# A Mathematical Study Of Cannibalism In the Fish-Plankton Model By Taking Into Account The Catching Effect 

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#### Abstract

In this work, we build a Plankton-Fish model with discrete weight structure for the fish population. This model will conserve the number of individuals and the biomass and both growth and reproduction of the fish will depend on the food ingested. Moreover the model allows cannibalism which means that the fish can eat the plankton but also other fishes. We will focus on a simple version with two weight classes or stages (larvae and adults) taking into account the fishing effect in the model and present some mathematical results concerning the solution existence, the stability of the equilibria of the model. Some numerical simulations are done to illustrate these results in the different exploited and inexploited areas.


Key words: Populations Dynamic, Fishing effort, Slow-fast systems, marine environment, cannibalism, predator-prey systems, exploited an non exploited area.
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## 1 Introduction

Fishing plays an important socio-economic role in most countries in maritime vocation. It is an essential lever in the development of these countries and contributes significantly to the growth objectives of the national economy including the deficit of the balance of payments, it also contributes to job creation and food needs satisfaction [1, 13]. Along with the high demand for fish products in the absence of adequate and coherent policies for sustainable management of exploitable resources, a situation of overfishing has settled in certain operating areas $[3,8]$.

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Some fish species are becoming increasingly rare. Due to high demand on the international market, the amount of fish caught in commercial value in certain operating areas declined by over $80 \%$ for some fish species.

At the present stage of knowledge, research on fish, raises a set of problems in terms of eco-biology and population dynamics. These issues related to biology and species ecology are all ambiguous areas for a comprehensive study and reliable use of existing knowledge to the applied level (management and planning). Current trends of the evolution of stocks of the species show worrying signs regarding the average size of fish caught, the reduction of catch per unit of fishing effort. It appears therefore that essential and relevant control mechanisms should be put in place in order to observe a biological rest which will enable the juveniles to reach the age of sexual maturity before being captured. The central problem of dynamics of this species is the declining of the inventories. The main causes of the problem are different among which there are the increase in fishnig effort, the lack of a management plan and layout, changing climatic and hydrological factors, the strong external demand etc.. And the fact of encountering these crises affects this sector and reflections should be undertaken to advocate the sustainable management of this resource.

It is in this context that we are interested in this work to study the dynamics of fish, taking into account the notion of cannibalism within this population. To deepen our study, we will introduce a structure of two classes, larvae and adults, for the fish population and also take into account their resource which is the plankton. Note that our study will focus on a mathematical model that we will build starting from the work of Gaucel [ $5,9,10]$ on the predator-prey models. It is about actually watching carefully the effect of cannibalism on the one hand and on the other hand looking at the effect of the fishery regarding the dynamics of the population in order to take adequate measures concerning the preservation of the species.

The paper is structured as follows. In section 2, we present the mathematical model which will be the subject of our study. Section 3 provides some mathematical results of the model in its general version. Also, we take a closer look at the cannibalism in the model in the section 4. Computational simulations are performed in section 5 and finally, in the last section, we end with some conclusion remarks and future works.

## 2 Model equations

In this part, we are interested in building the model that will be the subject of our study $[2,4,6,7]$. Actually it is about analysing the dynamic of the fish-plankton set to build up the model. To do this we make the following biological assumptions:

## Hypothesis 1:

(a) Larvae feed only on the resources, regardless of quantities of larvae and adults, [14].
(b) Adults spend a fraction of the time $\kappa$ to feed on the resources and a fraction $(1-\kappa)$ to eat each other. Predation on the plankton (resp. the cannibalism) is independent of the quantities of larvae and adults (resp. quantities of larvae and plankton).

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(c) Assimilated biomass of individuals that are not in the final class is used to grow and therefore enables the passage from a class to the following one.
(d) Assimilated biomass of individuals that are in the final class is used to lay. A fraction $\varepsilon$ of the laid eggs hatch and become individuals of the first class

Under the above assumptions, plankton dynamics system, larvae and adult fish is governed by the following figure:


Figure 1: Plankton, Larvae and Fish-adults model

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The state variables are: $P$ the plankton density, $A$ the fish adult density, and $L$ the larvae density. Let $m_{L}$ and $m_{A}\left(m_{L}<m_{A}\right)$ respectively larva and fish adult weigth. Simularly, we define $t_{a}$ the energy used to grow up and to reproduce, and $\varepsilon$ the fraction of eggs hatched. Again, for the construction of the running model we will opt for the selection of response functions Holling Type II and logistic growth of the plankton. So, according to Figure 1, we obtain the following system of three differential equations

$$
\left\{\begin{array}{l}
\frac{d P}{d t}=\varphi(P)-f_{L}(P) L-\kappa f_{A}(P) A, \quad P(0)=P_{0}>0  \tag{2.1}\\
\frac{d L}{d t}=b \kappa f_{A}(P) A+b(1-\kappa) f_{P}(A) A-\tau f_{L}(P) L-\mu_{L} L, \quad L(0)=L_{0}>0 \\
\frac{d A}{d t}=\tau f_{L}(P) L-\mu_{A} A-m_{f} A-\frac{(1-\kappa)}{m_{A}} f_{P}(A) A, \quad A(0)=A_{0}>0
\end{array}\right.
$$

with

- $f_{P}(A)=d_{A} \frac{A}{A_{h}+A}$, a function related to predation for the population of adult fish per time unit,
- $f_{L}(P)=d_{L} \frac{P}{P_{h}+P}$ is the energy acquired per time unit for the larvae on the plankton population,
- $f_{A}(P)=d_{A} \frac{P}{P_{h}+P}$ is the energy acquired per time unit for the fish adult on the plankton population,
- $\mu_{L}$ is the larva external mortality,
- $\mu_{A}$ is the adult external mortality,
- $\tau=\frac{t_{a}}{m_{A}-m_{L}}$ is the transfer-rate of the larva in adult population,
- $b=\frac{\varepsilon t_{a}}{m_{L}}$ is the fertility-rate of the adult population,
- $\varphi(P)=e\left(1-\frac{P}{K_{p}}\right) P$ is the growth function of the plankton quantity,
- $q$ the coefficient of catchability for the adults class ,
- $E_{f}$ the fishing effort for the adults class,
- $e$ is the net increase-rate of the plankton quantity,
- $m_{f}=q E_{f}$ is the peach mortality of the adults population,


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- $P_{h}$ is a constant half-saturation of predation on plankton,
- $A_{h}$ is a constant half-saturation of cannibalism,
- $d_{L}$ is the maximum digestion rate of the larvae,
- $d_{A}$ is the maximum digestion rate of the fish adults,
- $K_{p}$ is the carrying capacity of the plankton.


## 3 Mathematical primaries results

This part is devoted to the mathematical study of the model described in the previous section. It is about actually answering questions related to local existence, global solutions and also balancing a strictly positive components. To do this, we will make the following assumptions:
Hypothesis 2: The functions $f_{A}, f_{L}, f_{P}$ are strictly increasing and continuously differentiable on $\mathbb{R}_{+}$and verify $f_{i}(0)=0$ for $i \in\{P, L, A\}$.
Hypothesis 3: The function $\varphi$ is continuously differentiable on [ $0, K_{p}$ ], strictly positive on $] 0, K_{p}\left[\right.$ and verify: $\varphi(0)=\varphi\left(K_{p}\right)=0, \varphi^{\prime}(0)>0, \varphi^{\prime}\left(K_{p}\right)<0$
Hypothesis 4: The initial condition is taken in the domain $\Omega=\{(P, L, A) / P \in] 0, K_{p}[, L, A \in$ $\left.\mathbb{R}_{+}^{*}\right\}$
Hypothesis 5: All demographic parameters of (2.1) are positive constants.
Hypothesis 6: $\left.\left.t_{a}, \varepsilon \in\right] 0,1\right]$ and $\kappa \in[0,1]$
Let $\Omega:=] 0, K_{p}\left[\times \mathbb{R}_{+}^{*} \times \mathbb{R}_{+}^{*}, \bar{\Omega}\right.$ the closer of $\Omega$ and $\operatorname{Fr}(\Omega)$ its boundary, then $\widehat{\mu}_{A}=\mu_{A}+m_{f}$.
Proposition 3.1 The domain $\Omega$ is positively invariant by means of the positive semiwave generated by (2.1).

Proof: Using the precedent hypotheses, we have:

$$
\begin{gathered}
P \geq 0, L=0, A \geq 0 \Rightarrow \frac{d L}{d t}=b\left(\kappa f_{A}(P)+(1-\kappa) f_{P}(A)\right) A \geq 0 \\
P=K_{p}, L \geq 0, A \geq 0 \Rightarrow \frac{d P}{d t}=-\left(f_{L}\left(K_{p}\right) L-\kappa f_{A}\left(K_{p}\right)\right) A \leq 0 \\
P=0, L \geq 0, A \geq 0 \Rightarrow \frac{d P}{d t}=0 \\
\text { and } P \geq 0, L \geq 0, A=0 \Rightarrow \frac{d A}{d t}=\tau f_{L}(P) L \geq 0
\end{gathered}
$$

We will focus on the trajectories of our solutions.
Proposition 3.2 For any initial condition in $\Omega$ the continuing trajectory from this initial condition remains in a bounded domain included in $\Omega$ under the hypotheses (2) and (3).

Proof: We consider the total biomass of the system at an instant $t$ i.e the function $Q$ defined by:

$$
Q(t)=P(t)+m_{L} L(t)+m_{A} A(t)
$$

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The expression of the derivative of $Q$ is:

$$
\dot{Q}=\varphi(P)-\left(1-t_{a}\right)\left(f_{L}(P)\right) L-\left(1-\varepsilon t_{a}\right)\left(\kappa f_{A}(P)+(1-\kappa) f_{P}(A)\right) A-m_{L} \mu_{L} L-m_{A} \widehat{\mu}_{A} A
$$

As we have $\left.\left.t_{a}, \varepsilon \in\right] 0,1\right]$, we obtain $Q \leq \varphi(P)-m_{L} \mu_{L} L-m_{A} \widehat{\mu}_{A} A \leq \varphi_{\max }-\mu_{\min } Q+\mu_{\min } P$, with $\mu_{\min }=\min \left(\mu_{L}, \widehat{\mu}_{A}\right)$ and $\varphi_{\max }=\sup _{\left[0, K_{p}\right]} \varphi$. Finally, we deduce that $\mu_{\min } P \leq \mu_{\min } K_{p}$ and

$$
Q(t) \leq Q(0) e^{-\mu_{\min } t}+\left(\frac{\varphi_{\max }}{\mu_{\min }}+K_{p}\right)\left(1-e^{-\mu_{\min } t}\right)
$$

then

$$
\lim _{t \rightarrow \infty} \operatorname{Sup} Q(t) \leq \frac{\varphi_{\max }}{\mu_{\min }}+K_{p} .
$$

We introduce the following functions:

$$
\begin{gather*}
\phi(P, \kappa):=\widehat{\mu}_{A} \tau f_{L}(P)+\widehat{\mu}_{A} \mu_{L}-\kappa b \tau f_{L}(P) f_{A}(P)  \tag{3.1}\\
h(P):=1+\frac{\mu_{L}}{\tau f_{L}(P)} \tag{3.2}
\end{gather*}
$$

Proposition 3.3 The system (2.1) admits two trivial equilibria $E_{o}^{*}=(0,0,0)$ always locally unstable and $E_{p}^{*}=\left(K_{p}, 0,0\right)$ locally stable if and only if:

$$
\begin{equation*}
\phi\left(K_{p}, \kappa\right)>0 \tag{3.3}
\end{equation*}
$$

Moreover, if $E_{p}^{*}$ is locally stable, it is globally stable if at least one of the following sufficient conditions is verify:

$$
\begin{equation*}
\kappa f_{A}\left(K_{p}\right) \geq \widehat{\mu}_{A} m_{A} \tag{3.4}
\end{equation*}
$$

or

$$
\begin{equation*}
m_{A} b<h\left(K_{p}\right) \tag{3.5}
\end{equation*}
$$

Proof: We will look at the eigenvalues of the Jacobian matrix defined by

$$
J\left(E_{p}^{*}\right)=\left(\begin{array}{ccc}
\varphi^{\prime}\left(K_{p}\right) & -f_{L}\left(K_{p}\right) & -\kappa f_{A}\left(K_{p}\right) \\
0 & -\tau f_{L}\left(K_{p}\right)-\mu_{L} & \kappa b f_{A}\left(K_{p}\right) \\
0 & \tau f_{L}\left(K_{p}\right) & -\mu_{A}-m_{f}
\end{array}\right)
$$

Clearly $\varphi^{\prime}\left(K_{p}\right)<0$ is an eigenvalue of the Jacobian matrix. The other eigenvalues are those of the sub-matrix formed by the last two rows and columns of $J\left(E_{p}^{*}\right)$. The trace of this sub-matrix being negative, these two eigenvalues will have their real part strictly negative (and therefore the trivial balance will be locally asymptotically stable) if and only if its determinant is positive, that is to say, if and only if relation (3.3) is satisfied.
For the conditions of global stability, we will use two Lyapunov functions (3.6) and (3.8) defined and positive for couples $(L, A) \in \mathbb{R}_{+}^{2}$, domain which is invariant. The first Lyapunov function is:

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$$
\begin{equation*}
Q_{1}(t)=\widehat{\mu}_{A} L(t)+\kappa b f_{A}\left(K_{p}\right) A(t) \tag{3.6}
\end{equation*}
$$

We have

$$
\begin{gather*}
\frac{d Q_{1}}{d t}=(1-\kappa) \cdot b \cdot\left(\widehat{\mu}_{A}-\frac{\kappa}{m_{A}} f_{A}\left(K_{p}\right)\right) f_{P}(A) A+\left(\tau f_{L}(P)\left(\kappa b f_{A}\left(K_{p}\right)-\widehat{\mu}_{A}\right)-\mu_{L} \widehat{\mu}_{A}\right) L  \tag{3.7}\\
+\kappa b \widehat{\mu}_{A}\left(f_{A}(P)-f_{A}\left(K_{p}\right)\right) A
\end{gather*}
$$

- $(1-\kappa) \cdot b \cdot\left(\widehat{\mu}_{A}-\frac{\kappa}{m_{A}} f_{A}\left(K_{p}\right)\right) f_{P}(A) A$ is negative if $\kappa f_{A}\left(K_{p}\right) \geq \widehat{\mu}_{A} m_{A}$,
- $\left(\tau f_{L}(P)\left(\kappa b f_{A}\left(K_{p}\right)-\widehat{\mu}_{A}\right)-\mu_{L} \widehat{\mu}_{A}\right) L$ is stricty negative if $L \neq 0$ and $E_{p}^{*}$ is locally stable.
- Also $\kappa b \widehat{\mu}_{A}\left(f_{A}(P)-f_{A}\left(K_{p}\right)\right) A$ is stricly negative for $A \neq 0$ because $f_{A}$ is a strictly increasing function and $P$ is at values in $] 0, K_{p}[$.

It follows that if $E_{p}^{*}$ is locally stable and if the condition (3.4) holds, then the $\dot{Q}_{1}$ is strictly negative for $(L, A) \neq(0,0)$. We conclude that $(L, A)$ converges to $(0,0)$ and therefore $P$ converges to $K_{p}$ according to the plankton equation.
The second Lyapunov function that we will use is:

$$
\begin{equation*}
Q_{2}(t)=L(t)+b m_{A} A(t) \tag{3.8}
\end{equation*}
$$

and its derivative function is

$$
\begin{equation*}
\left.\frac{d Q_{2}}{d t}=\left(\kappa . b \cdot f_{A}(P) A-b \widehat{\mu}_{A} m_{A} A\right)+\left(\left(m_{A} b-1\right) \tau\right) f_{L}(P) A-\mu_{L}\right) L \tag{3.9}
\end{equation*}
$$

- As $f_{A}$ is strictly increasing, the first term in (3.9) is strictly negative if $A \neq 0$ and

$$
\kappa . f_{A}\left(K_{p}\right)<\widehat{\mu}_{A} m_{A} .
$$

- The second term is strictly negative if $L \neq 0$ and $b . m_{A}<h\left(K_{p}\right)$.

Here again, $\dot{Q}_{2}$ is then strictly negative for $(L, A) \neq(0,0)$ if $f_{A}\left(K_{p}\right)<\widehat{\mu}_{A} m_{A}$ (which is simply the negation of (3.4)) and if the relation (3.5) is satisfied. Consequently ( $L, A$ ) converges to $(0,0)$.

Remark 3.1 Condition (3.3) shows that if fish are too cannibalistic (that is to say if $\kappa$ is close enough to 0); then the $E_{p}^{*}$ equilibrium is locally stable. However, the numerical simulations show that this balance can be locally stable without being globally.

Proposition 3.4 For each solution in $] 0, K_{p}[$ of (3.10) equation, there exists a strict equilibrium positive to model (2.1).

$$
\begin{equation*}
\phi(P, \kappa)+l(P, \Phi(P, \kappa), \kappa)=0 \tag{3.10}
\end{equation*}
$$

with

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$$
\begin{equation*}
l(P, A, \kappa):=-\left(\frac{1-\kappa}{m_{A}}\right) f_{P}(A)\left(\tau f_{L}(P)\left(m_{A} b-1\right)-\mu_{L}\right) \tag{3.11}
\end{equation*}
$$

and $\Phi$ defined on $] 0, K_{p}[$ an unique $A>0$ satisfying:

$$
i(P, A, \kappa):=\tau \varphi(P)-\left(\widehat{\mu}_{A}+\left(\frac{1-\kappa}{m_{A}}\right) f_{P}(A)+\tau f_{A}(P)\right) A=0
$$

Proof: If $\dot{L}=0$ and $\dot{A}=0$, then

$$
b A \frac{\kappa f_{A}(P)+(1-\kappa) f_{P}(A)}{\tau f_{L}(P)+\mu_{L}}=A \frac{\widehat{\mu}_{A}+\frac{1-\kappa}{m_{A}} f_{P}(A)}{\tau f_{L}(P)}
$$

We deduce that there is a positive equilibrium if $(P, A) \in] 0, K_{p}\left[\times \mathbb{R}_{+}^{*}\right.$ satisfying:
$\phi(P, \kappa)+l(P, A, \kappa)=0$

## 4 Cannibalism study

In this section we will show that cannibalism may under certain conditions stabilize a positive balance which would be unstable without cannibalism. For that we will assume that plankton dynamics is faster than fish dynamics, $[4,5,10,11]$ and $\kappa \neq 0$.

### 4.1 Notations, symbols and hypotheses

Hypothesis 7: $0<P_{h}<K_{p}$
The set of triplets $(P, L, A)$ satisfying $\dot{P}=0$ is the slow manifold of fast-slow system. The cancellation of the dynamics of the plankton leads to the following algebraic equation:

$$
\begin{equation*}
\left(P^{2}+\left(P_{h}-K_{p}\right) P+\left(d_{L} L+\kappa d_{A} A-e P_{h}\right) \frac{K_{p}}{e}\right) P=0 \tag{4.1}
\end{equation*}
$$

Knowing that $K_{p}>P_{h}$, the slow manifold is made in the positive octant by three sub-manifolds $\mathcal{N}_{0}, \mathcal{N}_{\kappa}^{+}$and $\mathcal{N}_{\kappa}^{-}$parameterized by $L, A$ :

$$
\begin{gathered}
\mathcal{N}^{0}=\left\{(L, A, 0):(L, A) \in \mathbb{R}_{+}^{2}\right\} \\
\mathcal{N}_{\kappa}^{+}=\left\{\left(L, A, i_{+}(L, A, \kappa)\right):(L, A) \in \Omega_{\kappa}^{+}\right\} \\
\mathcal{N}_{\kappa}^{-}=\left\{\left(L, A, i_{-}(L, A, \kappa)\right):(L, A) \in \Omega_{\kappa}^{-}\right\}
\end{gathered}
$$

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with definitions domain:

$$
\begin{gathered}
\Omega_{\kappa}^{+}=\left\{(L, A) \in \mathbb{R}_{+}^{2}: \frac{d_{L}}{e} L+\kappa \frac{d_{A}}{e} A \leq \frac{1}{4 K_{p}}\left(K_{p}+P_{h}\right)^{2}\right\} \\
\left.\Sigma_{\kappa}=\left\{(L, A) \in \mathbb{R}_{+}^{2}: \frac{d_{L}}{e} L+\kappa \frac{d_{A}}{e} A \leq P_{h}\right)\right\} \varsubsetneqq \Omega_{\kappa}^{+} \\
\Omega_{\kappa}^{-}=\overline{\Omega_{\kappa}^{+} \backslash \Sigma_{\kappa}}
\end{gathered}
$$

and with $i_{ \pm}(L, A, \kappa)=\frac{1}{2}\left(K_{p}-P_{h} \pm \sqrt{\triangle(L, A, \kappa)}\right), \triangle(L, A, \kappa)=\left(K_{p}+P_{h}\right)^{2}-4\left(d_{L} L+\kappa d_{A} A\right) \frac{K_{p}}{e}$.
The fold of the slow manifold is defined as the intersection slow sheets $\mathcal{N}_{\kappa}^{+}$and $\mathcal{N}_{\kappa}^{-}$. The value of $P$ on this fold is independent of $L, A$ and is $\left(K_{p}-P_{h}\right) / 2$. The projection of this fold on the plane $P=0$ is the set of points: $\vartheta_{\kappa}^{N}=\left\{(L, A) \in \mathbb{R}_{+}^{2}: \frac{d_{L}}{e} L+\kappa \frac{d_{A}}{e} A=\frac{1}{4 K_{p}}\left(K_{p}+P_{h}\right)^{2}\right\}$.
The intersection between the sub-manifolds $\mathcal{N}_{\kappa}^{-}$and the sub-manifolds $\mathcal{N}^{0}$ is the set of triples $(L, A, 0)$ satisfying $(L, A) \in \vartheta_{\kappa}^{0}$ with $\vartheta_{\kappa}^{0}=\left\{(L, A) \in \mathbb{R}_{+}^{2}: \frac{d_{L}}{e} L+\kappa \frac{d_{A}}{e} A=P_{h}\right\}$

Throughout the remainder of this study, we will make the following assumption:

## Hypothesis 8

(1) $\mu_{L}+\tau d_{L}<\tau d_{A}$
(2) $\widehat{\mu}_{A}<b d_{L}$

### 4.2 Without cannibalism in the model

We will now study the model without cannibalism, that is to say we will take $\kappa=1$ and show that under certain conditions, the limit system can not have a positive equilibrium which is stable whereas the trajectories converge to a relaxation cycle, $[5,9,10,11,14]$.
Proposition 4.1 - If $\widehat{\mu}_{A}\left(\mu_{L}+\tau d_{L}\right)-b d_{A} \tau d_{L}>0$, then for $P_{h}$ sufficiently little, $(0,0)$ is the global attractor of the slow field on $\mathcal{N}_{1}^{+}$.

- If $\widehat{\mu}_{A}\left(\mu_{L}+\tau d_{L}\right)-b d_{A} \tau d_{L}<0$, then for $P_{h}$ sufficiently little, every trajectory leaves the manifold $\mathcal{N}_{1}^{+}$through the fold.
- If furthermore $\tau d_{A}>\mu_{L}+\tau d_{L}$ and $b d_{L}>\widehat{\mu}_{A}$, then for $P_{h}$ sufficiently little, every trajectory leaves the manifold $\mathcal{N}_{1}^{+}$through the fold transversally.
Proof: The slow fields on $\mathcal{N}_{1}^{+}$has for equation:

$$
\left\{\begin{aligned}
\frac{d L}{d t} & =-\mu_{L} L-\tau f_{L}\left(i_{+}(L, A)\right) L+b f_{A}\left(i_{+}(L, A)\right) A \\
\frac{d A}{d t} & =\tau f_{L}\left(i_{+}(L, A)\right) L-\widehat{\mu}_{A} A
\end{aligned}\right.
$$

We have for all $L, A: i_{+}(L, A) \geq \frac{1}{2}\left(K_{p}-P_{h}\right)>0$.
Consequently:

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$$
f_{L}\left(i_{+}(L, A)\right)=d_{L} \frac{i_{+}(L, A)}{P_{h}+i_{+}(L, A)} \in\left[d_{L} \frac{K_{p}-P_{h}}{K_{p}+P_{h}}, d_{L}\right]
$$

and

$$
f_{A}\left(i_{+}(L, A)\right)=d_{A} \frac{i_{+}(L, A)}{P_{h}+i_{+}(L, A)} \in\left[d_{A} \frac{K_{p}-P_{h}}{K_{p}+P_{h}}, d_{A}\right]
$$

We will now study the slow field on $\mathcal{N}_{1}^{+}$, considering $P_{h}$ as a parameter. In case $P_{h}=0$, this slow field is linear and verifies:

$$
\left\{\begin{align*}
\frac{d L}{d t} & =-\mu_{L} L-\tau d_{L} L+b d_{A} A  \tag{4.2}\\
\frac{d A}{d t} & =\tau d_{L} L-\widehat{\mu}_{A} A
\end{align*}\right.
$$

The trace of this linear system is positive and its determinant is given by: $\widehat{\mu}_{A}\left(\mu_{L}+\tau d_{L}\right)-b d_{A} \tau d_{L}$.

- If this determinant is positive, then $(0,0)$ is globally stable. In particular any compact $\mathbb{R}_{+}^{2}$ is included in the basin of attraction of the null equilibrium.
- If this determinant is strictly negative, then $(0,0)$ is a saddle point which variety stable is in the positive quadrant and the unstable variety is not there (this is immediate knowing that the positive quadrant is invariant).

Therefore, any trajectory will lead to infinity and thus end up leaving any compact $\mathbb{R}_{+}^{2}$. Moreover, the trajectory will leave the manifold $\mathcal{N}_{1}^{+}$transversely to the fold if the following quantity is positive for any $(L, A) \in \vartheta_{1}^{N}$ :

$$
\begin{equation*}
\frac{d_{L}}{e} \dot{L}+\frac{d_{A}}{e} \dot{A}=\frac{d_{L}}{e}\left(-\mu_{L}-\tau d_{L}+\tau d_{A}\right) L+\frac{d_{A}}{e}\left(b d_{L}-\widehat{\mu}_{A}\right) A \tag{4.3}
\end{equation*}
$$

Assumptions made in the third part of the proposition (4.1) ensure that this quantity is always strictly positive, $[5,9,10,11]$. These properties being structurally stable, if $P_{h}$ is small enough, they are still verified

### 4.3 Adding cannibalism in the model

We will now add cannibalism, meaning that we will, take values of $\kappa$ different from 1 . We assume that Hypothesis $\boldsymbol{8}$ is verified and $P_{h}$ little enough so that there will be existence of the relaxation cycle of the model in reduced without cannibalism, [5, 9, 10, 11].
Introduce the constant $\underline{\kappa}$ defined by $\underline{\kappa}=\frac{\widehat{\mu}_{A} \mu_{L}+\widehat{\mu}_{A} \tau d_{L}}{b \tau d_{L} d_{A}}$ which is strictly smaller than 1 according to the Hypothesis 8 .

Hypothesis 9: $\underline{\kappa}>m_{A} \widehat{\mu}_{A} / d_{A}$

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Proposition 4.2 Suppose assumptions (8) and (9) verified and $P_{h}=0$. Then there exists $\bar{\kappa}>\underline{\kappa}$ so that for all $\kappa \in] \underline{\kappa}, \bar{\kappa}\left[\right.$ the system admits a unique positive equilibrium on $\mathcal{N}_{\kappa}^{+}$.

Proof: To achieve this result, just look at when the isoclines of the slow system $\mathcal{N}_{\kappa}^{+}$have an intersection in the domain $\operatorname{int}\left(\Omega_{\kappa}^{+}\right),[10,11]$. For $P_{h}=0$ the slow system on $\mathcal{N}_{\kappa}^{+}$is written:

$$
\left\{\begin{aligned}
\frac{d L}{d t} & =b \kappa d_{A} A+b(1-\kappa) f_{P}(A) A-\mu_{L} L-\tau d_{L} L \\
\frac{d A}{d t} & =\tau d_{L} L-\widehat{\mu}_{A} A-\frac{1}{m_{A}}(1-\kappa) f_{P}(A) A
\end{aligned}\right.
$$

We can then write the equation of the two nullclines $\dot{L}=0$ and $\dot{A}=0$ this way:

$$
\left\{\begin{array}{l}
L=\frac{b \kappa d_{A}}{b d_{L}+\mu_{L}} A+\frac{b(1-\kappa)}{b d_{L}+\mu_{L}} f_{P}(A) A:=f_{1}(A) \\
L=\frac{\widehat{\mu}_{A}}{b d_{L}} A+\frac{(1-\kappa)}{b d_{L} m_{A}} f_{P}(A) A:=f_{2}(A)
\end{array}\right.
$$

These two nullclines intersect (not necessarily in the domain $\operatorname{int}\left(\Omega_{\kappa}^{+}\right)$) when $f_{1}(A)-f_{2}(A)=0$, that is to say that

$$
\begin{equation*}
\left(\frac{\kappa}{\underline{\kappa}}-1\right)+(1-\kappa)\left(\frac{1}{\underline{\kappa}}-\frac{d_{A}}{m_{A} \widehat{\mu}_{A}}\right) \frac{f_{P}(A)}{d_{A}}=0 \tag{4.4}
\end{equation*}
$$

- By setting $s(\kappa)=\frac{\kappa \kappa}{\underline{\kappa}}-1$ and $r(\kappa)=(1-\kappa)\left(\frac{1}{\underline{\kappa}}-\frac{d_{A}}{m_{A} \widehat{\mu}_{A}}\right)$, knowing that $\kappa>\underline{\kappa}$ and $\underline{\kappa}>m_{A} \widehat{\mu}_{A} / d_{A}$, we have $s(\kappa)>0$ and $r(\kappa)<0$.
- $f_{P}$ being a strictly function increasing, there is at most a value $A^{*}(\kappa)$ solution of equation (4.4).
- Furthermore $s(\kappa) \rightarrow 0$ when $\kappa$ tends to $\underline{\kappa}$ inferiory and $r(\kappa) \neq 0$ when $\kappa$ tends to $\underline{\kappa}$ inferiory.
- knowing that $f_{P}(0)=0$, this solution $A^{*}(\kappa)$ tends to 0 when $\kappa$ decrease to $\underline{\kappa}$.
- Therefore $L^{*}(\kappa)=f_{1}\left(A^{*}(\kappa)\right)$ tends also to 0 .

Finally, by the continuity of the value of the equilibrium regarding $\kappa$, there exists $\bar{\kappa}>\underline{\kappa}$ such that:

$$
\forall \kappa \in] \underline{\kappa}, \bar{\kappa}\left[, \quad 0<\frac{d_{L}}{e} L^{*}(\kappa)+\frac{d_{A}}{e} A^{*}(\kappa)<\frac{K_{p}}{4}\right.
$$

Theorem 4.1 Let us assume that Hypothesis 8 and Hypothesis 9 hold. So for $P_{h}$ suffi-

i) there exists a unique positive balance of $\mathcal{N}_{\kappa}^{+}$,

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ii) there exists $(\bar{L}, \bar{A}) \in \vartheta_{\kappa}^{N}$ so that the domain $V$ defined below is included in the attraction basin of this positive balance

$$
\left.\left.\left.V=[0, \bar{L}] \times[0, \bar{A}] \times] 0, K_{p}\right] \backslash\left(\{(0,0, P): P \in] 0, K_{p}\right]\right\} \cup \mathcal{N}_{\kappa}^{-}\right)
$$

Proof: We consider the following sets $D=\left\{\left(L, A, i_{+}(L, A, \kappa)\right):(L, A) \in \widehat{D}\right\}$ and $\widehat{D}:=$ $[0, \bar{L}] \times[0, \bar{A}]$
In addition the domain $D \subset \mathcal{N}_{\kappa}^{+}$is positively invariant by the slow field reduced system $\mathcal{N}_{\kappa}^{+}$, [9].
We show in the first place that the domain $D \backslash(0,0)$ is included in the attraction basin of the positive balance on $\mathcal{N}_{\kappa}^{+}$basin. For $P_{h}=0$, the Jacobian matrix of the slow field of limit system is the following:

$$
J_{\kappa}(L, A)=\left(\begin{array}{cc}
-\mu_{L}-\tau d_{L} & b d_{A} \kappa+b d_{A}(1-\kappa)\left(1-\frac{A_{h}^{2}}{\left(A_{h}+A\right)^{2}}\right) \\
\tau d_{L} & -\widehat{\mu}_{A}-d_{A} \frac{(1-\kappa)}{m_{A}}\left(1-\frac{A_{h}^{2}}{\left(A_{h}+A\right)^{2}}\right)
\end{array}\right)
$$

The divergence of the field is always negative, much more, according to the proposition(4.2) the domain $D$ is invariant. We conclude by Green's theorem $[9,10,11,12]$ that it is not possible to have any limit cycle or polycycle in $D$ and therefore any trajectory beginning in $D$ converges to a balance belonging to $D$. The trivial equilibrium is unstable and its stable variety having an empty intersection with $D$, there is necessarily convergence to the positive balance (except if the trajectory starts exactly at the point of the trivial balance trivial). For $P_{h}$ sufficiently small, these properties are verified and there is therefore yet convergence towards positive equilibrium.

The study of the fast dynamics in the volume $V$ enables to conclude. Indeed, two cases of situation may occur:

- The trajectory starts in the part of $V$ where $\mathcal{N}_{\kappa}^{+}$is attractive, so the trajectory arrives in $D$ for then converge toward equilibrium.
- The trajectory begins in part of $V$ where $\mathcal{N}_{0}$ is attractive, it then runs this manifold $\mathcal{N}_{0}$ for finally leave it when $(L, A) \in \Sigma_{\kappa} \subset \widehat{D}$ and go back up rapidly in $D$ and finally converge to the positive equilibrium.


## 5 Numerical simulations

In this part of the paper we will present a series of numerical simulations to illustrate the previous mathematical results. We will present the results of the numerical model with cannibalism without fishing to show how the dynamics can be stable. Then we will introduce the effect of fishing in the simulations and see that for some value of fishing effort, a stable equilibrium is possible. We will also highlight the conditions of illustrating the behavior of the dynamics indicating the extinction of the fish population.

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To illustrate the different behaviors of the model, we will use the parameters given in the following table:

| Parameters | $e$ | $K_{p}$ | $d_{A}$ | $b$ | $m_{L}$ | $d_{L}$ | $\tau$ | $q$ | $A_{h}$ | $P_{h}$ | $\varepsilon$ | $\mu_{A}$ | $\mu_{L}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Value used | 10 | 1000 | 300 | 8 | 5 | 10 | 16 | 1 | 10 | 2 | 1 | 1 | 1 |

Concerning the other parameters such as $E_{f}, \kappa$ we will carry out our numerical experiments by making them vary.

### 5.1 Numerical results of the model with cannibalism, without fishing

In this section we will present a series of numerical simulations to illustrate the previous results. We will present the numerical results of the model with cannibalism without fishing to show how the dynamics can be stable.
We will present the results we obtained in the numerical simulation while considering different values of $\kappa$ regarding cannibalism.

### 5.1.1 Numerical results for the model with cannibalism for $\kappa=0.6$

We will consider here that cannibalism value is $\kappa=0.6$. Figure 2 shows the evolution of the dynamics of the plankton (x2) and the dynamics of Larvae-Adults system (x1). This figure shows the existence of a solid and full area (figure) and an oscillation of the dynamics of the plankton. This result not only confirms the existence of different populations but also the noninfluence of cannibalism in fish population. So this simulation shows the existence of stable equilibrium. From a biological point of view, this means that when cannibalism is manifested with this value, the regeneration of the environment is always effective. Otherwise the transfer cycle of larvae into adults and the eggs laying of adult females are assured. It should be noted here that these results are in line with biological observations of the species. To further investigate the effect of cannibalism in the dynamics of behavior, we will increase its effect to observe the numerical results.

### 5.1.2 Numerical results for the model with cannibalism for $\kappa=0.45$

As we have said before, we increase the value materializing the effect of cannibalism in the simulation. Our goal is to deepen our study on the effect of cannibalism. To do this, we consider this time $\kappa=0.45$. Numerical simulation shows somehow similar behavior encountered previously. Meaning that the existence of solid and full area for the dynamics of Larvae-Adults system. Figure 3 confirms this behavior and this attests the existence of balance to the dynamic. We can say without doubt that if cannibalism is presented with this value, our biological system operates normally. From a biological point of view, cannibalism can not prevent the adults from reaching the maturity phase and thus a continuous egg-laying from the adult females. So the larvae will exist in the environment and then they will be transfered into adult by going through the class of fry (the intermediate status between the adult and larvae). Note again that these

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Figure 2: Distribution of populations of fish and plankton
results are in conformity with the biological observations on the phenomenon of cannibalism. Always in the interest of understanding the impact of cannibalism, we continue to vary the


Figure 3: Distribution of populations of fish and plankton
value of $\kappa$ to observe the behavior of the dynamics.

### 5.1.3 Numerical results for the model with cannibalism for $\kappa=0.3$

We continue our study by considering this time for the simulation, a value $\kappa=0.3$ for the cannibalism. The numerical simulation also shows the existence of a solid and full area for the dynamic modeling the larvae-adults system and an oscillation of the plankton dynamics. Figure 4 shows that a balance is installed after a transition phase for the three populations. Here again, this explains the existence of the population with this value for cannibalism. In fact what should be noted in this situation is that cannibalism, as in the previous case has no

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negative impact on the system's regeneration process. Biologically, this confirms the fact that the regeneration cycle is maintained for this species.


Figure 4: Distribution of populations of fish and plankton

Remark 5.1 These results show that cannibalism has no negative effect on this population and even on the contrary it helps to balance the environment and this is in perfect conformity with the biological observations.

Numerical simulation of the model without fishing in the area has shown very interesting results based on different values of cannibalism in the fish population. Thus, we saw that cannibalism has no effect on the dynamics of our model. This simply means that if there is extinction or remarkable decline of this species, that implies other factors affecting the regeneration of the environment. In order to contribute to resolve this concern, we will conduct our numerical study to an area where fishing is practised to watch the behaviour of the dynamics.

Regarding the remaining of our simulations, we consider a fishing area by varying the value of the fishing effort $\left(E_{f}\right)$ and also with the notion of cannibalism $(\kappa=0.45)$ in the fish population.

### 5.2 Numerical results of the model with cannibalism introducing fishing

We continue our numerical study, but now in fishing area to answer questions related to the existence and extinction of the fish population. Here we will present the main results obtained in our numerical simulations by considering different fishing areas. The goal in this part of the simulation is to find the conditions on fishing effort for which we will have an unbalance to our dynamic. So we will present a series of simulation for our model by varying the value of the fishing effort.

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### 5.2.1 Numerical results for the model with cannibalism for $E_{f}=0.23$

We start by looking at the dynamics of behavior by considering the value of the effort $E_{f}=0.23$. Figure 5 shows the existence of a solid and full area for the dynamics. Compared to the previous figures, we see a sleazy disturbance regarding the behavior of the dynamics and this is quite normal since in addition to cannibalism, we have fishing which is practiced in this area. What is interesting to notice here is the existence of solid and full area and this means that there is a convergence towards equilibrium and thus the existence of the fish population at long term. Those results allow us to say that even if the area is exploited with such value for fishing effort, despite the cannibalism, we see that the observed dynamics is very close to the one encountered in the case of a population living in an pristine area. This allows us to say that if the zone is exploited with this value for fishing effort, fish populations are not at risk since there will always be the existence of this species in this area.


Figure 5: Distribution of populations of fish and plankton

### 5.2.2 Numerical results for the model with cannibalism for $E_{f}=0.5$

We continue our numerical study by considering now a fish population living in an exploited area with a fishing effort $E_{f}=0.5$. Figure 6 shows the existence of solid and full area with a sleazy disruption compared to previous figure (figure 5), which enables us to say that there is coexistence of different classes of fish despite the presence of fishing and cannibalism within this population. Thus the phenomenon of regeneration system and transfert are respected. So, this result means that if fishing is done with effort $E_{f}=0,5$, there is not much risk since adult females reach their maturity phase to ensure the regeneration of the especies. The persistence of the convergence toward a center of these dynamics on the population of different classes, despite fishing, shows that the area is normally exploited even with cannibalism within the fish population. Now, if with these values of the fishing effort, we have a coexistence, what about regarding the behaviour of the dynamics when we consider an area operated with a value of fishing effort greater than 0.5 ?

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Figure 6: Distribution of populations of fish and plankton

### 5.2.3 Numerical results for the model with cannibalism for $E_{f}=0.87$

We consider for this simulation that the dynamics evolves in an exploited area. The goal is to understand how each population evolves when the area is exploited with $E_{f}=0.87$. Figure 7 shows the behavior of the dynamics of the fish population. Trivially, we notice from this figure, the existence of a vacuum in the zone materializing the cycle of relaxation of larvae-adult fish dynamics and therefore we can say that there is no persistence at long time of the dynamics. This is explained by the fact that the adult females decrease because of levies by fishing, and not because of cannibalism; necessarily, the number of egg-laying decreases considerably. This situation also explains the non regeneration of the system. This can be explained by the fact that, given the intense fishing, juvenile do not reach their maturity stage and therefore a decreasing of the number of adult fish according to a biological point of view.


Figure 7: Distribution of populations of fish and plankton

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### 5.2.4 Numerical results for the model with cannibalism for $E_{f}=1.15$

We continue our numerical study by considering this time that the fish population living in an area exploited with $E_{f}=1,15$ still with a goal to observe the extinction condition of the fish population. Figure 8 shows the behavior of the dynamics of the population. We notice from this figure an empty persistence in the relaxation area of the dynamics. Therefore we can say that there is a divergence of our dynamics in a long frame of time. This is explained by the fact that adult females reach with difficulty their mature phase because of the excessive levies by fishing. This situation also explains the non regeneration of the system and therefore those populations tend to disappear. Consequently, the transfer rate decreases very significantly and even if they reach the adult stage, the disturbance does not promote the regeneration of the system; because of the fishing but not the effect of cannibalism within this population. One can also say that those populations will extinct in finite time. As until now we do not have an


Figure 8: Distribution of populations of fish and plankton
extinction condition of the species, we continue our numerical analysis by increasing the value of fishing effort to watch the behavior of the fish dynamics.

### 5.2.5 Numerical results for the model with cannibalism for $E_{f}=2.18$

We will finish our numerical study by considering a fish population living in an area exploited with $E_{f}=2.18$ still with a goal to watching the extinction condition of the fish population. Figure 9 shows the behavior of the dynamics of the fish population. We notice from this figure high vacuum persistence in the relaxation area of the dynamic and therefore we can say that there is a divergence of the dynamics in a long time period. This is explained by the fact that adult females reach with difficulty their maturity phase because of the excessive levies by fishing. Perforce, the number of egg-laying decreases very considerably. This situation also explains the non regeneration of the system and therefore those populations tend to disappear. These results confirm that only fishing can cause this disturbance dynamics and not cannibalism. One can also say that those populations will extinct in finite time.

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For those types of exploited area, we talk about a severely exploited area.


Figure 9: Distribution of populations of fish and plankton

## 6 Conclusion

In this note, we were interested in the dynamics of fish population on diet on the plankton taking into account the aspect of cannibalism within fish population. The mathematical model associated to this dynamics is based on ODE systems. The mathematical results allowed us to show that despite cannibalism with fish population, equilibria with strictly positive components exist. We can say that cannibalism does not negatively influence the aquatic ecosystem. Also the numerical simulations allowed us to have an idea about the behavior of our dynamics based on different values of the fishing effort. However, it is obvious that when the area is exploited with some fishing effort value, an efficient management policy must be adopted; otherwise it is likely to be catastrophic for the fish population.

In our future works, we will continue our study, focusing on the impact of cannibalism in the fish population by considering many classes with the concept of spatial structure in the model in order to observe the diffusion of the population.

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