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NUMERICAL IMPLEMENTATION OF THE ADM FOR THE NONLINEAR SECOND KIND WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper we propose new idea for the implementation of the Adomian decomposition method to solve nonlinear weakly singular Volterra integral equations. This method represents the solution of proposed integral equations as a series generated by the Adomian decomposition method and coefficients are evaluated by the product integration technique. Some examples are prepared to show the efficiency and simplicity of the method.
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1. Introduction

Integral equations of Volterra type with weakly singular kernels arise in many modelling problems in mathematical physics and chemical reactions, such as stereology [13], heat conduction, crystal growth, electrochemistery, superfluidity [14], the radiation of heat from a semi infinite solid [15] and many other practical applications.We remark here that equations of this type have been the focus of many papers [5, 7, 8, 11] in recent years.
In this paper we consider nonlinear Volterra integral equations of the second kind,

$$u(t) = f(t) + \int_0^t p(t,s)K(t,s,u(s))ds, \quad s \in [0,1], \tag{1.1}$$

where $u(t)$ is the unknown function whose value is to be determined in the interval $0 < t < 1$. We are primarily interested in the case when $p(t,s)$ is unbounded in the region of integration, since equations of this type arise in a number of important practical applications. Typical forms of $p(t,s)$ are

$$\begin{aligned} p(x,t) &= |x-t|^{-\alpha},\ 0<\alpha<1, \\ &\ \ or \\ p(x,t) &= log(x-t). \end{aligned} \tag{1.2}$$

For the purpose of this paper we shall assume:

(a): $f(t)$ is bounded and continuous in $0 < t < 1$; $K(t,s,u)$ is bounded and continuous in t and s, for $0 \le s \le t \le 1$, and satisfies the Lipschitz condition

$$| K(t,s,u_1) - K(t,s,u_1) | \le L \mid u_1 - u_2 |;$$

(b): $\int_0^t p(t,s)ds \le M < \infty, \quad 0 \le t \le 1;$

(c): for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$, independent of t and α, such that

$$\int_\alpha^{\alpha+\delta} | p(t,s) | ds < \epsilon, \quad 0 \le \alpha \le t-\delta;$$

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Under these conditions (1.1) has a unique and continuous solution in $[0, 1]$.
For regular Volterra integral equations the smoothness of the kernel and of the forcing function$f(t)$ determines the smoothness of the solution on the closed interval $[0, 1]$. Whereas if we allow weakly singular kernels, then the resulting solutions are typically nonsmooth at the initial point of the interval. Some results concerning the behavior of the unique solutions of equations of type (1.1) are given in [8]. Note that the numerical solvability of weakly singular Volterra integral equations have been investigated, see for example [8].
In recent years the applications of the Adomian decomposition method (ADM) in mathematical problems has been developed by scientists. This method continuously transforms a complicated problem into a sequence of simpler problems which can be easily solved. The ADM solves successfully different types of linear and nonlinear equations in deterministic and stochastic fields [3, 4]. Application of ADM for solving different types of integral equations has been discussed by many authors [6, 10]. The objective of the present paper is to approximate the solution of equation (1.1) using a new strategy of product integration, in conjunction with ADM.
This paper is organized as follows. In section 2, some basic concepts of Adomian decomposition method (ADM) are presented. In section 3, we describe an algorithm based on product integration method and ADM for numerical solution of the nonlinear weakly singular Volterra integral equation (1.1). Section 4 is devoted to the numerical examples selected from the literature in connection with Volterra integral equations.

2. Adomian Decomposition Method

The Adomians decomposition method (ADM) is a solution method with a wide range of applications including the solution of algebraic, differential, integral and integro-differential equations or system of equations. This method was first introduced by Adomian. In the beginning of the 1980s. In this method the solution is considered as an rapidly converging, infinite series. The convergence of the method proved by Y. Cherrualt et al. [1].
In this work, the nonlinear Volterra integral equation of the second kind (*1.1*) is considered. The Adomian process gives

$$u(t) = \sum_{i=0}^{\infty} u_i(t) \tag{2.1}$$

and the nonlinear term $K(t, s, u(s))$ has the Adomian polynomial representation

$$K(t, s, u(s)) = \sum_{n=0}^{\infty} A_n(t, s, u_0, u_1, ..., u_n), \tag{2.2}$$

where the A_n are functions called Adomian's polynomials. We remark that the A_n are formally obtained from the relationship

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} K(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}. \tag{2.3}$$

For more details see [1, 2, 3]. Adomian polynomials are defined as

$$\begin{aligned}
A_0 &= K(u_0),\\
A_1 &= u_1 K'(u_0),\\
A_2 &= u_2 K'(u_0) + \frac{1}{2!} u_1^2 K''(u_0),\\
A_3 &= u_3 K'(u_0) + u_1 u_2 K''(u_0) + \frac{1}{3!} u_1^3 K''(u_0),\\
&\vdots
\end{aligned} \tag{2.4}$$

The author in [10] deduced another programmable formula for the Adomian polynomials:

$$A_n = K(S_n) - \sum_{j=0}^{n-1} A_j, \tag{2.5}$$

where the partial sum is $S_n = \sum_{i=0}^{n} u_i(t)$.
Substitutes (2.1) and (2.3) into (1.1) to obtain $u_n(t)$ recursively using

$$\begin{aligned} u_0(t) &= f(t) \\ u_{n+1}(t) &= \int_0^t p(t,s)A_n(t,s)dt, \quad n \geq 0. \end{aligned} \tag{2.6}$$

When the kernel of the integral equation is complicated or where calculating terms of the series $\sum_{i=0}^{\infty} u_i(t)$ is difficult or impossible analytically, the Adomian method needs modifications. We deal with this in the following section.

3. Description of Numerical Procedure

3.1. **Discretization of Problem.** According to the ADM, the solution of equation (1.1) may be derived using the series introduced as (2.1). Many authors used the zeroes of Chebyshev and Legendre orthogonal polynomials as collocation points, see [12]. Here we discretize equation (2.6) at the collocation nodes $\{t_i\}_{i=1}^{N} \bigcup \{0\}$, which yields using orthogonal Chelyshkov polynomials $P_{N,0}(t)$ on $[0,1]$ with the weight function 1, (see,[9]). These polynomials are defined as follows

$$P_{N,k}(t) = \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \binom{N+k+1+j}{N-k} t^{k+j}, \; k = 0, 1, ..., N. \tag{3.1}$$

The polynomials $P_{N,k}(t)$ have properties which are analogous to the properties of the classical orthogonal polynomials. In the family of orthogonal polynomials $\{P_{N,k}(t)\}_{k=0}^{N}$ every member has degree N with $N-k$ simple roots. Hence for every N, polynomial $P_{N,0}(t)$ has exactly N simple roots in $(0,1)$.
Using a quadrature which is based on $N+1$ nodal points $\{t_i\}_{i=1}^{N} \bigcup \{0\}$, and selecting collocation points to be the same as nodal points, then for $i = 0, 1, ..., N$, we have

$$u_{n+1}(t_i) = \int_0^{t_i} p(t_i, s)A_n(t_i, s)ds, \tag{3.2}$$

where $\{t_i\}_{i=1}^{N}$ are the roots of N^{th} degree polynomials $P_{N,0}(t)$.

3.2. **Product Integration Techniques.** To obtain an approximate solution to (3.2) we must replace the integral by a numerical quadrature, but since the integrand is unbounded, standard methods are not applicable. In such circumstances product integration is often employed. Thus, to approximate an integral of the form

$$I = \int_a^b \phi(t)f(t)dt,$$

where $\phi(t)$ is unbounded, one chooses a function$\bar{f}$ such that $\bar{f}(t) \simeq f(t)$, and such that

$$\int_a^b \phi(t)\bar{f}(t)dt$$

can be evaluated in a simple manner (for example, by analytical methods). In the present paper we use quadrature rules derived by approximating the integrand by piecewise polynomials of fixed degree, such as straight lines. In this way one obtains composite quadrature rules analogous to the trapezoidal method. In order to applying product trapezoidal method, we can rewrite equation (3.2) as a follow:

$$u_{n+1}(t_i) = \int_0^{t_i} p(t_i, s)A_n(t_i, s)ds = \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} p(t_i, s)A_n(t_i, s)ds, \tag{3.3}$$

The product trapezoidal method is constructed by approximating $A_n(t_i, s)$ by piecewise linear functions (for more details see Linz, [15]), in particular

$$A_n(t_i, s) = \frac{t - t_k}{t_{k-1} - t_k} A_n(t_i, t_{k-1}) + \frac{t - t_{k-1}}{t_k - t_{k-1}} A_n(t_i, t_k), \quad t_{k-1} \le t \le t_k \tag{3.4}$$

This leads to the integration formula

$$\int_0^{t_i} p(t_i, s)A_n(t_i, s)ds \simeq \alpha_{i,1}A_n(t_i, t_0) + \sum_{k=1}^{i-1} (\alpha_{i,k+1} + \beta_{i,k})A_n(t_i, t_k) + \beta_{i,i}A_n(t_i, t_i), \tag{3.5}$$

where

$$\alpha_{i,k} = \frac{1}{t_{k-1} - t_k} \int_{t_{k-1}}^{t_k} (t - t_k)p(t_i, s)ds, \tag{3.6}$$

$$\beta_{i,k} = \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} (t - t_{k-1})p(t_i, s)ds. \tag{3.7}$$

The numerical method for solving (1.1) is then

$$u_{n+1}(t_i) = \alpha_{i,1}A_n(t_i, t_0) + \sum_{k=1}^{i-1} (\alpha_{i,k+1} + \beta_{i,k})A_n(t_i, t_k) + \beta_{i,i}A_n(t_i, t_i), \tag{3.8}$$

Hence we obtain the following numerical values by ADM as

$$\begin{aligned}
u_0(t_i) = u_{0,i} &= f(t_i),\ i = 1, ..., N, \\
u_1(t_i) \cong u_{1,i} &= \alpha_{i,1}A_0(t_i, t_0) + \sum_{k=1}^{i-1} (\alpha_{i,k+1} + \beta_{i,k})A_0(t_i, t_k) + \beta_{i,i}A_0(t_i, t_i),\ i = 1, ..., N, \\
&\vdots \\
u_n(t_i) \cong u_{n,i} &= \alpha_{i,1}A_n(t_i, t_0) + \sum_{k=1}^{i-1} (\alpha_{i,k+1} + \beta_{i,k})A_n(t_i, t_k) + \beta_{i,i}A_n(t_i, t_i),\ i = 1, ..., N.
\end{aligned} \tag{3.9}$$

Therefore the approximation of $u(t_i)$ may be obtained using the M-term partial sum of the Adomian decomposition series solution as follows

$$u(t_i) \simeq \hat{u}_i = u_{0,i} + u_{1,i} + ... + u_{n,i},\ n = 1, ..., M. \tag{3.10}$$

4. Numerical Examples

We evaluate the efficiency of our method using some examples by comparing the numerical results with the analytical solution of the problem. To show the efficiency of the presented method we calculate the error norms which are defined in the following:

(1): Maximum error :

$$E = \|u - \hat{u}\| = \max_{0 \le i \le N} |u_i - \hat{u}_i| \tag{4.1}$$

(2): Root mean square error (RMS) :

$$E_{RMS} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [u_i - \hat{u}_i]^2} \tag{4.2}$$

where u_i denotes the exact solution and $\hat{u}_i$ denotes the approximate solution at the nodes $t_i,\ i = 0, 1, 2, ..., N$. We note that this error formula represents a reasonable measure of the accuracy. We consider Chelyshkov polynomials $\{P_{N0}(t)\}$ of different degrees ($N = 8, 16, 32, 64$) to examine the accuracy of our proposed algorithms of the previous section in the following examples. In addition in our computations, we consider a fixed $M = 15$.

Example 1. *Consider the following integral equation*

$$u(t) = \sqrt{t}(1 - \frac{t}{3}) + \frac{1}{4}\int_0^t \frac{u^2(s)}{\sqrt{t-s}}ds.$$

One may see that $u(t) = \sqrt{t}$ is the solution of this equation. Table 1 shows the errors for different values of N. Example 1 has been solved using the radial basis functions (RBF) method in [11]. The global error of this method is $3.59E-9$ with five basis functions. In comparison with this BFM method, the accuracy of the our proposed scheme is considerable and its running time is reasonable.

N	E	E_{RMS}
8	$1.9E-8$	$2.64E-9$
16	$7.88E-9$	$6.66E-10$
32	$5.52E-9$	$3.12E-10$
64	$4.39E-9$	$1.94E-10$

Table 1. Results of Example 1.

Example 2. *Consider the following integral equation*

$$u(t) = \sqrt{t} - \frac{3\pi t^2}{8} + \int_0^t \frac{u(s)^3}{\sqrt{t-s}}ds.$$

It can be investigated that $u(t) = \sqrt{t}$ is the solution of this equation. Table 2 shows the error between exact and approximate solutions at the nodes $x_i,\ i = 1, \cdots, N$, for different values of N. Example 2 is solved in [16] with variable transformation methods in combination with the trapezoidal quadrature rule and the absolute error between the exact and the approximate solution evaluated at the mesh points is presented. In comparisons with this method, our proposed method is very simple and the accuracy of the numerical results obtained with this method is considerable.

N	E	E_{RMS}
8	$6.82E-4$	$2.15E-4$
16	$1.93E-4$	$4.32E-5$
32	$5.18E-5$	$8.21E-6$
64	$1.34E-5$	$1.51E-6$

Table 2. Results of Example 2.

Example 3. *As the final example consider following nonlinear integral equation*

$$u(t) = t + \frac{11}{18}t^3 - \frac{1}{3}t^3 lnt + \int_0^t ln(t-s)u^2(s)ds.$$

One may show that following function is the solution of this equation $u(t) = t$. Table 3 shows the error approximation solutions at the nodes $x_i,\ i = 1, \cdots, N$ for different values of N. Khater et al. [12] have been solved the example 3, by Chebyshev polynomials expansion. Comparing results reported in [12] show that for $N = 128$ and 64 issued maximum errors of this problem are $O(10^{-6})$ and $O(10^{-9})$, respectively. Looking at Table 3, we can observe an improvement of the accuracy for $N = 64$ in the case of our algorithm respect to methods in [12].

N	E	E_{RMS}
8	$5.78E-5$	$2.21E-6$
16	$1.75E-6$	$1.69E-7$
32	$4.15E-8$	$1.18E-8$
64	$8.6E-10$	$7.92E-10$

Table 3. Results of Example 3.

5. Conclusion

In this work, a class of nonlinear weakly singular Volterra integral equations of the second kind is investigated by using an algorithm based on Adomian decomposition method and product integration approaches. The method of product integration is constructed with respect to a new family of orthogonal polynomials, named Chelyshcov polynomials. The new orthogonal polynomials keep distinctively of the classical orthogonal polynomials and give more accurate quadratures and hence better results.

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