can label P_5 with no label greater than 5. Thus $\lambda(P_5) = 5$.

Next we show $\lambda(P_n) = 5$ for $n > 5$. Let P_n be a path with more than 5 vertices. Since P_5 is a subgraph of P_n, we know $\lambda(P_n) \geq \lambda(P_5) = 5$. It is clear from the figure 4 we can repeat the labels in $P_n(3,0,4,1,5,0,4,1,5,0,4,1...)$. Thus $\lambda(P_n) \leq 4$, hence the proof. □

$$\lambda(P_n) = \begin{cases} 3 & \textit{for } n = 2 \\ 4 & \textit{for } n = 3 \textit{ and } n = 4 \\ 5 & \textit{for } n \geq 5. \end{cases} \tag{1}$$

Fig. 3. Path with 4 vertices

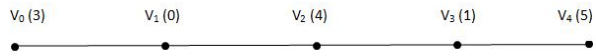

Fig. 4. Path with 5 vertices

3.2. Cycle:

If we join the start vertex with the end vertex of a path we get a cycle see figure 5.

Proposition 4. *Let C_n be a cycle of length n. Then $\lambda(C_n) = 6$ for all $n \geq 3$ and $n \in N$.*

Proof. : If $n < 4$, then it is easy to verify the result. Thus suppose that $n > 5$. For all $n > 5$, C_n must contain a P_5 as a subgraph. Hence $\lambda(C_n) \geq \lambda(P_5) = 5$. □

Now we are going to consider the $L(3,1)$ labeling of cycle C_n. Let $v_0, v_1, v_2, ..., v_{n-1}$ be the vertices of the cycle C_n where v_i is adjacent to v_{i+1} and v_0 is adjacent to v_{n-1}. The rule of labeling of cycle C_n are given below.

1. *If* $n \equiv 0(mod\ 3)$

$$f(v_i) = \begin{cases} 0, \ i \equiv 0(mod\ 3); \\ 3, \ i \equiv 1(mod\ 3); \\ 6, \ i \equiv 2(mod\ 3) \end{cases} \tag{2}$$

2. *If $n \equiv 1(mod\ 3)$, and cycle with multiple of 4 vertices only for $v_{n-4}, v_{n-3}, ..., v_{n-1}$ vertices, rest will follow the rule* (1).

$$f(v_i) = \begin{cases} 0, \ \textit{if } i = n-4; \\ 4, \ \textit{if } i = n-3; \\ 1, \ \textit{if } i = n-2\ ; \\ 5, \ \textit{if } i = n-1 \end{cases} \tag{3}$$

3. *If $n \equiv 2(mod\ 3)$, only for v_{n-2}, v_{n-1}, as follows.*

$$f(v_i) = \begin{cases} 2, \text{ if } i = n-2; \\ 5, \text{ if } i = n-1 \end{cases} \quad (4)$$

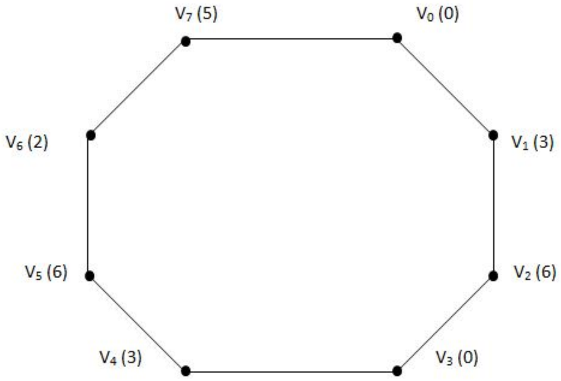

Fig. 5. Cycle with 8 vertices

3.3. Complete graph

Consider the complete graph on n vertices, K_n.

Proposition 5. *For complete graph K_n, $\lambda(K_n) = 3n-3$*

Proof. : Given K_n with vertices $v_1, v_2, ..., v_n$, the function $f : V(G) \to \{0, 1, 2, ...3n-3\}$ defined by $f(v_i) = 3i-3$ is a labeling of K_n. So, $\lambda(K_n) \leq 3n-3$. We claim we can't label K_n with just the numbers $0, 1, 2, ..., 3n-4$. Note that we have $3n-3$ labels that need to be assigned to n vertices. We can think of this as $n-1$ disjoint pairs of consecutive labels in which n vertices must be placed. By the pigeon hole principle, one of these pairs of consecutive labels must contain two vertices. However, since these two vertices are adjacent in K_n, this violates the labeling condition. Thus, $\lambda(K_n) = 3n-3$. □

3.4. Complete bipartite graph

Considering the Complete Bipartite Graph with two set of vertices $|V_1| = m$ and $|V_2| = n$ denoted by $K_{m,n}$.

Proposition 6. *For complete bipartite graph $K_{m,n}$, $\lambda(K_{m,n}) = m+n+1$.*

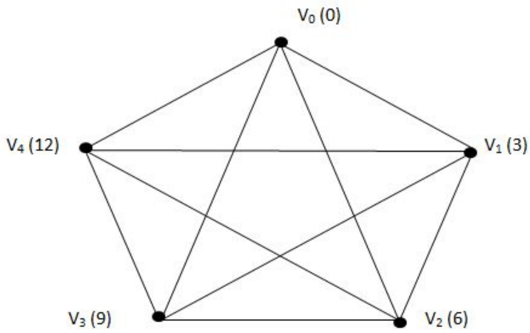

Fig. 6. Complete graph with 5 vertices

Proof. : Let $K_{m,n}$ be the complete bipartite graph with two set of vertices $|V_1| = m$ and $|V_2| = n$. It is clear that the vertices within a set is not connected, so each vertex in a particular set is at distance 2 where as any two vertices from different set are at distance 1. So if we labeling a particular vertex set $|V_1| = m$ start with a by following the rule of distance 2 remaining all vertices take the label $a+1, a+2, a+3, ..., a+m-1$. Next we are going to label another vertex set $|V_2| = n$, which are connected at distance 1. We can start label by $a+m-1+3$, because the vertex label with $a+m-1$ directly connected. Rest labeling proceed with $a+m-1+3+1, a+m-1+3+2, ..., a+m-1+n+2$. Now if we start labeling with 0 it will take minimum number of integer and it becomes $m+n+1$. So $\lambda(K_{m,n}) = m+n+1$. This explanation is illustrated in figure 7. □

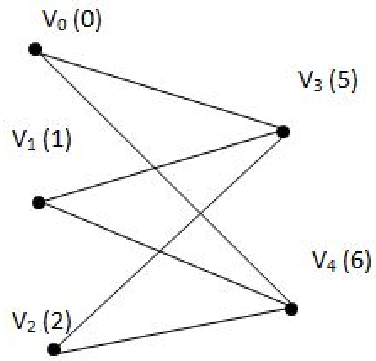

Fig. 7. Complete bipartite graph and its $L(3,1)$ labeling

3.5. Star graph

A star graph actually a bipartite graph with $K_{1,n}$, denoted by S_n.

Proposition 7. *For a star graph S_n, $\lambda(S_n) = n + 2$.*

Proof. : Star graph is a complete bipartite graph where two set of vertices $|V_1| = 1$ and $|V_2| = n$.Therefore, the result follows from proposition 6. □

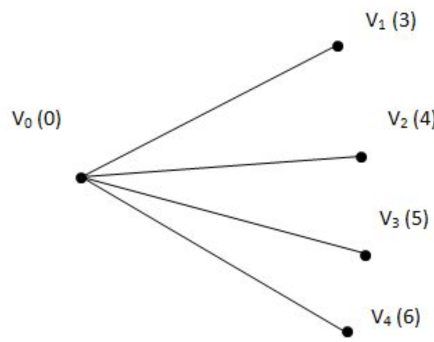

Fig. 8. Star graph

3.6. Bi-star graph

A bi-star graph we can obtain by joining the center (apex) vertices of two $K_{1,n}$ by an edge, denoted by $B_{n,n}$.

Proposition 8. *For a bi-star graph $B_{n,n}$, $\lambda(B_{n,n}) = n + 5$.*

Proof. : Bi-star graph is a two star graph whose apex connected by an edge, we already proof that for a star graph S_n, $\lambda(S_n) = n + 2$. So we have two $K_{1,n}$ connected by an edge, we can label a single $K_{1,n}$ by $a + n + 2$ that we have already shown in the proof of star graph. Another end of the edge can get at least $a + n + 3$ according to the rule of $L(3,1)$-labeling. If we start labeling by 0 it will take minimum labeling, so $\lambda(B_{n,n}) = n + 3$. □

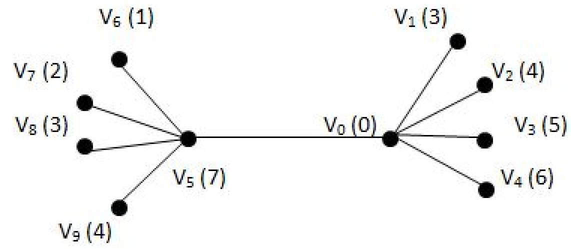

Fig. 9. Bi-star graph

3.7. n-ary tree

Basically a tree is a connected acyclic graph and an n-ary tree is a tree having maximum of n children.

Proposition 9. *For n-ary tree, $\lambda(G) \leq n + 5$.*

Proof. : Let $G = (V, E)$ be an n-ary tree.We can see carefully that every induced subgraph of tree is a sun graph. For a sun graph S_n, $\lambda(G) = n + 2$. From the figure 9 we can see v_0 is the root of the n-ary tree, $v_1, v_2, v_3, ..., v_{n-1}, v_n$ all these are the adjacent to the root vertex v_0, so there will be at least 3 label difference. If we label the root vertex by a then $v_1, v_2, v_3, ..., v_{n-1}, v_n$ will take the label $a + 3, a + 4, a + 5, ..., a + n + 1, a + n + 2$. We have to find the maximum label $\lambda(G)$ so we can consider the highest label node that is v_n which is labeled by $a + n + 2$.
Let there exist another n adjacent vertices of v_n are $v_{n+1}, v_{n+2}, v_{n+3}, ..., v_{2n-1}, v_{2n}$, only v_0 is the vertex which is at distance 2 obviously there should be at least 1 label difference. So we can repeat the label of all the vertices which is adjacent to root vertex except v_n. We can label the vertices v_{2n} by $a + n + 5$. Again we can repeat all the above except the label of v_{2n} for the all the predecessor of v_{2n} and the process will continue. If we consider the value of $a = 0$ then it will attain the minimum label and $\lambda(G) \leq n + 5$. Hence proof. □

4. Conclusion

In this paper, we consider the $L(3,1)$-labeling problem of some simple graphs and tree. The upper bound of $\lambda_{3,1}$ for path, cycle, complete graph, complete bipartite graph, star graph, bi-star graph and tree are provided. Basically $L(3,1)$-labeling is a special form of $L(h,k)$-labeling problem, in future we want to use this technique to label the family of intersection graph to

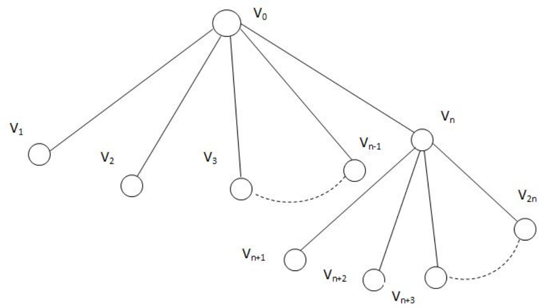

Fig. 10. n-ary tree

tune up the time complexity. Radio frequency assignment problem with minimum use of frequency with the restricted labeling like $L(3,1)$ is always inspire us to achieve the better result.

5. Acknowledgement

Financial support offered by the Department of Science and Technology, New Delhi, India (Ref. No.SB/S4/MS: 894/14) is thankfully acknowledged.

References

[1] Bertossi, A.A., Battiti, R., and Cavallaro, D.,(2001) A randomized saturation-degree heuristic for channel assignment in cellular radio networks. IEEE Trans. Vehicular Tech, vol.50, pp.364-374.

[2] Bertossi, A.A. and Bonuccelli, M.A., (1995)Code assignment for hidden terminal interference avoidance in multihop packet radio networks. IEEE/ACM Transactions on Networking (TON), vol.3, pp.441-449.

[3] Griggs, J.R. and Roger, Y.K., (1992)Labelling graphs with a condition at distance 2. SIAM J. Disc. Math, vol.5, nr.4, pp.586-595.

[4] Hale, W. K., (1980)Frequency assignment. Theory and applications. Proc. IEEE, vol.68, pp.1497-1514.

[5] Khan, N., Pal, M. and Pal, A., (2012)$L(0,1)$-Labelling of cactus graphs. Communications and Network, vol.4, pp.18-29.

[6] Panda, B.S. and Goel, P., (2011)$L(2,1)$-labeling of perfect elimination bipartite graphs. Discrete Applied Mathematics, vol.159, pp.1878-1888.

[7] Panda, B.S. and Goel, P., (2012)$L(2,1)$-labeling of dually chordal graphs and strongly orderable graphs. Information Processing Letters, vol.112, pp.552-556.

[8] Paul, S., Pal, M. and Pal, A., (2014)$L(2,1)$-labeling of interval graphs. J. Appl. Math. Comput, (DOI 10.1007/s 12190-014-0846-6).

[9] Paul, S., Pal, M. and Pal, A., (2014)$L(2,1)$-Labeling of Permutation and Bipartite Permutation Graphs. Math.Comput.Sci, (DOI 10.1007/s 11786-014-0180-2).

[10] Paul, S., Pal, M. and Pal, A., (2013)An efficient algorithm to solve L(0,1)-labelling problem on interval graphs. Advanced Modeling and Optimization, vol.15, nr.1, pp.31-43.

[11] Paul, S., Pal, M. and Pal, A., (2015)$L(0,1)$-labelling of permutation graphs. J Math Model Algor, (DOI 10.1007/s 10852-015-9280-5).

[12] Schwarza, C. and Troxellb, D.S., (2006)$L(2,1)$-labelings of Cartesian products of two cycles. Discrete Applied Mathematics, vol.154, pp.1522-1540.

[13] Shao, Z. and Solis-Oba, R., (2013)$L(2,1)$-labelings on the modular product of two graphs. Theoretical Computer Science, vol.487, pp.74-81.