AMO — Advanced Modeling and Optimization, Volume 18, Number 2, 2016

ON INEXTENSIBLE FLOWS OF NEW TYPE SURFACES WHICH GENERATED BY FIRST PRINCIPLE DIRECTION CURVE IN $\mathbb{E}^3$

VEDAT ASIL, GÜLDEN ALTAY AND TALAT KÖRPINAR

Abstract. In this paper, we study inextensible flows of new type surfaces which generated by first principle direction curve in Euclidean 3-space E^3. Finally, we obtain mean curvature and Gaussian curvature of these surfaces.

1. Introduction

The evolution of curves and surfaces has important applications in many fields such as computer vision, computer animation, and image processing. Also, many nonlinear phenomena in physics, chemistry and biology are described by dynamics of shapes, such as curves and surfaces. The time evolution of a curve or surface generated by its corresponding flow in $\mathbb{E}^3$. The flow of a curve or surface is said to be inextensible if, in the former case, the arclength is preserved, and in the latter case, if the intrinsic curvature is preserved. Physically, inextensible curve and surface flows are characterized by the absence of any strain energy induced from the motion. A piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications. Also, inextensible curve and surface flows arise in the context of many problems in computer vision and computer animation, [8].

The Frenet frame is generally known an orthonormal vector frame for curves. But, it does not always meet the needs of curve characterizations. In [9], there is a alternative moving frame and the Sabban frame, respectively. Then, they gave some new characterizations of the C-slant helix and prove that a curve of C-constant precession is a C-slant helix.

In this paper, we study inextensible flows of some types of the surfaces which generated by first principle direction curve in Euclidean 3-space $\mathbb{E}^3$. Furthermore, we give some new characterizations of this surfaces.

1991 *Mathematics Subject Classification.* Primary 53B25; Secondary 53C40.

Key words and phrases. Inxtensible flows, first principle direction curve, mean curvature, Gaussian curvature..

***AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

2. Preminaries

Denote by $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned}\mathbf{T}' &= \kappa\mathbf{N},\\ \mathbf{N}' &= -\kappa\mathbf{T}+\tau\mathbf{B},\\ \mathbf{B}' &= -\tau\mathbf{N}.\end{aligned}$$

Definition 2.1. *Let $\gamma(s)$ be a regular unit-speed curve in terms of $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$. The integral curves of $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are called the tangent direction curve, principal direction curve and binormal direction curve of $\gamma(s)$, respectively,* [9] .

The principal direction curve of $\gamma(s)$, $\alpha=\int\mathbf{N}(\mathbf{s})ds$ has a new frame as,

$$\begin{aligned}\mathbf{T}_1 &= \mathbf{N},\\ \mathbf{N}_1 &= \frac{\mathbf{N}'}{\|\mathbf{N}\|} = \frac{-\kappa\mathbf{T}+\tau\mathbf{B}}{\sqrt{\kappa^2+\tau^2}},\\ \mathbf{B}_1 &= \mathbf{T}_1\times\mathbf{N}_1 = \frac{\tau\mathbf{T}+\kappa\mathbf{B}}{\sqrt{\kappa^2+\tau^2}},\end{aligned}$$

where the tangent vector and the binormal vector of α are the principle normal vector and the unit Darboux vector of $\gamma(s)$, respectively. Also curvatures of α are

$$\kappa_1=\sqrt{\kappa^2+\tau^2},\ \tau_1=\sigma\kappa_1,$$

where σ is the geodesic curvature of $\mathbf{N}_1$. The first principle direction curve,

$$\alpha=\int\mathbf{N}(\mathbf{s})ds \text{ with the frame } \{\mathbf{T}_1 = \mathbf{N},\, \mathbf{N}_1 = \frac{\mathbf{N}'}{\|\mathbf{N}\|},\ \mathbf{B}_1 = \mathbf{T}_1\times\mathbf{N}_1\}.$$

The Frenet equations are satisfied

$$\begin{aligned}\mathbf{T}_1' &= \kappa_1\mathbf{N}_1,\\ \mathbf{N}_1' &= -\kappa_1\mathbf{T}_1+\tau_1\mathbf{B}_1,\\ \mathbf{B}_1' &= -\tau_1\mathbf{N}_1.\end{aligned}$$

Definition 2.2. *We can define the following one-parameter family of new type surface*

$$X(s,v,t)=\alpha(s,t)+f(v,t)\,\mathbf{T}_1(s,t),$$

where $\mathbf{T}_1(s,t)$ is unit tangent vector field of $\gamma(s,t)$.

Definition 2.3. *A surface evolution $\phi(s,u,t)$ is its flow $\dfrac{\partial\phi}{\partial t}$ are said to be inextensible if its first fundamental form $\{E,F,G\}$ satisfies*

$$\frac{\partial E}{\partial t}=\frac{\partial F}{\partial t}=\frac{\partial G}{\partial t}=0.$$

This definition states that the surface $\phi(s,u,t)$ is, for all time t, the isometric image of the original surface $\phi(s,u,t_0)$ defined at some initial time t_0. For a developable surface, $\phi(s,u,t)$ can be physically pictured as the parametrization of a waving flag. For a given surface that is rigid, there exists no nontrivial inextensible evolution.

3. Inextensible Flows of Some Types of Surfaces Generated by First Principle Direction Curve in $\mathbb{E}^3$

In this section, we characterize inextensible some types of surfces which is generated by first principal direction curve in Euclidean 3- space $\mathbb{E}^3$. Then, we obtain mean curvature and Gaussian curvature of these surfaces.

Theorem 3.1. *Let*

$$X(s,v,t) = \alpha(s,t) + f(v,t)\,\mathbf{T}_1(s,t) \tag{3.1}$$

in $\mathbb{E}^3$. *If the surface family* $X(s,v,t)$ *is inextensible, then*

$$\sigma = 1,\ f(v,t) = h(v) + g(t), \tag{3.2}$$

where $\sigma = \dfrac{\tau_1}{\kappa_1}$ *is the geodesic curvature of* $\mathbf{N}_1$.

Proof. If we take derivatives of the surface, which is given with the parametrization (3.1), we have

$$\begin{aligned} X_s &= (1 - f\kappa_1)\mathbf{T}_1 + f\tau_1\mathbf{B}_1, \\ X_v &= \frac{\partial f}{\partial v}\mathbf{N}_1. \end{aligned} \tag{3.3}$$

Then, components of the first fundamental form of the surface are

$$\begin{aligned} E &= (1 - f\kappa_1)^2 + f^2\tau_1^2, \\ F &= 0, \\ G &= f_v^2. \end{aligned} \tag{3.4}$$

So, derivatives of the coefficients of the first fundamental form are

$$\begin{aligned} \frac{1}{2}\frac{\partial E}{\partial t} &= -(1 - f\kappa_1)(\kappa_1\frac{\partial f}{\partial t} + f\frac{\partial \kappa_1}{\partial t}) \\ &\quad + (f\tau_1^2\frac{\partial f}{\partial t} + f^2\tau_1\frac{\partial \tau_1}{\partial t}), \end{aligned} \tag{3.5}$$

$$\frac{\partial F}{\partial t} = 0, \tag{3.6}$$

$$\frac{\partial G}{\partial t} = f_{vt}. \tag{3.7}$$

If the surface inextensible, we have

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0.$$

Then, from equations (3.5) and (3.7)

$$f(v,t) = h(v) + g(t),$$

$$\kappa_1 = \tau_1 = \frac{g'(t)}{h(v) + g(t)}.$$

So, the proof is complete.

Corollary 3.2. *The points of the surface family* $X(s,v,t)$ *are hyperbolic points.*

Proof. The unit normal vector field of the surface is

$$\mathbf{U} = \frac{1}{\sqrt{(1-f\kappa_1)^2 + f^2\tau_1^2}} \left(-\tau_1 f \mathbf{T}_1 + (1-f\kappa_1)\mathbf{B}_1\right). \tag{3.8}$$

Second derivatives of the surface are

$$\begin{aligned} X_{ss} &= -f\kappa_{1s}\mathbf{T}_1 + ((1-f\kappa_1)\kappa_1 - \tau_1^2 f)\mathbf{N}_1 + f\tau_{1s}\mathbf{B}_1 \\ X_{sv} &= f_v(-\kappa_1\mathbf{T}_1 + \tau_1\mathbf{B}_1) \\ X_{vv} &= f_{vv}\mathbf{N}_1. \end{aligned}$$

Then, components of the second fundamental form of the surface are

$$h_{11} = \frac{f^2\kappa_{1s}\tau_1 + f\tau_{1s}(1-f\kappa_1)}{\sqrt{(1-f\kappa_1)^2 + f^2\tau_1^2}}, \tag{3.9}$$

$$h_{12} = \frac{f_v\tau_1}{\sqrt{(1-f\kappa_1)^2 + f^2\tau_1^2}}, \tag{3.10}$$

$$h_{22} = 0. \tag{3.11}$$

Then, from (3.4) and (3.9)-(3.11) Gaussian curvature of the $X(s,v,t)$ is

$$K = -\frac{f_v^2\tau_1^2}{(1-f\kappa_1)^2 + f^2\tau_1^2}.$$

Corollary 3.3. *Let $X(s,v,t)$ be a surface which is given by equation (3.8) in $\mathbb{E}^3$. If*

$$\kappa_1 = const. \text{ and } \frac{\tau_1}{\kappa_1} = const., \tag{3.12}$$

then, $X(s,v,t)$ is a minimal surface.

Proof. From equations (3.4) and (3.6)-(3.8), the equation of mean curvature is

$$H = \frac{f(f\kappa_1'\tau_1 + \tau_1'(1-f\kappa_1))}{((1-f\kappa_1)^2 + f^2\tau_1^2)^{3/2}}. \tag{3.13}$$

So, if the surface $X(s,v)$ is minimal, then

$$\kappa_1 = const. \text{ and } \frac{\tau_1}{\kappa_1} = const.$$

Theorem 3.4. *Let*

$$\psi(s,v,t) = \alpha(s,t) + f(v,t)\mathbf{B}_1(s,t) \tag{3.14}$$

in $\mathbb{E}^3$. If the surface family $\psi(s,v,t)$ is inextensible, then

$$f(v,t) = h(v) + g(t)$$

and

$$\tau_1 = cnst. \text{ or } g'(t)\tau_1 + \tau_{1t}(h(v) + g(t)) = 0,$$

where $\sigma = \frac{\tau_1}{\kappa_1}$ is the geodesic curvature of $\mathbf{N}_1$.

Proof. If we take derivatives of the surface, which is given with the parametrization (3.14), we have

$$\psi_s = \mathbf{T}_1 - f\tau_1\mathbf{N}_1, \qquad (3.15)$$
$$\psi_v = f_v\mathbf{B}_1.$$

Then, components of the first fundamental form of the surface are

$$E = 1 + f^2\tau_1^2, \quad F = 0, \quad G = f_v^2. \qquad (3.16)$$

So, derivatives of the coefficients of the first fundamental form are

$$\frac{1}{2}\frac{\partial E}{\partial t} = (f\tau_1^2\frac{\partial f}{\partial t} + f^2\tau_1\frac{\partial \tau_1}{\partial t}), \qquad (3.17)$$

$$\frac{\partial F}{\partial t} = 0, \qquad (3.18)$$

$$\frac{\partial G}{\partial t} = f_{vt}. \qquad (3.19)$$

If the surface inextensible, we have

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0.$$

Then, from equations (3.5) and (3.7)

$$f\,(v,t) = h\,(v) + g\,(t)\,,$$

$$\tau_1 = 0$$

or

$$g'\,(t)\,\tau_1 + \tau_{1t}(h\,(v) + g\,(t)) = 0.$$

So, the proof is complete.

Corollary 3.5. *$\psi\,(s,v,t)$ which is given by equation (3.14) in $\mathbb{E}^3$. Then, the Gaussian curvature of the surface $\psi\,(s,v,t)$ is given by*

$$K = -\frac{\tau_1^2}{(1+f^2\tau_1^2)^2}.$$

Proof. The unit normal vector field of the surface is

$$\mathbf{U} = -\frac{1}{\sqrt{1+f^2\tau_1^2}}\,(\tau_1 f\mathbf{T}_1 + \mathbf{N}_1)\,. \qquad (3.20)$$

Second derivatives of the surface are

$$\psi_{ss} = -f\kappa_1\tau_1\mathbf{T}_1 + (\kappa_1 - f\tau_{1s})\mathbf{N}_1 - f\tau_1^2\mathbf{B}_1$$
$$\psi_{sv} = -f_v\tau_1\mathbf{N}_1$$
$$\psi_{vv} = f_{vv}\mathbf{B}_1.$$

Then, components of the second fundamental form of the surface are

$$h_{11} = -\frac{f^2\kappa_1\tau_1^2 + \kappa_1 - f\tau_{1s}}{\sqrt{1+f^2\tau_1^2}}, \tag{3.21}$$

$$h_{12} = \frac{f_v\tau_1}{\sqrt{1+f^2\tau_1^2}}, \tag{3.22}$$

$$h_{22} = 0. \tag{3.23}$$

Corollary 3.6. *$\psi(s,v,t)$ which is given by equation (3.14) in $\mathbb{E}^3$. Then, the mean curvature of the surface $\psi(s,v,t)$ is given by*

$$H = -\frac{f^2\kappa_1\tau_1^2 + \kappa_1 - f\tau_1'}{(1+f^2\tau_1^2)^{3/2}}.$$

Proof. It is obviousliy from equations (3.16), (3.21)- (3.23).

References

[1] P. Alegre , K. Arslan, A. Carriazo , C. Murathan and G. Öztürk, Some Special Types of Developable Ruled Surface, Hacettepe Journal of Mathematics and Statistics, 39 (3) (2010), 319– 325.

[2] M.P. Carmo: Differential Geometry of Curves and Surfaces, Pearson Education, 1976.

[3] R. A. Hussien, S. G. Mohamed, Generated Surfaces via Inextensible Flows of Curves in $\mathbb{R}^3$, Journal of Applied Mathematics, 2016 (Appear),

[4] S. Izumiya and N.Tkeuchi, New special curves and developable surfaces, Turk J. Math., 28 (2004), 153-163 .

[5] F. Kanbay, Bonnet ruled Surfaces, Acta Mathematica Sinica , 21 (3) (2005), 623- 630.

[6] T. Körpınar,V. Asil, S. Bas, Characterizing Inextensible Flows of Timelike Curves According to Bishop Frame in Minkowski Space, Journal of Vectorial Relativity Vol 5 (4) (2010), 18-25.

[7] T. Körpınar,V. Asil, S. Bas, On Characterization Inextensible flows of Curves According to Bishop frame in E^3, Revista Notas de Matematica, 7 (2011), 37-45.

[8] DY. Kwon , FC. Park, DP Chi, Inextensible flows of curves and developable surfaces, Appl. Math. Lett. 18 (2005), 1156-1162.

[9] B. Uzunoğlu, İ. Gök and Y. Yaylı, A new approach on curves of constant precession, Applied Mathematic sand Computation 275, (2016), 317–323.

[10] Z. Soyuçok, The Problem of Non- Trivial Isometries of Surfaces Preserving Principal Curvatures, Journal of Geometry, 52 (1995), 173- 188.

Firat University, Department of Mathematics,23119, Elaziğ, Turkey, Muş Alparslan University, Department of Mathematics,49250, Muş, Turkey

E-mail address: vedatasil@gmail.com, guldenaltay23@gmail.com, talatkorpinar@gmail.com