The Energy of a Hypersurface

Ayşe Altın*

Abstract

In this paper, we compute the energy of a unit normal vector field on a hypersurface M in (n+1)-dimensional manifold $\overline{M}$. We show that the energy of a unit normal vector field may be expressed in terms of the principal curvatures functions of M. To this end we define the energy of the hypersurface.

Keywords and Phrases : Energy, Energy of a unit normal vector field, Energy of a Hypersurface, Sasaki metric.

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1 Introduction

In [1], we compute the energy of a Frenet vector field and the pseudo-angle between Frenet vectors for a given non-null curve C in semi Euclidien space of signature (n, ν). We observe that the energy and pseudo-angle may be expressed in terms of the curvature functions of C. In [2], we calculated the energy of a unit normal vector field on the surface M in R^3 and we show that the energy may be expressed in terms of the Gaussian curvature, mean curvature and area of M. We achieved that energy of a unit normal vector field is invariant on the orthonormal basis of the tangent space of M. We will forget the constant term of area of M and we
define the energy of the surface. Using this definition we calculated the energy of a domain on surface.

In this paper, we compute the energy of a unit normal vector field on a hypersurface M in (n+1)-dimensional manifold $\overline{M}$ with respect to the basis. We show that the energy of a unit normal vector field may be expressed in terms

*Hacettepe University, Faculty of Science, Department of Mathematics, 06550 Beytepe Ankara Turkey.
Tel.:+90 0312 297 78 50; Fak: +90 0312 299 20 17.
E-mail address: ayse@hacettepe.edu.tr
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of the principal curvatures functions of M. The principal curvatures functions on M are independent from the choice of the basis. To this end we define the energy of the hypersurface. Using this definition we calculated the energy of a hypersurface.

2 Preliminaries

Definition 2.1. Let (M, g) be a hypersurface of a (m + 1)-dimensional Riemannian manifold $(\overline{M}, \overline{g})$ and $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$, let

$$A_N : \Gamma(TM) \to \Gamma(TM), \quad A_N(\zeta) = -\overline{\nabla}_\zeta N \tag{1}$$

where N is a unit normal vector field on M and $\Gamma(TM)$ is set of tangent vector fields. A_N is called Weingarter funtamental tenser.
If $A_N = \lambda$ then M is called totally umbilical hypersurface where $\lambda \in C^\infty(M; R)$.

For a unit normal vector N of M at a point p, A_N is self-adjoint. Hence there exist orthonormal vectors $\{e_1, ..., e_n\}$ of M at p which are the eigenvectors of A_N , that is,

$$A_N e_i = k_i e_i \tag{2}$$

for real numbers k_i. We call the eigenvalues k_i are call the principal curvatures and the eigenvectors e_i the principal directions of M at p [5].

Definition 2.2. The energy of a differentiable map
$f : (M, <,>) \to (N, h)$ between Riemannian manifolds is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_M (\sum_{a=1}^{n} h(df(e_a), df(e_a))\upsilon \tag{3}$$

where υ is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (see for example [3], [6]).

In [6], the energy of a unit vector field on a Riemannian manifold M is defined as the energy of the mapping $X : M \to T^1M$, where $\pi : T^1M \to M$ is the bundle projection and the unit tangent bundle T^1M is equipped with the restriction of the Sasaki metric.

Proposition 2.1. The connection map $K : T(T^1M) \to T^1M$ verifies the following conditions.

1) $\pi \circ K = \pi \circ d\pi$ and $\pi \circ K = \pi \circ \widetilde{\pi}$, where $\widetilde{\pi} : T(T^1M) \to T^1M$ is the tangent bundle projection.

2) For $\omega \in T_xM$ and a section $\xi : M \to T^1M$, we have

$$K(d\xi(\omega)) = \nabla_\omega \xi$$

where ∇ is the Levi-Civita covariant derivative (see [4]).

Definition 2.3. For $\eta_1, \eta_2 \in T_\xi(T^1M)$ define

$$g_{\mathcal{S}}(\eta_1, \eta_2) =< d\pi(\eta_1), d\pi(\eta_2) > + < K(\eta_1), K(\eta_2) > . \tag{4}$$

This gives a Riemannian metric on TM. Recall that $g_{\mathcal{S}}$ is called *the Sasaki metric*. The metric g_s makes the projection $\pi : T^1M \to M$ a Riemannian submersion (see [4]).

3 The energy of the unit normal vector field of a hypersurface

Theorem 3.1. Let $(M, <,>)$ be a hypersurface of a (n + 1)-dimensional Riemannian manifold $(\overline{M}, <,>)$ and $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Let N be unit normal vector field of M. Then we have the energy of N.

$$\mathcal{E}(N) = \frac{1}{2}\int_M \sum_{i,j=1}^{n} (a_{ij}^2)\upsilon + \frac{n}{2}V(M)$$

where a_{ij} are real-valued functions υ is the volume form in M, $V(M)$ is volum of M.

Proof. Let $\{e_1, ..., e_n\}$ be an local orthonormal basis of the tangent space, N be unit normal vector field of M and NM be normal bundle. Thus we have $N : (M, <,>) \to (NM, g_{\mathcal{S}})$ where $NM = \bigcup_{p\in M} N_pM$, N_pM denotes generated by N and $g_{\mathcal{S}}$ is the restriction of the Sasaki metric on the tangent bundle TM.

Now, let $\pi : NM \to M$ be the bundle projection and the Levi-Civita connection map $K : T(NM) \to NM$. By using equation (3) we obtain the energy of N is

$$\mathcal{E}(N) = \frac{1}{2}\int_M (\sum_{i=1}^{n} g_{\mathcal{S}}(dN(e_i), dN(e_i)))\upsilon. \tag{5}$$

From (4) we get

$$g_{\mathcal{S}}(dN(e_i), dN(e_i)) =< d\pi(dN(e_i)), d\pi(dN(e_i)) > + < K(dN(e_i)), K(dN(e_i)) > .$$

Since N is a section we have $d(\pi)\circ d(N) = d(\pi\circ N) = d(id_M) = id_{TM}$. By Proposition 2.1, we also have that $K(dN(e_i)) = \overline{\nabla}_{e_i}N$, giving

$$g_{\mathcal{S}}(dN(e_i), dN(e_i)) =< e_i, e_i > + < \overline{\nabla}_{e_i}N, \overline{\nabla}_{e_i}N) > .$$

For hypersurface $\overline{\nabla}_{e_i}N = -A_N e_i$, $\forall p \in M$, because of (1), we get,

$$g_{\mathcal{S}}(dN(e_i), dN(e_i)) =< e_i, e_i > + < A_N e_i, A_N e_i > .$$

Using these results in (5) we get

$$\mathcal{E}(N) = \frac{1}{2}\int_M \sum_{i=1}^{n} < A_N e_i, A_N e_i > \upsilon + \frac{n}{2}V(M). \tag{6}$$

On the other hand, since $A_N e_i \in \Gamma(TM)$ we have that

$$A_N e_i = \sum_{j=1}^{n} a_{ij} e_j$$

where a_{ij} are real-valued functions and $< A_N e_i, A_N e_i >= \sum_{j=1}^{n} a_{ij}^2$
From (6) we get,

$$\mathcal{E}(N) = \frac{1}{2}\int_M (\sum_{i,j=1}^{n} a_{ij}^2)\upsilon + \frac{n}{2}V(M).$$

Corollary 3.1. If $k_1, ..., k_n$ are principal curvatures functions on M, then we have

$$\mathcal{E}(N) = \frac{1}{2}\int_M \sum_{j=1}^{n} k_j^2 \upsilon + \frac{n}{2}V(M).$$

Proof. Using equations (2) in (6), we obtain result.

The principal curvatures functions on M are independent from the choice of the basis $\{e_1, ..., e_n\}$, therefore the energy of a unit normal vector field invariant on the each orthonormal basis of the tangent space of M. We may ignore the constant term of $V(M)$ and we can give the following definition.

Definition 3.1. The integral

$$\frac{1}{2}\int_M \sum_{j=1}^{n} k_j^2 \upsilon$$

is called the energy of surface M or the integral

$$\frac{1}{2}\int_M \sum_{i=1}^{n} < A_N e_i, A_N e_i > \upsilon$$

is called the energy of surface M with respect to the basis $\{e_1, ..., e_n\}$ and is denoted by $\mathcal{E}(M)$.

Example 1. Energy of hyperplain in Euclidian space is zero.

Example 2. Let $M = S_r^n = \{(x_1, ..., x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = r^2, \}$ be the Hypershape in $(n+1)$- dimensional Euclidian space and N be unit normal vector field of M then

$$A_N X = -\frac{1}{r}X, \quad for \quad any \quad X \in \Gamma(TM).$$

Using equation (5) we obtain the energy of M is

$$\mathcal{E}(M) = \frac{1}{2}\int_M \sum_{i=1}^{n} < A_N e_i, A_N e_i > \upsilon = \frac{n}{2r^2}V(M).$$

Example 3. Let M be the totally umbilical hypersurface and N be unit normal vector field of M then $A_N = \lambda$ and

$$\mathcal{E}(M) = \frac{n}{2}\int_M \lambda^2 \upsilon.$$

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