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A linearized Peaceman-Rachford splitting method with applications to constrained image deblurring problems

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Abstract. In this paper, we apply the Peaceman-Rachford splitting method (PRSM) to solve the problem of constrained image deblurring corrupted by Gaussian noise. To speed up PRSM, we linearize its two subproblems and obtain the closed-form solutions. Compared with PRSM, the resulting new method is matrix-inversion free. The global convergence of the new algorithm is proved via the analytic framework of contractive type methods. Numerical comparisons with alternating direction methods (ADMs) illustrate that our proposed algorithm is efficient and promising.

Keywords. Linearized Peaceman-Rachford splitting method; Image deblurring; Global convergence.

AMS(2000) Subject Classification: 90C25, 90C30

1 Introduction

In this paper, we consider the problem of deblurring digital images under Gaussian noise. Given original image concatenated into an n-vector $\bar{x} \in \mathcal{R}^n$, and let $A \in \mathcal{R}^{n\times n}$ be a blurring operator (integral operator) acting on the image. Let $\omega \in \mathcal{R}^n$ be the Gaussian noise added onto the image. The observed image $c \in \mathcal{R}^n$ can be modeled by $c = A\bar{x} + \omega$. The constrained image deblurring problem (CIDP$_\lambda$) is to recover $\bar{x}$ from c, which can be depicted as

$$\min_{l\le x\le u} \frac{1}{2}\|Ax-c\|^2 + \frac{\lambda^2}{2}\|Bx\|^2, \tag{1}$$

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where $l, u \in \mathcal{R}^n_+$; $B \in \mathcal{R}^{n\times n}$ is a regularization operator (differential operator); $\lambda^2 \in \mathcal{R}$ is the regularization parameter; $x \in \mathcal{R}^n$ is the restored image; $\|\cdot\|$ denotes the 2-norm. The box constraints $0 \le x \le u$ represent the dynamic range of the image (e.g., $l_i = 0$ and $u_i = 255$ for an 8-bit gray-scale image). Throughout this paper, we assume that the solution set of (1) is nonempty.

By introducing an auxiliary variable $y \in \mathcal{R}^n$, we can change CIDP$_\lambda$ to the equivalent form

$$\begin{aligned} \min\ & \frac{1}{2}\|Ax-c\|^2 + \frac{\lambda^2}{2}\|By\|^2 \\ \text{s.t. } & -x+y=0, \\ & x \in \mathcal{R}^n, y \in \Omega, \end{aligned} \tag{2}$$

where $\Omega = \{y | l \le y \le u\}$.

Obviously, (2) is a separable convex minimization model whose variables are subject to some linear constraints and two addition simple constraints. Thus, all the numerical methods which can solve the separable convex programming is applicable to (2), such as the alternating direction methods (ADMs)[1,2], the Peaceman-Rachford splitting methods (PRSMs)[3,4], and the split Bregman methods [5,6], etc. In this paper, we focus our attention on the Peaceman-Rachford splitting method. Let the augmented Lagrangian function of (2) be

$$\mathcal{L}_{\mathcal{A}}(x, y; \xi) = \frac{1}{2}\|Ax-c\|^2 + \frac{\lambda^2}{2}\|By\|^2 - \langle \xi, y-x\rangle + \frac{\beta}{2}\|y-x\|^2,$$

where $\xi \in \mathcal{R}^n$ is the Lagrangian multiplier and $\beta > 0$ is a penalty parameter. Then the iterative scheme of PRSM [3,4] for (2) reads as

$$\begin{cases} x^{k+1} = \text{argmin}_{x\in\mathcal{R}^n}\mathcal{L}_{\mathcal{A}}(x, y^k; \xi^k), \\ \xi^{k+\frac{1}{2}} = \xi^k - \alpha\beta(y^k - x^{k+1}), \\ y^{k+1} = \text{argmin}_{y\in\Omega}\mathcal{L}_{\mathcal{A}}(x^{k+1}, y; \xi^{k+\frac{1}{2}}), \\ \xi^{k+1} = \xi^{k+\frac{1}{2}} - \alpha\beta(y^{k+1} - x^{k+1}), \end{cases} \tag{3}$$

where the parameter $\alpha \in (0, 1)$ is an underdetermined relaxation factor introduced by He et al.[4] to ensure the global convergence of PRSM. Applying the Peaceman-Rachford splitting method (PRSM) to (2), we can get the following iterative scheme

$$\begin{cases} x^{k+1} = (A^\top A + \beta I)^{-1}(A^\top c + \beta y^k - \xi^k), \\ \xi^{k+\frac{1}{2}} = \xi^k - \alpha\beta(y^k - x^{k+1}), \\ y^{k+1} = \text{argmin}_{y\in\Omega}\left\{\frac{\lambda^2}{2}\|By\|^2 + \frac{\beta}{2}\|y - x^{k+1} - \frac{\xi^{k+\frac{1}{2}}}{\beta}\|^2\right\}, \\ \xi^{k+1} = \xi^{k+\frac{1}{2}} - \alpha\beta(y^{k+1} - x^{k+1}). \end{cases} \tag{4}$$

Obviously, the second subproblem in (4) does not have closed-form solution when $B \ne I$. Furthermore, although the first subproblem of (4) admits the closed-form solution, it needs to compute the matrix inversion $(A^\top A+\beta I)^{-1}$. As discussed in [7], the above matrix inversion can be solved exactly by two FFTs (including one inverse FFT) when A is a spatially-invariant blur under circulant or reflective boundary condition assumption. However, for some applications such as magnetic resonance imaging, the matrix may be extremely difficult to invert.

To solve the above issues, we linearize the two subproblems of the classical PRSM (3), and propose two linearized PRSMs for (2). All the subproblems of the proposed method have closed-form solutions and are free from any matrix inversion.

The rest of this paper is organized as follows. In Section 2, we present a linearized Peaceman-Rachford splitting method for solving (2). In Section 3, we prove the global convergence of the derived method. In Section 4, numerical comparisons with ADMs are carried out to confirm the effectiveness of our method. Finally, some concluding remarks are given in Section 5.

2 Linearized PRSM

In this section, we present a linearized PRSM for solving (2). Firstly, it is easy to see that, for fixed y^k, ξ^k, the minimization of $\mathcal{L}_{\mathcal{A}}(x,y;\xi)$ with respect to x can be formulated by

$$\begin{aligned} & \text{argmin}_{x\in\mathcal{R}^n}\mathcal{L}_{\mathcal{A}}(x,y^k;\xi^k) \\ = \ & \text{argmin}_{x\in\mathcal{R}^n}\left\{\frac{1}{2}\|Ax-c\|^2+\frac{\beta}{2}\|x-y^k+\frac{\xi^k}{\beta}\|^2\right\}. \end{aligned}$$

Now we linearize the first quadratic term $\frac{1}{2}\|Ax-c\|^2$ at the current point x^k and add a proximal term, i.e.,

$$\frac{1}{2}\|Ax-c\|^2 \approx \frac{1}{2}\|Ax^k-c\|^2+(Ax^k-c)^\top A(x-x^k)+\frac{\tau}{2}\|x-x^k\|^2,$$

where τ is the parameter of the proximal term and $\tau>\rho(A^\top A)$. Here $\rho(A^\top A)$ denotes the largest eigenvalues of $A^\top A$. Then, the first subproblem of PRSM is transformed into

$$\text{argmin}_{x\in\mathcal{R}^n}\left\{(Ax^k-c)^\top Ax+\frac{\tau}{2}\|x-x^k\|^2+\frac{\beta}{2}\|x-y^k+\frac{\xi^k}{\beta}\|^2\right\}. \tag{5}$$

Thus taking derivative on the above problem with respect to x, forcing the result to zero and letting the stationary point be x^{k+1}, we have

$$x^{k+1}=\frac{1}{\beta+\tau}[\tau x^k+\beta y^k-\xi^k-A^\top(Ax^k-c)]. \tag{6}$$

Secondly, for fixed $x^{k+1}, \xi^{k+\frac{1}{2}}$, the minimization of $\mathcal{L}_{\mathcal{A}}(x,y;\xi)$ with respect to y can be formulated by

$$\begin{aligned} & \text{argmin}_{y\in\Omega}\mathcal{L}_{\mathcal{A}}(x^{k+1},y;\xi^{k+\frac{1}{2}}) \\ = \ & \text{argmin}_{y\in\Omega}\left\{\frac{\lambda^2}{2}\|By\|^2+\frac{\beta}{2}\|y-x^{k+1}-\frac{\xi^{k+\frac{1}{2}}}{\beta}\|^2\right\}. \end{aligned}$$

Similarly, we linearize the first quadratic term of the above problem at the current point y^k, and get

$$y^{k+1} = \text{argmin}_{y\in\Omega} \left\{ \lambda^2[(By^k)^\top B(y-y^k) + \frac{\upsilon}{2}\|y-y^k\|^2] + \frac{\beta}{2}\|y-x^{k+1}-\frac{\xi^{k+\frac{1}{2}}}{\beta}\|^2 \right\},$$

where υ is also a proximal parameter. The optimality condition of the above problem leads to the following variational inequality

$$(y'-y^{k+1})^\top\{\lambda^2[B^\top By^k + \upsilon(y-y^k)] + \beta(y-x^{k+1}-\frac{\xi^{k+\frac{1}{2}}}{\beta})\} \geq 0, \quad \forall y' \in \Omega.$$

Then, y^{k+1} can be given explicitly by

$$y^{k+1} = P_\Omega \left\{ \frac{1}{\lambda^2\upsilon+\beta}[-\lambda^2 B^\top By^k + \lambda^2\upsilon y^k + \xi^{k+\frac{1}{2}} + \beta x^{k+1}] \right\}, \tag{7}$$

where $P_\Omega(\cdot)$ denotes the projection operator onto Ω under the Euclidean norm. Based on (6) and (7), we get a linearized PRSM for (2) with p=2, whose full steps can be described as follows.

Algorithm 1 A linearized PRSM for (2)

Input $A, B, c, \lambda, \alpha, \beta$ and $\tau > \rho(A^\top A), \upsilon > \rho(B^\top B)$. Initialize $(x, y; \xi) = (x^0, y^0; \xi^0), k = 0$.

while "not converged", **do**

(1) Compute x^{k+1} according to (6).

(2) Compute $\xi^{k+\frac{1}{2}} = \xi^k - \alpha\beta(y^k - x^{k+1})$.

(3) Compute y^{k+1} according to (7).

(4) Compute $\xi^{k+1} = \xi^{k+\frac{1}{2}} - \alpha\beta(y^{k+1} - x^{k+1})$.

(5) $k = k + 1$.

end while

Output x^{k+1}.

Remark 2.1 Setting $R = \tau I_n - A^\top A$, we have

$$\begin{aligned} & \text{argmin}_{x\in\mathcal{R}^n}\{\mathcal{L}_\mathcal{A}(x, y^k; \xi^k) + \frac{1}{2}\|x-x^k\|_R^2\} \\ = & \text{argmin}_{x\in\mathcal{R}^n} \left\{ \frac{1}{2}\|Ax-c\|_2^2 + \frac{\beta}{2}\|x-y^k+\frac{\xi^k}{\beta}\|_2^2 + \frac{1}{2}\|x-x^k\|_R^2 \right\} \\ = & \text{argmin}_{x\in\mathcal{R}^n} \left\{ (Ax^k-c)^\top Ax + \frac{\tau}{2}\|x-x^k\|_2^2 + \frac{\beta}{2}\|x-y^k+\frac{\xi^k}{\beta}\|_2^2 \right\}, \end{aligned}$$

and the last expression is just (5). Therefore, linearizing the x's quadratic terms in $\mathcal{L}_\mathcal{A}(x, y^k; \xi^k)$ is equivalent to adding a proximal term $\frac{1}{2}\|x-x^k\|_R^2$ on it. Similarly, we have that linearizing the y's quadratic terms in $\mathcal{L}_\mathcal{A}(x^{k+1}, y^k; \xi^{k+\frac{1}{2}})$ is equivalent to adding a proximal term $\frac{1}{2}\|y-y^k\|_S^2$ with $S = \upsilon I - B^\top B$ on it.

3 Global convergence

In this section, we establish the global convergence of our proposed linearized PRSM. Firstly, we give a general model which includes (2) as a special case.

$$\min\{\theta_1(x_1)+\theta_2(x_2)|A_1x_1+A_2x_2=d, x_1\in\mathcal{X}_1, x_2\in\mathcal{X}_2\}, \tag{8}$$

where $A_i\in\mathcal{R}^{l\times n_i}(i=1,2)$, $d\in\mathcal{R}^l$ and $\mathcal{X}_i\subset\mathcal{R}^{n_i}(i=1,2)$ are nonempty closed convex sets, $\theta_i:\mathcal{R}^{n_i}\to\mathcal{R}(i=1,2)$ are convex but not necessarily smooth functions.

In fact, (2) is a special case of (8) by setting $x_1=x, x_2=y$, $\theta_1(x_1)=\frac{1}{2}\|Ax-c\|_2^2$, $\theta_2(x_2)=\frac{\lambda^2}{2}\|By\|_2^2$, $A_1=-I_n, A_2=I_n, d=0$, and $\mathcal{X}_1=\mathcal{R}^n, \mathcal{X}_2=\Omega$.

Now, we present a proximal PRSM to solve the general model (8). Let the augmented Lagrangian function of (8) be

$$\mathcal{L}_{\mathcal{A}}(x_1,x_2;\xi)=\theta_1(x_1)+\theta_2(x_2)-\langle\xi, A_1x_1+A_2x_2-d\rangle+\frac{\beta}{2}\|A_1x_1+A_2x_2-d\|^2,$$

where $\xi\in\mathcal{R}^l$ is the Lagrangian multiplier and $\beta>0$ is a penalty parameter.

Algorithm 2 A proximal PRSM for (8)

Input the parameters α,β and two positive semidefinite matrices $R,S\in\mathcal{R}^{l\times l}$. Initialize $(x_1,x_2;\xi)=(x_1^0,x_2^0;\xi^0), k=0$.

while "not converged", **do**

(1) Compute x_1^{k+1} by $\mathrm{argmin}_{x_1\in\mathcal{X}_1}\{\mathcal{L}_{\mathcal{A}}(x_1,x_2^k;\xi^k)+\frac{1}{2}\|A_1(x_1-x_1^k)\|_R^2\}$.

(2) Compute $\xi^{k+\frac{1}{2}}=\xi^k-\alpha\beta(A_1x_1^{k+1}+A_2x_2^k-d)$.

(3) Compute x_2^{k+1} by $\mathrm{argmin}_{x_2\in\mathcal{X}_2}\{\mathcal{L}_{\mathcal{A}}(x_1^{k+1},x_2;\xi^{k+\frac{1}{2}})+\frac{1}{2}\|A_2(x_2-x_2^k)\|_S^2\}$.

(4) Compute $\xi^{k+1}=\xi^{k+\frac{1}{2}}-\alpha\beta(A_1x_1^{k+1}+A_2x_2^{k+1}-d)$.

(5) $k=k+1$.

end while

Output x_1^{k+1}, x_2^{k+1}.

From Remark 2.1, to prove the global convergence of Algorithm 1, we only need to prove the global convergence of Algorithm 2. Firstly, we define some auxiliary variables: $u=(x_1,x_2)$, $w=(u,\xi)$ and $\theta(u)=\theta_1(x_1)+\theta_2(x_2)$. Then, by invoking the first-order optimality condition for convex programming, we can reformulate problem (8) as the following mixed variational inequality problem (denoted by MVI$(\mathcal{W},F,\theta)$): Finding a vector $w^*\in\mathcal{W}$ such that

$$\theta(u)-\theta(u^*)+(w-w^*)^\top F(w^*)\geq 0,\quad \forall w\in\mathcal{W}, \tag{9}$$

where $\mathcal{W} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{R}^l$, and

$$F(w) = \begin{pmatrix} -A_1^\top \xi \\ -A_2^\top \xi \\ A_1 x_1 + A_2 x_2 - d \end{pmatrix}. \tag{10}$$

Now, we give the global convergence of Algorithm 2.

Theorem 3.1. For any fixed $\alpha \in (0,1)$ and $\beta > 0$, the sequence $\{(x_1^{k+1}, x_2^{k+1}; \xi^{k+1})\}$ generated by Algorithm 3 from any starting point $(x_1^0, x_2^0; \xi^0)$ converges to a solution of MVI($\mathcal{W}, F, \theta$)), and the corresponding sequence $\{(x_1^{k+1}, x_2^{k+1})\}$ converges to a solution of (13).

Proof. It proof is similar to that of Theorem 3.1 in [8]. This completes the proof.

Theorem 3.2. For any fixed $\alpha \in (0,1)$ and $\beta > 0$, suppose $\{(x^k, y^k; \xi^k)\}$ is the the sequence generated by Algorithm 1 from any starting point $(x^0, y^0; \xi^0)$. Then the corresponding sequence $\{(x^k, y^k)\}$ converges to a solution of (2).

Proof. From $\tau > \rho(A^\top A)$ and $\upsilon > \rho(B^\top B)$, we have that the two matrices $R = \tau I_n - A^\top A$ and $S = \upsilon I_n - B^\top B$ are both positive semidefinite. Then, by Remark 2.1 and Theorem 3.1, we get the conclusion of this theorem. The proof is complete.

Proof. From $\tau > \rho(\lambda^2 B^\top B + \gamma A^\top A)$, we have that the matrix $T = \tau I_n - (\lambda^2 B^\top B + \gamma A^\top A)$ is positive semidefinite. Then, by Remark 2.2 and Theorem 3.1, we get the conclusion of this theorem. The proof is complete.

Remark 3.1. When the matrix inversion $(A^\top A + \beta I)^{-1}$ is easy to compute, then we can replace the iterate x^{k+1} in Algorithm 1 by $x^{k+1} = (A^\top A + \beta I)^{-1}(A^\top c + \beta y^k - \xi^k)$, which is just the x iterate equation in (4). Obviously, the resulting method is also globally convergent, because it is equivalent to setting $R = 0$ in Algorithm 3.

4 Numerical experiments

In this section, we test our proposed linearized PRSM on the image deblurring problem (1), and compare the proposed Algorithm 1 with the famous ADM in [3,4]. We implement all the algorithms with codes written in Matlab 7.10. The testing is performed on a ThinkPad notebook with Pentium(R) Dual-Core CPU T4400@2.2GHz, 2GB of memory. To assess the restoration performance qualitatively, we use the peak signal to noise ratio (PSNR) defined as

$$\text{PSNR} = 20\text{log}_{10}\frac{x_{\max}}{\sqrt{\text{Var}(x,\bar{x})}} \quad \text{with} \quad \text{Var}(x,\bar{x}) = \frac{\sum_{j=1}^{n^2}[\bar{x}(j) - x(j)]^2}{n^2}.$$

Here $\bar{x}$ is the true image, and $\bar{x}_{\max}$ is the maximum possible pixel value of the image. Furthermore, the stopping criterion of all the tested methods is

$$\frac{|\mathcal{J}^{k+1} - \mathcal{J}^{k}|}{|\mathcal{J}^{k}|} \leq 10^{-5},$$

where $\mathcal{J}^k$ is the objective function value of (1) at the kth iteration.

The test images are 256-by-256 ($l_i = 0, u_i = 255$ for all $i = 1, 2, \ldots, n$) images as shown in Figure 1: Text, Heart, Cameraman and Lena. According, $n = 65,536$ in model (1) for these images. The blurring matrix A is chosen to be the out-of-focus blur and the matrix B is taken to be the gradient matrix. The observed image c is expressed as $c = A\bar{x} + \omega$, where ω is the Gaussian or impulse noise. Here, we employ the Matlab scripts: `A=fspecial('average',alpha)` and `c=imfilter(x,A,'circular','conv')` $+\omega$, in which `alpha` is the size of the kernel. In the experiment, we apply Algorithm 1 and ADM to solve model

Figure 1: The original test images: Text, Heart, Cameraman and Lena

(1) with Gaussian noise and $\lambda = 0.16$. Here, we set $\omega = \eta *$ `randn(n,n)`, and η is the level of noise. For Algorithm 1, we set $\alpha = 0.9, \beta = 0.1$, $\tau = 1.01 \cdot \rho(A^\top A), \upsilon = 1.01 \cdot \rho(B^\top B)$. For ADM, we also set $\beta = 0.1$. All iterations start with the blurred images. For each test case, we repeat the experiment three times and report the average performance in Table 1. We report the CPU time (in seconds), the number of iterations (Niter) required for the whole deblurring process.

Table 1: Comparison of Algorithm 1 with ADM

Image	`alpha`	η	Algorithm 1			ADM		
			Time	Iter	PSNR	Time	Iter	PSNR
Text	3	3	1.05	13	25.02	1.19	16	25.02
	3	5	0.92	13	24.91	1.36	17	24.90
Heart	3	3	0.94	16	33.46	1.20	20	33.46
	3	5	1.09	16	31.73	1.20	22	31.72
Cameraman	5	3	1.44	19	25.47	1.48	21	25.47
	5	5	1.25	19	25.05	1.62	22	25.05
Lena	5	3	1.51	19	27.80	1.53	21	27.80
	5	5	1.26	20	27.08	1.37	22	27.08

The numerical results in Table 1 indicate that both methods reach almost the same restored PSNR, and the restored PSNR by Algorithm 1 is always the same or a litter higher than that by ADM. In addition, Algorithm 1 is always faster than ADM, and the number of iterations of Algorithm 1 is always smaller than that of ADM. Thus, Algorithm 1 is more efficient and robustness than the famous ADM.

5 Conclusions

In this paper, we have proposed a linearized PRSM, which is free from any matrix inversion. Under standard assumption, it is global convergent. The numerical results reported indicate that the proposed method is quite efficient and promising.

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