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THE PEDAL CONE SURFACE CONSISTING PARALLEL CURVES ACCORDING TO BISHOP FRAME IN $\mathbb{E}^3$

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Abstract. In this paper, we study pedal surface consisting parallel curves of any curve according to Bishop frame. We characterize B-pedal cone surface of the developable ruled surface. Finally, we examine the pedal cone surface of a parallel curve and give new examples for this characterizations.

1. Introduction

To α is a curve in plane, there exist two curves $\mathbf{P}_+ = \alpha(s) + t\mathbf{N}(s)$ and $\mathbf{P}_- = \alpha(s) - t\mathbf{N}(s)$ at a given distance t. But these cuves is not easy to characterize in 3-dimensional space. Then, Chrastinova developed a new construction in [2]. This construction is carried over the three-dimensional space and as a result, two parallel curves are obtained as well. Additionally author study parallel helices in three-dimensional space.

The Frenet frame is constructed for the curve of continuously differentiable non-degenerate curves. However, curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, we need an alternative frame in $\mathbb{E}^3$. Therefore, in [1], Bishop has defined a new frame for a curve called Bishop frame which is well defined even if the second derivative of the curve in Euclidean 3-space is zero, [15]. Then, it has been the subject of numerous studies. For example, in [8,13] the outhors introduced new characterizations of parallel curves according to Bishop frame and Yilmaz studied new version of Bishop frame and application to spherical images in [17].

In this paper, we obtain new pedal cone surface consisting parallel curves according to Bishop frame in $\mathbb{E}^3$. Firstly, we summarize properties Bishop frame and Frenet frame which are parameterized by arc-length parameter s and the basic concepts on curves and ruled surfaces. The Finally, we give examples of this surface according to Bishop frame in $\mathbb{E}^3$.

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2. Preliminaries

In plane let a smooth curve $\alpha(s) = (x(s), y(s))$, where s is the arc-length and its unit tangent and unit normal vectors are $\mathbf{T}(s)$ and $\mathbf{N}(s)$, respectively. Then, we get

$$\mathbf{P}_+(S_+) = \alpha(s) + t\mathbf{N}(s) \text{ and } \mathbf{P}_-(S_-) = \alpha(s) - t\mathbf{N}(s),$$

where $S_\pm = S_\pm(s)$ and $S_\pm$ denotes the length along $\mathbf{P}\pm$, at the distance t. Determining the length $S_\pm$, we can write

$$\frac{dS_\pm}{ds} = 1 \pm t\kappa,$$

where κ is the curvature of $\alpha(s)$, [2, 10].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}\mathbf{T}' &= \kappa_1\mathbf{M}_1 + \kappa_2\mathbf{M}_2,\\ \mathbf{M}_1' &= -\kappa_1\mathbf{T},\\ \mathbf{M}_2' &= -\kappa_2\mathbf{T},\end{aligned}$$

where $\theta(s) = \arctan\frac{\kappa_2}{\kappa_1}$, $\tau(s) = \dot{\theta}(s)$ and $\kappa(s) = \sqrt{\kappa_1^2 + \kappa_2^2}$.

Definition 2.1. *Let C be a curve in $\mathbb{E}^3$ and O be a fixed point not on C. The locus of the foots of perpendicular drawing from O to the tangents of C with respect to O as origin are called the pedal curve of C with respect to O*, [14].

We denote a surface $\mathbf{M}$ in $\mathbb{E}^3$ by

$$\mathbf{M}(s,t) = (m_1(s,t),\ m_2(s,t),\ m_3(s,t)).$$

Let U be the standard unit normal vector field on a surface $\mathbf{M}$ defined by

$$\mathbf{U} = \frac{\mathbf{M}_s \wedge \mathbf{M}_t}{\|\mathbf{M}_s \wedge \mathbf{M}_t\|},$$

where $\mathbf{M}_s = \partial\mathbf{M}(s,t)/\partial s,\ \mathbf{M}_t = \partial\mathbf{M}(s,t)/\partial t$, respectively.

A ruled surface is a surface swept out by a straight line L moving along a curve α. The various positions of the generating line L are called the ruling of the surface. Such a surface thus always has a parametrization in ruled form

$$X(u,v) = \alpha(u) + v\delta(u),$$

where we call α the base curve, δ the director curve. Alternatively we may vizualize δ as a vector field on α. Frequently it is necessary to restrict v to some interval, so the ruling may not be entire straight lines.

On the other hand, a ruled surface is a surface generated by the motion of a straight line δ along α. Furthermore, if α is a closed curve, then this surfaces is called closed ruled surface, [9].

Definition 2.2. *Let M be a smooth, convex surface in $\mathbb{E}^3$ and O be a point not on M. If X is teh position vector of a point $\mathbf{P}$ on M with respect to O as* *origin and N is the inner unit normal vector of the surface at $\mathbf{P} \in M$, then support function h of M is defined by*

$$h = -\langle X, N\rangle,$$

where $\langle , \rangle$ is the usual metric in $\mathbb{E}^3$, [4].

Definition 2.3. *The surface $\hat{M}$ with the position vector*

$$\hat{X} = -hN$$

of an arbitrary point $\hat{\mathbf{P}}$ on the tangent plane $T_M(\mathbf{P})$ of M with respect to O as origin is called the pedal surface of M with respect to O.

Geometrically, we can construct the pedal surface $\hat{M}$ as follows:

We draw tangent plane $T_M(\mathbf{P})$ and we get the normal to that plane from O. The normal contracts the plane $T_M(\mathbf{P})$ at a point $\hat{\mathbf{P}}$. The locus of all points $\hat{\mathbf{P}}$ for all $\mathbf{P} \in M$ will give the pedal surface, [4].

Definition 2.4. *Let α be a regular curve with parametrized by arc-length, $\hat{\alpha}$ is a parallel curve of α and M, $\hat{M}$ be herhangi two ruled surface. If the base curve $\hat{\alpha}$ of $\hat{M}$ is the pedal of the base curve α of M, the $\hat{M}$ is called the pedal cone surface of the developable ruled surface M,* [4].

In the rest of the paper, assume that $S = s$ and Frenet Frame, curvature and torsion, Bishop Frame, Bishop curvatures of α with respect to arc-length parameter s denote $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, κ, τ, $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$, κ_1, κ_2 and Frenet Frame, curvature and torsion, Bishop Frame, Bishop curvatures of $\mathbf{P}$ with respect to arc-length parameter s denote $\{\tilde{\mathbf{T}}, \tilde{\mathbf{N}}, \tilde{\mathbf{B}}\}$, $\tilde{\kappa}$, $\tilde{\tau}$, $\left\{\tilde{\mathbf{T}}, \tilde{\mathbf{M}}_1, \tilde{\mathbf{M}}_2\right\}$, $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, respectively.

3. The Pedal Cone Surface of Parallel Curves in $\mathbb{E}^3$

Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length and $\mathbf{P}$ its a parallel curve. We obtained that for any parallel curve with Bishop frame;

$$\mathbf{P}(s) = \alpha(s) + \mu(s)\,\mathbf{M_1}(s) + \eta(s)\,\mathbf{M_2}(s),$$

where

$$\mu = \frac{1}{\kappa_1} - \frac{2\kappa_2 \tan\theta - \kappa_1\kappa_2 C}{2\kappa_1^2 \sec^2\theta} \text{ and } \eta = \frac{2\tan\theta - \kappa_1 C}{2\kappa_1 \sec^2\theta}.$$

On the other hand, ruled surface in $\mathbb{E}^3$ can be written as

$$M : \psi(s, v) = \mathbf{P}(s) + v\tilde{\mathbf{T}}(s), \tag{3.1}$$

where $\mathbf{P}(s)$ is the base curve and $\tilde{\mathbf{T}} = (1 - \kappa_1\mu - \kappa_2\eta)\mathbf{T} + \mu'\mathbf{M_1} + \eta'\mathbf{M_2}$ is tangent at $\mathbf{P}$, [8].

Let M be a developable ruled surface given by (3.1) in $\mathbb{E}^3$. Since the tangent plane is constant along ruling of M, it is clear that the pedal of M is a curve. Thus, for the pedal of M, we can write

$$\hat{\mathbf{P}}(s) = \mathbf{P}(s) + Q(s)\,\tilde{\mathbf{T}}(s), \quad \left\|\tilde{\mathbf{T}}(s)\right\| = \|\mathbf{P}'(s)\| = 1, \tag{3.2}$$

where Q is the distance between the points $\mathbf{P}(s)$ and $\hat{\mathbf{P}}(s)$, [4].

Theorem 3.1. *Let $\alpha : I \rightarrow \mathbb{E}^3$ be a regular curve with parametrized by arc-length, $\mathbf{P}$ be parallel curve of α and $\hat{\mathbf{P}}$ be the pedal curve of $\mathbf{P}$. Give $\mathbf{P}$ be base curve of M ruled surface and $\hat{\mathbf{P}}$ be base curve of $\hat{M}$ ruled surface. If $\hat{M}$ is B-pedal cone surface of the developable ruled surface M,* then

$$\begin{aligned}\hat{M} &= \alpha + (Q - Q\kappa_1\mu - Q\kappa_2\eta + \phi_2\phi_7\eta' - \phi_3\phi_7\mu')\mathbf{T} \\ &\quad + (Q\mu' + \mu + \phi_3\phi_7 - \phi_1\phi_7\eta' - \phi_3\phi_7\kappa_1\mu - \phi_3\phi_7\kappa_2\eta)\mathbf{M}_1 \\ &\quad + (Q\eta' + \eta - \phi_2\phi_7 - \phi_1\mu'\phi_7 + \phi_2\phi_7\kappa_1\mu + \phi_2\phi_7\kappa_2\eta)\mathbf{M}_2,\end{aligned}$$

where

$$\begin{aligned}\phi_7(s,v) &= [(\phi_2\eta' - \phi_3\mu')^2 + (\phi_3 - \phi_1\eta' - \phi_3\kappa_1\mu - \phi_3\kappa_2\eta)^2 + (-\phi_2 \\ &\quad - \phi_1\mu' + \phi_2\kappa_1\mu + \phi_2\kappa_2\eta)^2]^{-1}[\phi_2\phi_4\eta' - \phi_3\phi_4\mu' - \phi_1\phi_5\eta' \\ &\quad + \phi_3\phi_5 - \phi_3\phi_5\kappa_1\mu - \phi_3\phi_5\kappa_2\eta + \phi_1\phi_6\mu' - \phi_2\phi_6 + \phi_2\phi_6\kappa_1\mu \\ &\quad + \phi_2\phi_6\kappa].\end{aligned}$$

Proof. We can rewrite the equations (3.1) and (3.2) as

$$\hat{\mathbf{P}} = \alpha + (Q - Q\kappa_1\mu - Q\kappa_2\eta)\,\mathbf{T} + (Q\mu' + \mu)\,\mathbf{M}_1 + (Q\eta' + \eta)\,\mathbf{M}_2 \tag{3.3}$$

and

$$\psi = \alpha + (v - v\kappa_1\mu - v\kappa_2\eta)\,\mathbf{T} + (v\mu' + \mu)\,\mathbf{M}_1 + (v\eta' + \eta)\,\mathbf{M}_2. \tag{3.4}$$

Differanting ruled surface ψ according to the parameters s and v, we have by

$$\psi_s(s,v) = \phi_1(s,v)\,\mathbf{T}(s) + \phi_2(s,v)\,\mathbf{M}_1(s) + \phi_3(s,v)\,\mathbf{M}_2(s), \tag{3.5}$$

where

$$\begin{aligned}\phi_1 &= 1 - v\kappa_1'\mu - v\kappa_1\mu' - v\kappa_2'\eta - v\kappa_2\eta' - v\kappa_1\mu' - \kappa_1\mu - v\kappa_2\eta' - \kappa_2\eta, \\ \phi_2 &= v\kappa_1 - v\kappa_1^2\mu - v\kappa_1\kappa_2\eta + v\mu'' + \mu', \\ \phi_3 &= v\kappa_2 - v\kappa_1\kappa_2\mu - v\kappa_2^2\eta + v\eta'' + \eta'\end{aligned}$$

and

$$\psi_v(s,v) = (1 - \kappa_1\mu - \kappa_2\eta)\mathbf{T} + \mu'\mathbf{M_1} + \eta'\mathbf{M_2}. \tag{3.6}$$

Since

$$\psi(s,v) - \hat{\mathbf{P}}(s) = \phi_4(s,v)\,\mathbf{T}(s) + \phi_5(s,v)\,\mathbf{M}_1(s) + \phi_6(s,v)\,\mathbf{M}_2(s),$$

where

$$\begin{aligned}\phi_4 &= v - v\kappa_1\mu - v\kappa_2\eta - Q + Q\kappa_1\mu + +Q\kappa_2\eta, \\ \phi_5 &= v\mu' - Q\mu', \\ \phi_6 &= v\eta' - Q\eta',\end{aligned}$$

then

$$\begin{aligned}\left(\psi - \hat{\mathbf{P}}, \psi_s, \psi_v\right) &= \phi_2\phi_4\eta' - \phi_3\phi_4\mu' - \phi_1\phi_5\eta' + \phi_3\phi_5 - \phi_3\phi_5\kappa_1\mu \\ &\quad - \phi_3\phi_5\kappa_2\eta + \phi_1\phi_6\mu' - \phi_2\phi_6 + \phi_2\phi_6\kappa_1\mu + \phi_2\phi_6\kappa_{2.}\end{aligned} \tag{3.7}$$

It is clear that, the rullings of the ruled surface $\hat{M}$ are pass through the fixed point O. Since the base curve $\hat{\mathbf{P}}$ of $\hat{M}$ is the pedal of the base curve $\mathbf{P}$ of M, $\hat{M}$ is the *B-pedal cone surface* of the developable ruled surface M.

From (3.3), (3.4) and (3.7), we can write *B-pedal cone surface* of the developable ruled surface M by

$$\begin{aligned}\hat{M} &= \alpha + (Q - Q\kappa_1\mu - Q\kappa_2\eta)\,\mathbf{T} + (Q\mu' + \mu)\,\mathbf{M}_1 + (Q\eta' + \eta)\,\mathbf{M}_2 \\ &+ [(\phi_2\eta' - \phi_3\mu')^2 + (\phi_3 - \phi_1\eta' - \phi_3\kappa_1\mu - \phi_3\kappa_2\eta)^2 + (-\phi_2 \\ &- \phi_1\mu' + \phi_2\kappa_1\mu + \phi_2\kappa_2\eta)^2]^{-1}[\phi_2\phi_4\eta' - \phi_3\phi_4\mu' - \phi_1\phi_5\eta' \\ &+ \phi_3\phi_5 - \phi_3\phi_5\kappa_1\mu - \phi_3\phi_5\kappa_2\eta + \phi_1\phi_6\mu' - \phi_2\phi_6 + \phi_2\phi_6\kappa_1\mu \\ &+ \phi_2\phi_6\kappa][(\phi_2\eta' - \phi_3\mu')\mathbf{T} + (\phi_3 - \phi_1\eta' - \phi_3\kappa_1\mu - \phi_3\kappa_2\eta)\mathbf{M}_1 \\ &+ (-\phi_2 - \phi_1\mu' + \phi_2\kappa_1\mu + \phi_2\kappa_2\eta)\mathbf{M}_2].\end{aligned}$$

Example 3.2. Let us consider a unit speed curve in $\mathbb{E}^3$ by

$$\alpha = \alpha(s) = (\cos s, \sin s, 5).$$

One can calculate its Frenet-Serret apparatus as the following

$$\begin{aligned}\mathbf{T} &= (-\sin s, \cos s, 0), \\ \mathbf{N} &= (-\cos s, -\sin s, 0), \\ \mathbf{B} &= (0, 0, 1).\end{aligned}$$

Then, the curvatures of α is given by

$$\kappa = 1 \text{ and } \tau = 0.$$

Putting,

$$\theta = c,$$

where $c \in \mathbb{R}$, $\theta(s) = \int\limits_0^s \tau(s)\,ds$. If we choose $\theta = \frac{\pi}{4}$, then we can write the Bishop frame of α by

$$\begin{aligned}T &= (-\sin s, \cos s, 0), \\ M_1 &= \frac{\sqrt{2}}{2}(-\cos s, -\sin s, -1), \\ M_2 &= \frac{\sqrt{2}}{2}(-\cos s, -\sin s, 1).\end{aligned}$$

If $\mathbf{P}$ is parallel curve of α which is a regular curve with parametrized by arc-length, then

$$\mathbf{P} = \left(-\frac{\sqrt{2}}{2}\cos s, -\frac{\sqrt{2}}{2}\sin s, -\frac{5\sqrt{2}}{2}\right).$$

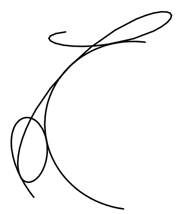
In this case, we can write M ruled surface as

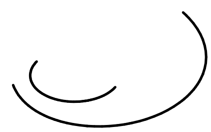
$$\psi = \left(-\frac{\sqrt{2}}{2}\cos s, -\frac{\sqrt{2}}{2}\sin s, -\frac{5\sqrt{2}}{2}\right) + v\left(\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, 0\right).$$

Then, pedal curve of ψ is

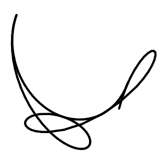
$$\hat{\mathbf{P}} = \left(-\frac{\sqrt{2}}{2}\cos s + \frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\sin s - \frac{\sqrt{2}}{2}\cos s, -\frac{5\sqrt{2}}{2}\right),$$

where $Q(s) = 1$ is the distance between the points $\mathbf{P}(s)$ ile $\hat{\mathbf{P}}(s)$.

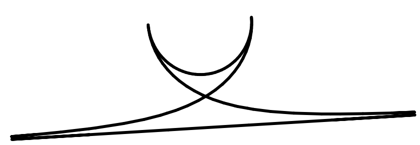




$Q(s) = \cos(s^2)$ $\quad$ $Q(s) = \sqrt{s^2+1}$



$Q(s) = \sin(s^2)$



$Q(s) = \tan(s)$

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