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BIHARMONIC CURVES IN LIE GROUP WITH BI-INVARIANT METRIC

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Abstract. In this paper, we study biharmonic curves in Lie group. We give some characterizations for curvatures of a biharmonic curve in Lie group.

1. Introduction

On the other hand, a smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study biharmonic curves in Lie group. We give some characterizations for curvatures of a biharmonic curve in Lie group.

2. Preliminaries

Let $\mathbb{G}$ be a Lie group with a bi-invariant metric $\langle , \rangle$ and ∇ be the Levi-Civita connection of Lie group $\mathbb{G}$. If g denotes the Lie algebra of $\mathbb{G}$ then we know that g is isomorphic to $T_e\mathbb{G}$ where e is neutral element of $\mathbb{G}$. If $\langle , \rangle$ is a bi-invariant metric on $\mathbb{G}$ then we have

$$\langle \mathbf{X}, [\mathbf{Y}, \mathbf{Z}] \rangle = \langle [\mathbf{X}, \mathbf{Y}], \mathbf{Z} \rangle$$

and

$$D_{\mathbf{x}}\mathbf{Y} = \frac{1}{2}[\mathbf{X}, \mathbf{Y}]$$

for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}$ be an arc-lenghted cure and $\{\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, we write that any two vector fields $\mathbf{W}$ and $\mathbf{Z}$ along the

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curve α as $\mathbf{W}=\sum_{i=1}^{n} w_i\mathbf{X}_i$ and $\mathbf{Z}=\sum_{i=1}^{n} z_i\mathbf{X}_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions, [4]. Also the Lie bracket of two vector fields W and Z is given

$$[\mathbf{W},\mathbf{Z}]=\sum_{i=1}^{n} w_i z_i\left[\mathbf{X}_i,\mathbf{X}_j\right]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha'}\mathbf{W}$ is given as follows

$$D_{\alpha'}\mathbf{W}=\dot{\mathbf{W}}+\frac{1}{2}\left[\mathbf{T},\mathbf{W}\right]$$

where $T=\alpha'$ and $\dot{W}=\sum_{i=1}^{n}\dot{w}_i X_i$ or $\dot{W}=\sum_{i=1}^{n}\frac{dw}{dt}X_i$. Note that if W is the left-invariant vector field to the curve α then $\dot{W}=0$, [1,4,9,11,12].

Let G be a three dimensional Lie group and $(\mathbf{T},\mathbf{N},\mathbf{B},\kappa,\tau)$ denote the Frenet apparatus of the curve α, and calculate $\kappa=\left\|\dot{\mathbf{T}}\right\|$.

Definition 2.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}$ be a parametrized curve with the Frenet apparatus $(\mathbf{T},\mathbf{N},\mathbf{B},\kappa,\tau)$ then*

$$\tau_{\mathbb{G}}=\frac{1}{2}\left\langle\left[\mathbf{T},\mathbf{N}\right],\mathbf{B}\right\rangle$$

or

$$\tau_{\mathbb{G}}=\frac{1}{2\kappa^2\tau}\left\langle\ddot{\mathbf{T}},\left[\mathbf{T},\dot{\mathbf{T}}\right]\right\rangle+\frac{1}{4\kappa^2\tau}\left\|\left[\mathbf{T},\dot{\mathbf{T}}\right]\right\|^2$$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}$ be an arc-lenght parametrized unit speed curve in three dimensional Lie groups. The curve α is called a Frenet curve of osculating order 3 if its derivatives $\alpha'(s),\alpha''(s),\alpha'''(s),\alpha''''(s)$ are linearly dependent and $\alpha'(s),\alpha''(s),\alpha'''(s),\alpha''''(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order 3 one can associate an orthonormal 3−frame $\mathbf{T},\mathbf{N},\mathbf{B}$ along α such that($\alpha'(s)=T$) called the Frenet frame and functions $\kappa,\tau : I \to \mathbb{R}$ called the Frenet curvatures, such that the Frenet formulas in three dimensional Lie groups are defined

$$
\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N}\\
\nabla_{\mathbf{T}}\mathbf{N}&=-\kappa\mathbf{T}+(\tau-\tau_{\mathbb{G}})\mathbf{B}\\
\nabla_{\mathbf{T}}\mathbf{B}&=(\tau_{\mathbb{G}}-\tau)\mathbf{N}
\end{aligned}
\tag{2.1}
$$

where ∇ is the Levi-Civita connections of Lie group $\mathbb{G}$, [4,9].

Proposition 2.2. *Let $\mathbb{G}$ be a 3-dimensional Lie group with a bi-invariant metric. Then, it is one of the Lie groups $SO(3)$, $\mathbb{S}^3$ or a commutative group and the following statements hold (see [4], [8]):*

(i) *If $\mathbb{G}$ is $SO(3)$, then $\tau_G=\frac{1}{2}$.*
(ii) *If $\mathbb{G}$ is $\mathbb{S}^3\cong SU(2)$, then $\tau_G=1$.*
(iii) *If $\mathbb{G}$ is a commutative group, then $\tau_G=0$.*

3. Biharmonic curves in Lie group

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R\left(\mathbf{T}, D_{\mathbf{T}}\mathbf{T}\right)\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Theorem 3.1. *Let $\gamma : I \longrightarrow \mathbb{G}$ be a non-geodesic curve on $\mathbb{G}$ parametrized by arc length. Then γ is a non-geodesic biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + (\tau - \tau_G)^2 &= \tau_G^2, \\ (\tau - \tau_G)' &= -\tau_G^2. \end{aligned} \tag{3.2}$$

Proof. From (3.1), we obtain

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + (\tau - \tau_G)^2 &= R(\mathbf{T}, \mathbf{N}, \mathbf{T}, \mathbf{N}), \\ (\tau - \tau_G)' &= R(\mathbf{T}, \mathbf{N}, \mathbf{T}, \mathbf{B}). \end{aligned} \tag{3.4}$$

A direct computation using (2.3) yields

$$R(\mathbf{T}, \mathbf{N}, \mathbf{T}, \mathbf{N}) = \tau_G^2, \quad R(\mathbf{T}, \mathbf{N}, \mathbf{T}, \mathbf{B}) = -\tau_G^2.$$

These, together with (3.4), complete the proof of the theorem.

Corollary 3.2. *Let $\gamma : I \longrightarrow \mathbb{G}$ be a non-geodesic curve on $\mathbb{G}$ parametrized by arc length. Then γ is a non-geodesic biharmonic curve if and only if*

$$\tau_G = Ce^{-\frac{s}{2}}, \tag{3.5}$$

where C is constant of integration.

Proof. Using (3.2), we have (3.5).

Corollary 3.3. *Let $\gamma : I \longrightarrow \mathbb{G}$ be a non-geodesic curve on $\mathbb{G}$ parametrized by arc length. Then γ is a non-geodesic biharmonic curve if and only if*

$$[\mathbf{T}, \mathbf{N}] = 2Ce^{-\frac{s}{2}}\mathbf{B},$$

where C is constant of integration.

As a consequence of the theorem 3.1, we have

Corollary 3.4. *Let γ be a unit speed curve with the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in the Abellian Lie group $\mathbb{G}$. Then, γ is not a biharmonic curve.*

Corollary 3.5. *Let γ be a unit speed curve with the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in $\mathbb{S}^3$. Then, γ is a biharmonic curve if and only if*

$$\tau = -s + P + 1,$$

where P is constant of integration.

Corollary 3.6. *Let γ be a unit speed curve with the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in $SU(2)$. Then, γ is a biharmonic curve if and only if*

$$\tau = -s + P + 1,$$

where P is constant of integration.

Corollary 3.7. *Let γ be a unit speed curve with the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in $SO(3)$. Then, γ is a biharmonic curve if and only if*

$$\tau = -\frac{1}{4}s + \frac{1}{2} + Q,$$

where Q is constant of integration.

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