First-Fit EFL

Suresh Dara*[1], Vance Faber†[2], Suresh M. Hedge‡[1] and Noah Streib§[2]

[1]Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Mangalore, India
[2]Center for Computing Sciences, Bowie, MD

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Abstract

In this note, we consider the on-line version of the Erdős-Faber-Lovász (EFL) conjecture for hypergraphs.

1 Notation

Let $H = (V, E)$ be a hypergraph: a set of subsets E called *edges* of a set V called *vertices* (see, for example, [Berge, 1989]). We often write $n = |V|$ and $m = |E|$. The *rank* of an edge e, denoted $r(e)$, is the cardinality of e. We let the minimum rank of an edge in E be ρ and the maximum rank be P. If all edges have the same rank, we say the hypergraph is *uniform*, or perhaps *ρ-uniform*. If H is 2-uniform, then H is a *graph*. If the intersection of an two distinct edges has at most one vertex, we call the hypergraph *linear*.

2 Coloring

A (proper) coloring of the edges of a hypergraph is a function γ from the edges of the hypergraph into a set Γ, called *colors*, such that $\gamma(e) = \gamma(f)$ only if e and f are disjoint. We let $q(H)$, called the *chromatic index of H*, be the cardinality of the smallest Γ for which there is a coloring.

*suresh.dara@gmail.com
†vance.faber@gmail.com
‡smhegde59@gmail.com
§nsstrei@super.org

3 EFL

We consider the following version of the EFL conjecture.

Conjecture 1 (EFL)**.** *Let $H = (V, E)$ be a linear hypergraph with $\rho = 2$ and $n = |V|$. Then $q(H) \leq n$.*

EFL has been discussed at length in the literature (see, for example, [Romero and Sánchez-Arroyo, 2007]). It has been shown that there is a constant C such that for each fixed rank larger than C, there can only be finitely many uniform counterexamples to EFL (see [Faber, 2016]; this paper also discusses more general bounds on the chromatic index of linear hypergraphs). In the present note, we address the evidence for an ostensibly stronger conjecture involving on-line coloring of H.

Definition 1. A *first-fit* coloring of a hypergraph H is a coloring created by taking the edges of H one at a time under some fixed ordering and coloring them with the first available color (according to some fixed total ordering on the elements of Γ). We let $\phi(H)$ be the largest number of colors needed by any ordering of the edges.

We remark that others have considered this problem. It is known that there exist H for which $\phi(H) > n$. However, all known examples have an edge of rank 2 [Berman, 2016].

Conjecture 2 (First-fit EFL)**.** *For a hypergraph H, if $\rho(H) \geq 3$ then $\phi(H) \leq n$.*

4 Computational Evidence for First-fit EFL

We have performed a computer exploration of first-fit algorithms both on randomly generated orderings and specifically chosen orderings for uniform H of rank 3 for n up to 51. Note that we feel that rank 3 is the worst case because of the results for coloring in [Faber, 2010] where it was shown to be easier to color when the minimum rank of H grows. We tabulate the results from 100,000 randomly generated hypergraphs[1] for each order in Figure 1.

As a check on the density of outliers, we also ran 1,000,000 hypergraphs for each $n \in \{23, 25, 27, 29, 31\}$ and the results did not change. However, if

[1]For the random generation, we start with a set $E = \emptyset$ and U as the set of all 3-tuples of $\{1, 2, \ldots, n\}$. We then choose a random element of U to put into E, and then remove all elements of U that intersect some element of E at least twice. We repeat this process until U is empty, at which time E is the edge set of H.

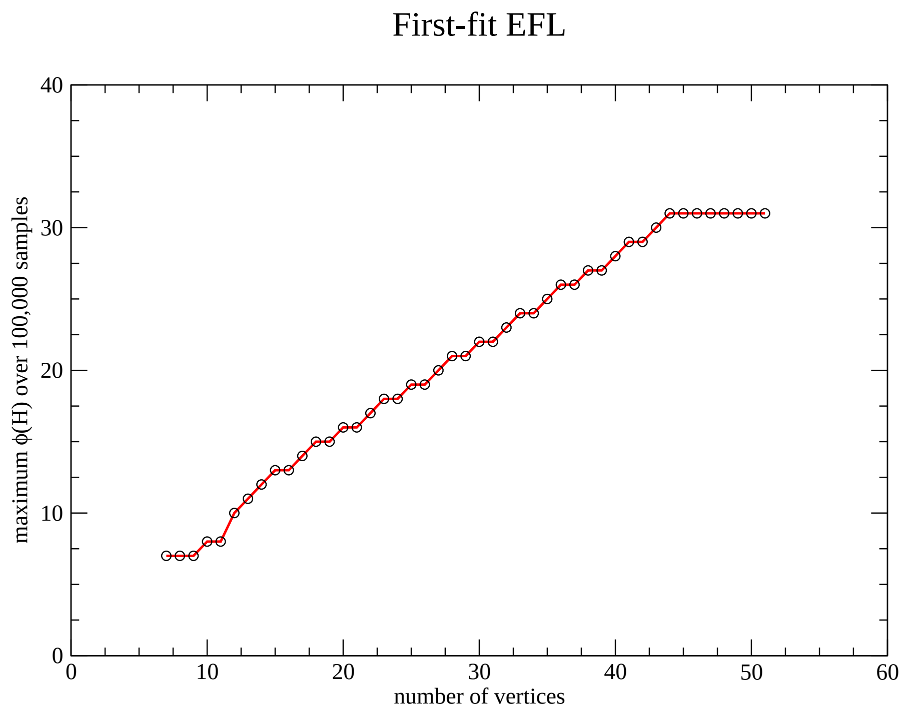

Figure 1: Each point represents the maximum value of $\phi(H)$ over 100,000 randomly generated 3-uniform hypergraphs H with n vertices.

we input Steiner Triple Systems instead of random graphs, the worst case colorings used more colors. We found that for H an STS(15) (i.e. a Steiner Triple System on 15 vertices), we could have $\phi(H) = 15$, but never found anything larger. For example, consider the coloring of an STS(15) given in Table 1.

When first-fit sees the edges in the order of the color classes (i.e. the edges of color 1 first, then those of color 2, etc.), then it will use 15 colors.

5 Similar Conjectures

It is possible that the list coloring version of EFL is true. That is, suppose $H = (V, E)$ is a hypergraph, and we assign to each edge $e \in E$ a set $\Gamma(e)$ of possible colors to give to e. Then a *list coloring of* H is an function γ such that γ is a coloring in which $\gamma(e) \in \Gamma(e)$ for each $e \in E$. We say that H has list chromatic index $q_L(H)$ if H can be list colored from any Γ with $|\Gamma(e)| \le q_L$ for each $e \in E$.

color	edges of STS(15)
1	$(2,7,12),(6,9,10),(0,1,8)$
2	$(0,2,6),(4,9,14),(3,10,11)$
3	$(6,8,12),(1,2,9),(0,4,7)$
4	$(2,4,8),(0,3,9),(1,6,11)$
5	$(5,9,12),(7,8,10),(3,4,6),(0,11,14)$
6	$(8,9,11),(1,4,5),(3,12,14)$
7	$(2,3,5),(1,10,12),(6,7,14)$
8	$(0,5,10),(1,3,7),(4,11,12)$
9	$(5,8,14),(0,12,13)$
10	$(3,8,13),(5,7,11),(2,10,14)$
11	$(5,6,13)$
12	$(1,13,14)$
13	$(2,11,13)$
14	$(7,9,13)$
15	$(4,10,13)$

Table 1: Ordering of edges for an STS(15) in which first-fit uses 15 colors.

Conjecture 3 (List EFL)**.** *Let $H = (V, E)$ be a linear hypergraph with $|V| = n$. Then $q_L(H) \leq n$.*

It should be noted that in [Faber, 2016], we showed that for many H with rank two edges, the truth of the first-fit EFL conjecture for hypergraphs with rank greater than two would imply the truth of EFL for H. In other cases, this implication would follow from a variant of the Vizing List Coloring Conjecture (see [Faber, 2016]) for graphs, namely that $q_L(G)$ is at most the maximum degree plus one. If H is 2-uniform, then List EFL is true for H (see [Haggkvist and Janssen, 1997]).

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