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ANALYSIS AND CONTROL OF SEIR EPIDEMIC MODEL VIA SLIDING MODE CONTROL

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Abstract. In this paper, we have decided to analyze SEIR epidemic model by piecewise control function dependent on threshold policy for diseases management strategy. In this study, E_c is determined as a critical value and based on this threshold have been defined and then the timing has been specified for triggering intervention measures, when the number of exposed individual exceeds a threshold level. The solution of this model finally approaches regular/sliding equilibrium point, and this result shows that outbreak is not possible.

Key words: Sliding mode control, Threshold policy, SEIR epidemic model, outbreak

1. Introduction

Mathematical models play an important role in epidemiology and control of diseases. In order to, predict the spread of infectious diseases, many epidemic models have been studied and analyzed in recent years. However, in most researches on epidemic systems, assume that the disease in Chronic is negligible and susceptible (S) individual becomes infectious (I) and later recovers (R). Based on these assumptions, models are called as SI, SIR, SIRS, etc. There are other models that assume, a susceptible individual first goes through a latent period(said exposed or E class) after infections, before becoming infectious, these models are called SEIR, SEIRS, etc. Thus perusal of these models is important for control of infectious disease spread. In recent years, several strategies to control spreading infectious diseases have been designed with a variety of factors. For example, several studies have examined variation of

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the basic reproduction number [6, 9, 17, 32, 25] or long-term dynamics on control measures. Also they consider variable periods of seasonality [23, 13, 21]. Pandemic influenza outbreak [5, 7, 26] has also been studied and survey has been done on intervention measures. Although such studies provide vital information, they do not consider situations when multiple outbreaks are possible. Several analyzes have suggested that individuals reactively reduced their contact rates during a pandemic [20, 28, 26, 27]. Some studies investigated global dynamics of an SEIR epidemic with prevention of vaccinate [29] and some researchers check out global stability of an SEIR epidemic model with age-dependent latency and relapse [16]. Some studies have been analyzed stability of a quarantined epidemic model with latent and breaking-out over the internet[12]. Also, this study has not considered timing for triggering intervention measures [33]. Proposed a mathematical model of SIR epidemic model and they controlled outbreaks of emerging infectious diseases by sliding mode control. The purpose of this paper is to control outbreaks of emerging SEIR epidemic model by sliding mode control. Some studies consider population of varying size with immigration of infective [26]. Other studies, consider optimal control on SEIR model with immigration of infective [10].
In this article, researchers investigated SEIR epidemic model and control it by a piecewise control function concerning threshold policy for disease management strategy and determined time for administer control action that depend on critical value. The overall objective is to develop a systematic way for control of infectious disease and attain a globally stable equilibrium. To illustrate our ideas, SEIR epidemic model has been considered by the following system of ordinary differential equations. This model is a description of threshold policy. In SEIR epidemic model, population is shared into four groups, and perused as the dynamics of susceptible (S), infected (I), exposed (E) and recovered individuals (R).

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \beta SI - \mu S \\ \frac{dE(t)}{dt} = \beta SI - \mu E - \sigma(1 - f\varepsilon)E \\ \frac{dI(t)}{dt} = \sigma(1 - f\epsilon)E - (\mu + \gamma)I \\ \frac{dR(t)}{dt} = \gamma I - \mu R, \end{cases} \tag{1}$$

where

$$\epsilon = \begin{cases} 0 & \alpha(E) = E - E_c < 0 \\ 1 & \alpha(E) = E - E_c > 0. \end{cases} \tag{2}$$

Model (1) description of threshold policy (TP) which is referred to as on-off control. TP leads to a variable structure system with two

Parameter	Description
β	transmission coefficient
μ	natural death rate
γ	recovery rate
$\frac{1}{\sigma}$	period of latent
f	control intensity
E_c	critical value for exposed group

TABLE 1. Parameters and descriptions of model (1)

distinct structure that separated by threshold level.
We only consider first three equations of model (1) because $S + E + I + R = 1$ and when specified S,E,I we can obtain R [8]. We denote the structure without intervention ($\epsilon = 0$) by FS (free system) and structure with intervention ($\epsilon = 1$) by CS (control system). The solutions of system model (1) bounded by 1, then attraction region for this system without considering recovered individuals is as follows [11],

$$D = \{(S, E, I) \in \mathbb{R}^3 | S(t) + E(t) + I(t) \leq 1\}. \tag{3}$$

In this section all the possible equilibria with their stability have been considered. This equilibria divided into two groups; natural and sliding and each of them may be virtual or real.

We obtain disease-free equilibrium that is $(1, 0, 0)$, with specified jacobian matrix and obtain eigenvalue for this point. Issue the disease-free equilibrium point is asymptotically stable [15].
For FS, the basic reproduction number given by [11],

$$\mathcal{R}_{01} = \frac{\sigma\beta}{(\mu+\sigma)(\mu+\gamma)}.$$

Also, the basic reproduction number have been defined for CS as fallow

$$\mathcal{R}_{02} = \frac{\sigma(1-f)\beta}{(\mu+\sigma(1-f))(\mu+\gamma)} ,$$

Now, the endemic state, equilibrium point of FS given by

$$E^1 = (\frac{1}{\mathcal{R}_{01}}, \frac{\mu(\mu+\gamma)}{\beta\sigma}(\mathcal{R}_{01} - 1), \frac{\mu}{\beta}(\mathcal{R}_{01} - 1)).$$

If $\mathcal{R}_{01} > 1$ then E^1 is globally asymptotically stable,
because eigenvalues of $E^1 :\ -0.0002+0.0041i, -0.0002-0.0041i, -0.2144$ are negative.
In endemic state, equilibrium point for CS is obtained as follows,

$$E^2 = (\tfrac{1}{\mathcal{R}_{02}}, \tfrac{\mu(\mu+\gamma)}{\beta\sigma(1-f)}(\mathcal{R}_{02}-1), \tfrac{\mu}{\beta}(\mathcal{R}_{02}-1)).$$

If $\mathcal{R}_{02} > 1$ then E^2 is globally asymptotically stable. For calculating eigenvalue, suppose one of them $\lambda = -\mu$, for obtaining remainder eigenvalues, The following process has been performed. If
$\mathbb{I}$ be indicative identity matrix,

$$det(\lambda\mathbb{I} - J) = 0$$

$$\Rightarrow \lambda^3 + (\mu\mathcal{R}_{02} + 2\mu + \sigma + \gamma)\lambda^2 + \mu\mathcal{R}_{02}(2\mu + \sigma + \gamma)\lambda + \mu(\mathcal{R}_{02} - 1)$$

$$(\mu + \sigma)(\mu + \gamma) = 0.$$

Since in more cases σ and γ are greater than μ and $\mu\mathcal{R}_{02}$ then one of solution approximate $\lambda \sim -(\sigma + \gamma)$ and one solution with second degree as follows to be left, [11]

$$\lambda^2 + \mu\mathcal{R}_{02}\lambda + \tfrac{\gamma\sigma}{\sigma+\gamma}\mu(\mathcal{R}_{02} - 1) \approx 0$$

$$\Longrightarrow \triangle = (\mu\mathcal{R}_{02})^2 - 4(\tfrac{\gamma\sigma}{\sigma+\gamma}\mu(\mathcal{R}_{02} - 1)).$$

Now if $\triangle \geq 0$ then E^2 is stable node and if $\triangle < 0$, E^2 is spiral stable.
Moreover, if we choice $0 < f < 1$ then $\mathcal{R}_{02} > 1$ is satisfied and eigenvalues of E^2 are complex with negative real part; therefore, E^2 is asymptotically stable.
Define H_1 and H_2 as follows;

$$H_1 = \tfrac{\mu}{\mu+\sigma(1-f)} - \tfrac{\mu(\mu+\gamma)}{\beta\sigma(1-f)}, \qquad H_2 = \tfrac{\mu}{\mu+\sigma} - \tfrac{\mu(\mu+\gamma)}{\beta\sigma}.$$

Now we have several cases: If $E_c < H_1$ then the endemic state of FS is virtual equilibrium, that denoted by E_V^1 and the endemic state of control system is regular equilibrium point that denoted by E_R^2.
Now, if $E_c > H_2$, FS has regular equilibrium point that is denoted by E_R^1 and CS has virtual equilibrium point denoted by E_V^2.
When $H_1 < E_c < H_2$ then we have virtual equilibrium points for both theFS and CS. Therefore, we need to defined sliding domain for region $H_1 < E_c < H_2$ and obtain sliding equilibrium.
At first we examine existence of the sliding mode and then sliding region have been defined . The manifold η is define as follows,

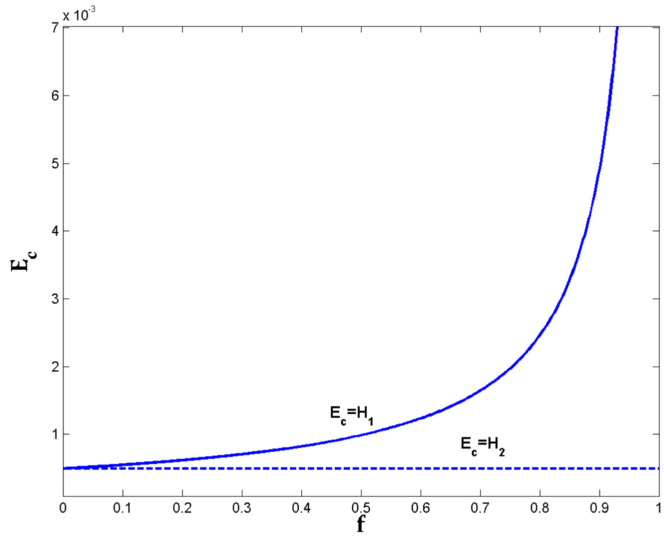

FIGURE 1. Bifurcation set for system (1) with respect to the control intensity f and threshold level E_c.

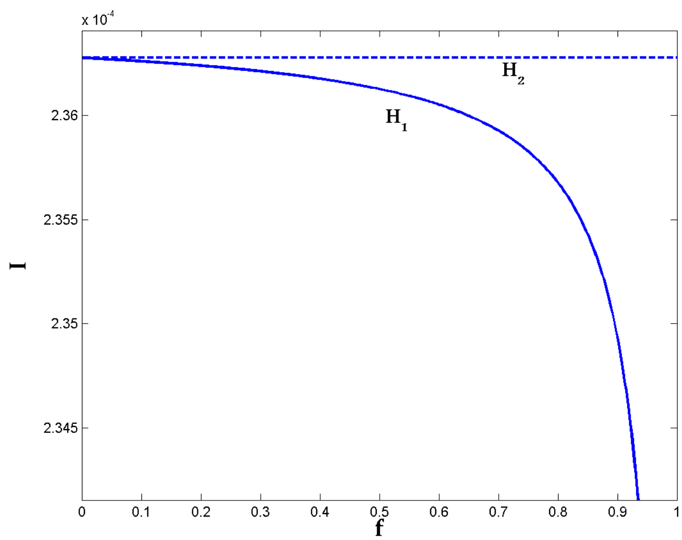

FIGURE 2. Bifurcation set for system (1) with respect to the control intensity f and change in number of individual infectious I.

$$\eta = \{(S, E, I) \in \mathbb{R}^3 : \alpha(E) = 0\},$$

that it is discontinue surface between two different structures of the system. Therefore, sliding domain has been defined as follow,

$$\Omega = \{(S, E, I) \in \mathbb{R}^3 : \tfrac{(\mu+\gamma)(\mu+\sigma)}{\beta\sigma} \leq S \leq \tfrac{(\mu\gamma)(\mu+\sigma(1-f))}{\beta\sigma(1-f)},$$
$$E = E_c, \tfrac{\mu\sigma}{(\mu+\gamma)(\mu+\sigma)} - \tfrac{\mu}{\beta} \leq I \leq \tfrac{\mu\sigma(1-f)}{(\mu+\gamma)(\mu+\sigma(1-f))} - \tfrac{\mu}{\beta}\}$$

For obtaining equivalent control in this system ε have been calculated and replace to another equation. Therefore, $\dot{\alpha} = \dot{E} = 0$

$$\Longrightarrow \beta SI - \mu E - \sigma(1 - f\varepsilon)E = 0$$

$$\Longrightarrow \varepsilon = \tfrac{1}{f}(1 - \tfrac{\beta SI - \mu E}{\sigma E}),$$

now if ε and $E = E_c$ replace into equations system (1) then gives the system dynamic on the switching surface,

$$\begin{cases} \dot{S} = \mu - \beta SI - \mu S \\ \dot{I} = \beta SI - \mu E_c - (\mu + \gamma)I. \end{cases} \tag{4}$$

Therefore, we have,

$E_s = (S^*, E_c, I^*)$ such that $I^* = -\frac{E_c\mu}{\gamma+\mu-\beta S}$ and $S^* = \frac{\mu}{\mu+\beta I}$

which is equilibrium point of equations (4) and it is locally asymptotically stable on switching surface, moreover, it is sliding equilibrium point for system (1). If

$\frac{(\mu+\gamma)(\mu+\sigma)}{\beta\sigma} < S^* < \frac{(\mu+\gamma)(\mu+\sigma(1-f))}{\beta\sigma}$

and $\frac{\mu\sigma}{(\mu+\gamma)(\mu+\sigma)} - \frac{\mu}{\beta} < I^* < \frac{\mu\sigma(1-f)}{(\mu+\sigma(1-f))(\mu+\gamma)} - \frac{\mu}{\beta}$,

then sliding equilibrium point E_s belongs to sliding domain.
Now, for analysing globally stability and behavior of system (1) we need to obtain attraction region and discover relation between sliding domain and attraction region. Expression between these regions have been investigated.

2. Global behavior

In this section asymptotically behavior of system (1) have been considered to achieve stability for this model. We consider this model with different control intensity f and the threshold level E_c. The result of this ascertainment is given by following theorems.

Theorem 1. *E_R^2 is globally asymptotically stable if*

$E_c > \frac{\mu}{\mu+\sigma(1-f)} - \frac{\mu(\mu+\gamma)}{\beta\sigma(1-f)}$.

Proof. We replace S and I with maximum value of them in sliding domain and E_c replace with above value. Then we have $\dot{I} > 0$ and $\dot{S} > 0$ so, when trajectories hit the sliding domain Ω, the state vector starts to move to the right end of the sliding domain along the sliding domain. □

Claim 1. The trajectory initiating from right end of sliding domain will not hit the sliding domain again.

Claim 2. No limited cycle surround the regular equilibrium E_R^2 and sliding mode domain.
Denote the right-hand side of three equations of model (1) by f_1, f_2 and f_3. For $E < E_c$, choosed $D = \frac{1}{EI}$ as a Dulac function [22] then we have,

$\frac{\partial(Df_1)}{\partial S} + \frac{\partial(Df_2)}{\partial E} + \frac{\partial(Df_3)}{\partial I} = \frac{\beta}{E} - \frac{\beta S}{E^2} - \frac{\sigma(1-f)}{I^2} < 0.$

Therefore, according to Bendixson-Dulac theorem [22] there is no limit cycle that surrounds the equilibrium E_R^2 and sliding mode domain.
In this case if we choose $0 < f < 0.9999$ and $E_c = 0.00001$ then E^2 is asymptotically stable with complex eigenvalues that they have negative real part. Also, if f be as previous and $E_c = 0.0001$, E^2 is asymptotically stable again.
combination of Claim 1, Claim 2, and local stability of E_R^2 implies that it is globally asymptotically stable.

Theorem 2. *The sliding equilibrium E_s is globally stable if*

$H_2 < E_c < H_1$.

Proof. By choosing the value $0 < f < 1$ and $E_c = 0.00001$ inequality $H_2 < E_c < H_1$ is satisfied and E_s (sliding equilibrium) has eigenvalues as follows:

$-0.000040395190097 + 0.000000119460617i$

$0.219956196209510 + 0.074604651500485i$
$-0.398583347844065 - 0.074595331823543i.$

So E_s has two complex eigenvalues with negative real part and one complex eigenvalue with positive real part then it is asymptotically stable. Also by using a analogous manner to Claim 2 in Theorem 1 we can prove that no limit cycle surrounds the sliding domain. Therefore, E_S is globally stable.

□

Theorem 3. *The equilibrium E_R^1 is globally stable if $E_c < H_2$.*

Proof. If we choose $E_c \leq 0.0001$, inequality $E_c < H_2$ is possible and E^1 has complex eigenvalues with negative real parts therefore, E^1 is asymptotically stable. Using alike style theorem 1 we can prove that E_R^1 is globally stable. □

Conclusion

In this paper SEIR epidemic model have been studied then shown how we can prevent of outbreak of infectious diseases. Piecewise control function has been applied depending on threshold policy to diseases management strategy. In the case that two equilibrium points of free system and control system, to be real, the model will be stable in two points and in the case that both of equilibrium points be virtual then we obtain sliding equilibrium point and showed that this point is stable. In general, control on exposed individual was defined and different modes were investigated. Finally, we found if critical value for exposed individual was defined then we can determine when intervention on system occurred. This process utilized by piecewise control function. Theorems in previous section show that in different cases one of equilibrium points of free system, control system and sliding equilibrium point is stable. Therefore, outbreak seems impossible.

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