AMO - Advanced Modeling and Optimization, Volume 18, Number 1, 2016

A new modified approximate proximal-extragradient type method for monotone variational inequalities

Ali Ou-yassine, **Abdellah Bnouhachem**[1] and **Fatimazahra Benssi**

Laboratoire d'Ingénierie des Systémes et Technologies de l'Information, ENSA,
Ibn Zohr University, Agadir, BP 1136, Morocco.

Abstract. In this paper, we propose a new approximate proximal point algorithm (APPA). The proposed method uses a new searching direction which differs from the other existing APPAs. Under some mild conditions, we show that the proposed method is globally convergent. The results presented in this paper extend and improve some well-known results in the literature.

Key words. Variational inequality; inexact proximal point algorithm; projection method.

2000 AMS Subject Classification: 49J40, 65N30

[1]Corresponding author E-mail: babedallah@yahoo.com (Abdellah Bnouhachem).

***AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

1 Introduction

Let R^n be a finite dimensional Euclidean space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, Let C be a nonempty closed convex subset of R^n and F be a continuous monotone mapping from R^n into itself. We consider the problem of finding a vector $x^* \in C$ such that

$$(x - x^*)^T F(x^*) \geq 0, \qquad \forall x \in C, \tag{1.1}$$

which is called the classical variational inequality, denoted by VI(F, C). It is worth to mention that the solution set C^* of VI(F, C) is not empty.
In recent years, variational inequality theory has witnessed an explosive growth in theoretical advances, algorithmic developments and applications across all disciplines of pure and applied sciences. This theory provides us a unified, novel and innovative treatment of unilateral, free, moving, obstacle, and equilibrium problems arising in economics, finance, transportation, elasticity, optimization, operations research and structural analysis, see, for example, [1-17] and the references therein.

The proximal point algorithm (PPA) is recognized as a powerful numerical approach and effective algorithm for solving (1.1). It was first introduced by Martinet [10] and further refined and extended by Rockafellar [14] to a more general setting, including convex programs, convex-concave saddle point problems, and variational inequality problems. The classical iterate scheme of PPA for solving problem (1.1) is as follows. Let $\lambda_{min} > 0$ and $\lambda_k \subset [\lambda_{min}, +\infty)$. For given $x^k \in C$ and λ_k, let x_*^{k+1} be the solution of following strongly monotone variational inequality:

$$\text{(PPA)} \qquad \text{Find} \quad x \in C \quad \text{such that} \quad (x^{'} - x)^T F_k(x) \geq 0, \qquad \forall x^{'} \in C \tag{1.2}$$

where

$$F_k(x) = (x - x^k) + \lambda_k F(x). \tag{1.3}$$

The new iterate x^{k+1} of the exact version of PPA is taken by $x^{k+1} = x_*^{k+1}$.
Let $P_C(.)$ denote the projection onto C in the Euclidean norm, it is well known that the problem (1.2) is equivalent to solving the following equation:

$$x^{k+1} = P_C[x^k - \lambda_k F(x^{k+1})]. \tag{1.4}$$

Notting that the new iteration x^{k+1} can not be calculated directly by (1.4) because it is an implicit scheme. This difficulty makes the application of PPA impractical in many cases. Therefore, instead of solving it exactly, it becomes important to develop the PPA in order to find an approximate solutions of (1.4). In 1976, Rockafellar [13, 14] set up the fundamental convergence analysis for the approximate proximal point algorithm (APPA) to a general maximal monotone operator. The new iterate x^{k+1} of Rockafellar's APPA is requested to satisfy the following condition:

$$\|x^{k+1} - x_*^{k+1}\| \leq \nu_k, \qquad \sum_{k=0}^{\infty} < +\infty, \tag{1.5}$$

or

$$\|x^{k+1} - x_*^{k+1}\| \leq \nu_k \|x^k - x^{k+1}\|, \qquad \sum_{k=0}^{\infty} < +\infty. \tag{1.6}$$

Since x_*^{k+1} is unknown, which makes the application of this APPA very difficult. Recently, many searching directions are developed to find a new correction step. And to ensure the convergence different suitable extended less restrictive relaxed inexactness restriction are used in each method.

The main target of this paper is to develop an algorithm for solving variational inequalities. More precisely, a new iterate is obtained by searching the optimal step size along a new descent direction which generalize different existing descent directions. We adopt the same inexactness restriction used in [17]. Global convergence of the proposed method is proved under some mild conditions.

2 Iterative method and some properties

In this section, we suggest and analyze a new modified approximate proximal-extragradient type method for solving variational inequality (1.1). The following lemma provides some basic properties of the projection onto C.

Lemma 2.1 *Let $P_C(.)$ denote the projection of R^n onto C. Then, we have the following inequalities.*

$$(x - P_C(x))^T(y - P_C(x)) \le 0, \qquad \forall y \in C, \forall x \in R^n; \tag{2.1}$$

$$\|P_C(x) - P_C(y)\| \le \|x - y\|, \qquad \forall x, y \in R^n; \tag{2.2}$$

$$\|P_C(x) - y\|^2 \le \|x - y\|^2 - \|x - P_C(x)\|^2, \qquad \forall x \in R^n, y \in C. \tag{2.3}$$

Lemma 2.2 ([5], p. 267) *Let $\lambda > 0$, then x^* solves VI(C, F) if and only if*

$$x^* = P_C[x^* - \lambda F(x^*)].$$

Denote

$$e(x, \lambda) = x - P_C[x - \lambda F(x)]. \tag{2.4}$$

From Lemma 2.2, it is clear that x is solution of VI(F, C) if and only if x is a zero point of the function $e(x, \lambda)$.

We propose the following iterative scheme for solving (1.1).

Algorithm 2.1.

For given $x^0 \in C$ and $\lambda_{min} > 0$, the sequence $\{x^k\}$ is generated by the iterative schemes:
Step 1. Find an approximate solution of (1.2), i.e., find y^k in the sense that

$$y^k \approx P_C[x^k - \lambda_k F(y^k)], \tag{2.5}$$

under the following inexactness restriction:

$$\Delta(y^k) \le \nu(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2), \qquad \nu < 1, \tag{2.6}$$

where

$$\tilde{y}^k = P_C[x^k - \lambda_k F(y^k)], \tag{2.7}$$

and

$$\Delta(y^k) = 2(\zeta^k)^T F_k(y^k) - \|\zeta^k\|^2, \quad \text{with} \quad \zeta^k = y^k - \tilde{y}^k. \tag{2.8}$$

Step 2. Compute the new iterate

$$x^{k+1}(\alpha) = P_C[x^k - \alpha(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k))], \qquad \theta_1, \theta_2 \geq 0. \tag{2.9}$$

How to choose a suitable step length α to force convergence will be discussed later.

Remark 2.1 *As special cases, we can obtain some well-known results.*

(a) If $\theta_1 = 1$ and $\theta_2 = 0$, we obtain (BYY-correction) step of the method proposed in [9].

(b) If $\theta_1 = 0$ and $\theta_2 = 1$, we obtain (second step) of the three step APPA method proposed in [17].

(c) If $\theta_1 = 1$, $\theta_2 = 0$ and $\alpha = 1$, we obtain (SS-correction step) of the method proposed in [15] by using another inexactness restriction $\Delta(y^k) \leq \nu(\|x^k - y^k\|^2, \nu < 1$, which is different from (2.6).

Proposition 2.1 *[9] For y^k, $\tilde{y}^k$, $\Delta(y^k)$ and ζ^k defined in (2.8), we have*

$$\|x^k - \tilde{y}^k\|^2 - \lambda_k(\zeta^k)^T F(y^k) = \frac{1}{2}\{(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2) - \Delta(y^k)\}. \tag{2.10}$$

The following theorem concerns how to choose the step size α.

Theorem 2.1 *For given x^k, $\tilde{y}^k$ and ζ^k are defined by (2.7) and (2.8) respectively, and $\lambda_k > 0$, let $y^k \in C$ be an approximate solution of (1.2) in the sense of (2.5) and $x^{k+1}(\alpha)$ the new iterate be given by (2.9). Then for any $\alpha > 0$, we have*

$$\Theta(\alpha) := \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \geq \Psi(\alpha) \tag{2.11}$$

where

$$\Psi(\alpha) = 2(\theta_1 + \theta_2)\alpha\{\|x^k - \tilde{y}^k\|^2 - \lambda_k(\zeta^k)^T F(y^k)\} - (\theta_1 + \theta_2)^2\alpha^2\|x^k - \tilde{y}^k\|^2. \tag{2.12}$$

Proof. Since $x^{k+1}(\alpha) = P_C[x^k - \alpha(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k))]$ and $x^* \in C$, it follows from (2.3) that

$$\begin{aligned} \|x^{k+1}(\alpha) - x^*\|^2 &\leq \|x^k - \alpha(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k)) - x^*\|^2 \\ &\quad -\|x^k - \alpha(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k)) - x^{k+1}(\alpha)\|^2 \end{aligned} \tag{2.13}$$

Consequently, using the definition of $\Theta(\alpha)$, we get

$$\begin{aligned} \Theta(\alpha) &\geq \|x^k - x^*\|^2 + \|x^k - x^{k+1}(\alpha) - \alpha(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k))\|^2 \\ &\quad -\|x^k - x^* - \alpha(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k))\|^2 \\ &= \|x^k - x^{k+1}(\alpha)\|^2 + 2\alpha(x^{k+1}(\alpha) - x^*)^T(\theta_1(x^k - \tilde{y}^k) + \theta_2\lambda_k F(y^k)) \\ &= \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - x^*)^T(x^k - \tilde{y}^k) + 2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - x^*)^T F(y^k). \\ &= \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - \tilde{y}^k + \tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) \\ &\quad +2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - x^*)^T F(y^k) \\ &= \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - \tilde{y}^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha(\tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) \\ &\quad +2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - x^*)^T F(y^k) \\ &= \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - x^k + x^k - \tilde{y}^k)^T(x^k - \tilde{y}^k) \\ &\quad +2\theta_1\alpha(\tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) + 2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - x^*)^T F(y^k) \\ &= \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - x^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha\|x^k - \tilde{y}^k\|^2 \\ &\quad +2\theta_1\alpha(\tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) + 2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - x^*)^T F(y^k). \end{aligned} \tag{2.14}$$

Since $x^* \in C^*$, using the monotonicity of F, we have

$$(y^k - x^*)^T F(y^k) \geq (y^k - x^*)^T F(x^*) \geq 0 \tag{2.15}$$

and consequently

$$(x^{k+1}(\alpha) - x^*)^T F(y^k) \geq (x^{k+1}(\alpha) - y^k)^T F(y^k). \tag{2.16}$$

Since $\tilde{y}^k = P_C[x^k - \lambda_k F(y^k)]$ and $x^* \in C$, it follows from (2.1) that

$$\{x^k - \lambda_k F(y^k) - \tilde{y}^k\}^T(\tilde{y}^k - x^*) \geq 0,$$

and thus

$$(\tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) \geq (\tilde{y}^k - x^*)^T \lambda_k F(y^k). \tag{2.17}$$

Using (2.15), we have

$$(\tilde{y}^k - x^*)^T \lambda_k F(y^k) \geq (\tilde{y}^k - y^k)^T \lambda_k F(y^k). \tag{2.18}$$

From (2.17) and (2.18), we get

$$2\theta_1\alpha(\tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) \geq 2\theta_1\alpha\lambda_k(\tilde{y}^k - y^k)^T F(y^k). \tag{2.19}$$

On the other hand, since $\tilde{y}^k = P_C[x^k - \lambda_k F(y^k)]$ and $x^{k+1}(\alpha) \in C$, it follows from (2.1) that for any $\alpha > 0$, we have

$$2\theta_2\alpha\{x^{k+1}(\alpha) - \tilde{y}^k\}^T\{x^k - \lambda_k F(y^k) - \tilde{y}^k\} \leq 0. \tag{2.20}$$

Applying (2.19), (2.16) to the right-hand-side of (2.14), we get

$$\begin{aligned}
\Theta(\alpha) \geq\ & \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - x^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha\|x^k - \tilde{y}^k\|^2 \\
& + 2\theta_1\alpha(\tilde{y}^k - x^*)^T(x^k - \tilde{y}^k) + 2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - x^*)^T F(y^k) \\
\geq\ & \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - x^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha\|x^k - \tilde{y}^k\|^2 \\
& + 2\theta_1\alpha\lambda_k(\tilde{y}^k - y^k)^T F(y^k) + 2\theta_2\alpha\lambda_k(x^{k+1}(\alpha) - y^k)^T F(y^k).
\end{aligned} \tag{2.21}$$

Adding (2.20) to the right-hand-side of (2.21), we obtain

$$\begin{aligned}
\Theta(\alpha) \geq\ & \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_1\alpha(x^{k+1}(\alpha) - x^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha\|x^k - \tilde{y}^k\|^2 \\
& + 2\theta_1\alpha\lambda_k(\tilde{y}^k - y^k)^T F(y^k) + 2\theta_2\alpha(x^{k+1}(\alpha) - \tilde{y}^k)^T(x^k - \tilde{y}^k) - 2\theta_2\alpha\lambda_k(y^k - \tilde{y}^k)^T F(y^k) \\
=\ & \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_2\alpha(x^{k+1}(\alpha) - \tilde{y}^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha(x^{k+1}(\alpha) - x^k)^T(x^k - \tilde{y}^k) \\
& + 2\theta_1\alpha\|x^k - \tilde{y}^k\|^2 - 2(\theta_1 + \theta_2)\alpha\lambda_k(y^k - \tilde{y}^k)^T F(y^k)
\end{aligned}$$

$$\begin{aligned}
&= \|x^k - x^{k+1}(\alpha)\|^2 + 2\theta_2\alpha\{(x^{k+1}(\alpha) - x^k) + (x^k - \tilde{y}^k)\}^T(x^k - \tilde{y}^k) \\
&\quad +2\theta_1\alpha(x^{k+1}(\alpha) - x^k)^T(x^k - \tilde{y}^k) + 2\theta_1\alpha\|x^k - \tilde{y}^k\|^2 - 2(\theta_1+\theta_2)\alpha\lambda_k(y^k - \tilde{y}^k)^T F(y^k) \\
&= \|x^k - x^{k+1}(\alpha)\|^2 + 2(\theta_1+\theta_2)\alpha\{x^{k+1}(\alpha) - x^k\}^T(x^k - \tilde{y}^k) + 2(\theta_1+\theta_2)\alpha\|x^k - \tilde{y}^k\|^2 \\
&\quad -2(\theta_1+\theta_2)\alpha\lambda_k(y^k - \tilde{y}^k)^T F(y^k) \\
&= \|(x^k - x^{k+1}(\alpha)) - (\theta_1+\theta_2)\alpha(x^k - \tilde{y}^k)\|^2 - (\theta_1+\theta_2)^2\alpha^2\|x^k - \tilde{y}^k\|^2 \\
&\quad +2(\theta_1+\theta_2)\alpha\|x^k - \tilde{y}^k\|^2 - 2(\theta_1+\theta_2)\alpha\lambda_k(y^k - \tilde{y}^k)^T F(y^k) \\
&\geq 2(\theta_1+\theta_2)\alpha\{\|x^k - \tilde{y}^k\|^2 - \lambda_k(y^k - \tilde{y}^k)^T F(y^k)\} - (\theta_1+\theta_2)^2\alpha^2\|x^k - \tilde{y}^k\|^2.
\end{aligned}$$

The proof is completed. □

3 Convergence analysis

Now, we mainly focus on investigating the convergence of the proposed method. The following theorem plays a crucial role in the convergence of the proposed method.

Theorem 3.1 *For given $x^k \in C$ and $\lambda_k > 0$, let $y^k \in C$ be an approximate solution of (1.2) in the sense of (2.5) and the new iterate x^{k+1} be given by (2.9). Then, we have*

$$\|x^{k+1}(\alpha) - x^*\|^2 \leq \|x^k - x^*\|^2 - \tau\left(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2\right) \tag{3.1}$$

where

$$\tau = (\theta_1+\theta_2)\frac{(1-\nu)^2}{4}.$$

Proof. Substituting (2.10) into (2.12), we have

$$\Psi(\alpha) = (\theta_1+\theta_2)\alpha\{(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2) - \Delta(y^k)\} - (\theta_1+\theta_2)^2\alpha^2\|x^k - \tilde{y}^k\|^2.$$

Since $\Psi(\alpha)$ is a quadratic function of α, it reaches its maximum at

$$\alpha_k^* = \frac{(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2) - \Delta(y^k)}{2(\theta_1+\theta_2)\|x^k - \tilde{y}^k\|^2}, \tag{3.2}$$

with

$$\Psi(\alpha_k^*) = \frac{(\theta_1+\theta_2)}{2}\alpha_k^*\{(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2) - \Delta(y^k)\}. \tag{3.3}$$

Under inexactness restriction (3.2), it follows from (3.2) and (3.3) that

$$\alpha_k^* \geq \frac{1-\nu}{2(\theta_1+\theta_2)} \quad \text{and} \quad \Psi(\alpha_k^*) \geq \frac{(1-\nu)^2}{4(\theta_1+\theta_2)}\{(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2)\}. \tag{3.4}$$

It follows from (2.11) and (3.4) that

$$\|x^{k+1}(\alpha) - x^*\|^2 \quad \leq \quad \|x^k - x^*\|^2 - (\theta_1+\theta_2)\frac{(1-\nu)^2}{4}\left(\|x^k - y^k\|^2 + \|x^k - \tilde{y}^k\|^2\right).$$

And we get the assertion of this theorem. □

The convergence of the proposed method can be proved by using similar arguments to [9]. Hence the proof is omitted.

Theorem 3.2 *[9] The sequence $\{x^k\}$ generated by the proposed method converges to some some x^∞ which is a solution of (1.1).*

4 Conclusions

In this paper, we proposed a new modified approximate proximal point algorithm for solving variational inequalities. The proposed method generates the new iterate by searching the optimal step size along a new descent direction which can be viewed as a refinement and improvement some well-known results in the literature. Global convergence of the proposed method is proved under mild assumptions.

References

[1] A. Bnouhachem, M.H. Xu, X.L. Fu and S. Zhaohan, Modified extragradient method for solving variational inequalities, Comp. Math. Appl. 57(2) (2009) 230-239.

[2] A. Bnouhachem, X.L. Fu, M.H. Xu and S. Zhaohan, New extragradient-type methods for solving variational inequalities, Appl. Math. Comput. 216(8) (2010) 2430-2440.

[3] A. Bnouhachem and M.A. Noor, An interior proximal point algorithm for nonlinear complementarity problems, Nonlinear Analysis-Hybrid Systems 4(3) (2010) 371-380.

[4] A. Bnouhachem, M.A. Noor and S. Zhaohan, An approximate proximal point algorithm for nonlinear complementarity problems, Hacettepe J. Math. Statistics 41(1) (2012) 103117.

[5] D.P. Bertsekas, J.N. Tsitsiklis, Parallel and Distributed Computation, Numerical Methods, Prentice Hall, Englewood Cliffs, NJ, 1989.

[6] J. Eckstein, Nonlinear proximal point algorithms using Bregman functions, with applications to convex programming, Math. Oper. Res. 18 (1993) 202-226.

[7] J. Eckstein, Approximate iterations in Bregman-function-based proximal algorithms, Math. Prog. 83 (1998) 113-123.

[8] B.S. He, L.Z. Liao, Improvements of some projection methods for monotone nonlinear variational inequalities, J. Optim. Theory Appl. 112 (2002) 111-128.

[9] B.S. He, Z.H. Yang, X.M. Yuan, An approximate proximal-extragradient type method for monotone variational inequalities, J. Math. Anal. Appl. 300 (2) (2004) 362-374.

[10] B. Martinet, Determination approche dun point fixe dune application pseudo-contractante, C. R. Acad. Sci. Paris 274 (1972) 163-165.

[11] M.A. Noor, Extragradient method for pseudomonotone variational inequalities, J. Optim. Theory Appl. 117 (2003) 475-488.

[12] M.A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199- 277.

[13] R.T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, Math. Oper. Res. 1 (1976) 97-116.

[14] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877-898.

[15] M.V. Solodov, B.F. Svaiter, Error bounds for proximal point subproblems and associated inexact proximal point algorithms, Math. Prog. Ser. B. 88 (2000) 371-389.

[16] N. Xiu and J.Z. Zhang, On finite convergence of proximal point algorithms for variational inequalities, J. Math. Anal. Appl. 312 (2005) 148-158.

[17] M. Li, X.M. Yuan, An APPA-based descent method with optimal step-sizes for monotone variational inequalities, European J. Oper. Res. 186 (2008) 486-495.