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Linear multistep method of order-six for the integration linear and nonlinear initial value problems of ODEs†

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Abstract

In this work, a sixth-order linear multistep method (LMM) is constructed for the numerical integration of linear and nonlinear second order initial value problems of ordinary differential equations. This method, which depend on certain algebraic parameters is developed following an extension of higher order derivatives and step length. The analysis of the basic properties of our method is examined and found to be zero-stable, symmetric and consistent. Error and step-length control is carried out by using Richardson extrapolation procedure. Extensive numerical results demonstrate increased accuracy with the same computational effort when compared with similar sixth and higher order formulas.

Subject classification: **65L05, 65L20**

Keywords: Initial-value problems; Stability; Consistent; nonlinear; Error Constant; Implicit methods; Symmetric.

1 Introduction

In this paper, we shall be concerned primarily with the numerical integration of differential system of the form

$$\frac{d^2y}{dx^2} = f(x, y(x), \frac{dy}{dx}), \quad y(x_0) = y_0, \quad y'(x_0) = y_1, f(x, y, y') \in \mathbb{R}^n. \tag{1.1}$$

The numerical solution of second order differential equations of type (1.1) has been the subject of great activity in areas of scientific research, we shall investigate in this paper the

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class of the above problems whose solution exhibits an oscillatory character. The result of this activity are methods which can be applied to many problems in quantum mechanics, radio-active process, celestial mechanics, nuclear physics, astrophysics, electronics, airflow and transverse motion.

For the approximate solution, many numerical techniques that have been designed for the integration of (1.1), (see, for example [1], [6], [11], [15], [18], [20] and references therein) are capable of solving first order problems of type

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \tag{1.2}$$

by reduction of (1.1) to first order systems. The approach of reducing such equation to a system of first order equations results to serious challenges in computation and wastage of computer time which actually lead some of these methods to be inefficient and cumbersome to implement, [7].

These reasons perhaps motivated some scholars, (see, for example[9], [8], [17],[21, 22, 23]) to adopt direct discretization methods for the integration of special second-order equation of the form

$$\frac{d^2y}{dx^2} = f(x, y(x)), \tag{1.3}$$

that is, differential equations for which the function f is independent from the first derivative of y [26]. The aim of this paper is to obtain practical and efficient implicit predictor-corrector method as well as local error estimations that allow the implementation of this method in a variable step code with a minimum computational cost.

The paper is organized as follows: In section 2, four-step implicit method of order 6 is constructed. Error constant, consistency, symmetry and zero-stability are discussed in section 3. Finally, in section 4, we examine some of the computational aspects and present numerical results in comparison with existing methods.

2 Construction of the method.

We consider the following general mth- order implicit linear multistep methods:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2.1}$$

operationally defined in [12] as

$$\rho(E) y_n = h^m \delta(E) f_n \tag{2.2}$$

where $\rho(E)$ and $\delta(E)$ represent the first and second characteristic polynomials of (2.1), α_j and β_j are real constants with constraints

$$\alpha_k \neq 0, \quad | \alpha_0 | + | \beta_0 | \neq 0. \tag{2.3}$$

In order to remove the arbitrary constant in (2.1), we shall always assume that $\alpha_k = 1$ since we can always divide the coefficients of (2.1) with α_k. Setting $m = 2$, $j = 0, ..., 4$ in (2.1), we obtain the following four-step implicit method

$$\begin{aligned}&\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \alpha_{k-2} y_{n+k-2} + \alpha_{k-3} y_{n+k-3} + \alpha_{k-4} y_{n+k-4} \\ &= h^2[\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \beta_{k-2} f_{n+k-2} + \beta_{k-3} f_{n+k-3} + \beta_{k-4} f_{n+k-4}]\end{aligned} \tag{2.4}$$

where $y_{n+i} = y(x + ih)$ with $i = 0, 1, ..., 4$, $f_{n+i} = y''(x + ih)$ with $i = 0, 1, ..., 4$, h is the step size.

We associate the principal term of the local truncation error (PLTE) given as T_{n+k} to method (2.1)

$$T_{n+k} = y_{n+k} - \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2.5}$$

Terms y_{n+k}, y_{n+j} and f_{n+j} are expanded in Taylors series about the point (x_n, y_n) and collected in equal powers of h to yield

$$\begin{aligned} T_{n+k} &= (1 - \sum_{j=0}^{k-1} \alpha_j) y_n + (k - \sum_{j=0}^{k-1} j\alpha_j) h y_n^1 + \\ &\quad (\frac{k^2}{2!} - \sum_{j=0}^{k-1} \frac{j^2}{2!}\alpha_j - \sum_{j=0}^{k} \beta_j) h^2 y_n^{(2)} + \\ &\quad (\frac{k^3}{3!} - \sum_{j=0}^{k-1} \frac{j^3}{3!}\alpha_j - \sum_{j=0}^{k} j\beta_j) h^3 y_n^{(3)} + \cdots + \\ &\quad (\frac{k^p}{p!} - \sum_{j=0}^{k-1} \frac{j^p}{p!}\alpha_j - \sum_{j=0}^{k} \frac{j^{(p-2)}}{(p-2)!}\beta_j) h^p y_n^{(p)} + O(h^{p+1}) \end{aligned} \tag{2.6}$$

which in compact form means

$$T_{n+k} = C_0 y_n + C_1 h y_n^1 + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + \cdots + C_{p+2} h^{p+2} y^{(p+2)} + O(h^{p+3}) \tag{2.7}$$

where

$$\begin{aligned} C_0 &= 1 - \sum_{j=0}^{k-1} \alpha_j y_n \cdots \\ C_p &= (\frac{k^p}{p!} - \sum_{j=0}^{k-1} \frac{j^p}{p!}\alpha_j - \sum_{j=0}^{k} \frac{j^{(p-2)}}{(p-2)!}\beta_j) h^p y_n^{(p)} + O(h^{p+1}). \end{aligned} \tag{2.8}$$

with $k=4$ and $\alpha_k=\alpha_4$ in (2.5), we obtain the following system of equations

$$
\begin{aligned}
1 &= \alpha_0+\alpha_1+\alpha_2+\alpha_3\\
4 &= \alpha_1+2\alpha_2+3\alpha_3\\
\frac{16}{2} &= \frac{1}{2!}(\alpha_1+4\alpha_2+9\alpha_3)+(\beta_0+\beta_1+\beta_2+\beta_3+\beta_4)\\
\frac{64}{6} &= \frac{1}{3!}(\alpha_1+8\alpha_2+27\alpha_3)+(\beta_1+2\beta_2+3\beta_3+4\beta_4)\\
\frac{256}{24} &= \frac{1}{4!}(\alpha_1+16\alpha_2+81\alpha_3)+\frac{1}{2!}(\beta_1+4\beta_2+9\beta_3+16\beta_4)\\
\frac{1024}{120} &= \frac{1}{5!}(\alpha_1+32\alpha_2+243\alpha_3)+\frac{1}{3!}(\beta_1+8\beta_2+27\beta_3+64\beta_4)\\
\frac{4096}{720} &= \frac{1}{6!}(\alpha_1+64\alpha_2+729\alpha_3)+\frac{1}{4!}(\beta_1+16\beta_2+81\beta_3+256\beta_4)\\
\frac{16384}{5040} &= \frac{1}{7!}(\alpha_1+128\alpha_2+2817\alpha_3)+\frac{1}{5!}(\beta_1+32\beta_2+243\beta_3+1024\beta_4)\\
\frac{65536}{40320} &= \frac{1}{8!}(\alpha_1+256\alpha_2+6561\alpha_3)+\frac{1}{6!}(\beta_1+64\beta_2+729\beta_3+4096\beta_4) \qquad (2.9)
\end{aligned}
$$

Equation (2.9) is represented in the form AX=B and solved with MATLAB package to obtain the following parameters:

$$\alpha_0=-1,\alpha_1=4,\alpha_2=-6,\alpha_3=4,\beta_0=-\frac{1}{12},\beta_1=\frac{8}{12},\beta_2=-\frac{18}{12},\beta_3=\frac{8}{12},\beta_4=\frac{1}{12} \qquad (2.10)$$

substituting (2.10) into (2.4), we have a symmetric four-step scheme

$$y_{n+4}=4y_{n+3}-6y_{n+2}+4y_{n+1}-y_n+\frac{h^2}{12}\{f_{n+4}+8f_{n+3}-18f_{n+2}+8f_{n+1}-f_n\} \qquad (2.11)$$

In order to use formula (2.11) for the integration of initial value problems (1.1), four important factors are considered

(a) the need to generate the starting values $y_{n+j}, j=0(1)4$ and their corresponding derivatives $y''_{n+j}, j=0(1)4$, by employing Taylor series expansion about x_n for y_{n+j} and its derivative y'_{n+j} up to the order of method (2.11) in attempt to avoid the initial data error in our computation.

$$
\begin{aligned}
y_{n+j} &= y(x_n+jh)=y(x_n)+(jh)y'(x_n)+\cdots+\frac{jh^p}{p!}y^p(x_n)+O(h^{p+1})\\
y'_{n+j} &= y'(x_n+jh)=y'(x_n)+(jh)y''(x_n)+\cdots+\frac{jh^p}{p!}y^{p+1}(x_n)+O(h^{p+2}),
\end{aligned}
$$

this is achieved by the adoption of predictor corrector mode denoted by PEC meaning Predict, Evaluate and Correct. The mode is described as

$$\begin{aligned}
P : y_{n+j} \quad &, \quad j = 0(1)4 \\
E : y''_{n+j} \quad &= \quad y''(t_{n+j}, y_{n+j}), j = 0(1)4 \\
C : y_{n+4} \quad &= \quad 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n \\
&\quad + \frac{h^2}{12}\{f_{n+4} + 8f_{n+3} - 18f_{n+2} + 8f_{n+1} - f_n\}
\end{aligned}$$

The error estimate is obtained from

$$Error = \frac{y_{n+4}^{(s+1)} - y_{n+4}^{(s)}}{y_{n+4}^{(s)} - y_{n+4}^{(s-1)}}$$

the iteration terminated whenever $Error < Tolerance$.

(b) the choice of appropriate stepsize h

(c) the need to solve implicit system of equation (2.11), now

$$y_{n+4} = \mathbb{A} + \frac{h^2}{12}\mathbb{G}(y_{n+4}) \tag{2.12}$$

where
$\mathbb{A} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n$
$\mathbb{G}(y_{n+4}) = y''_{n+4} + 8y''_{n+3} - 18y''_{n+2} + 8y''_{n+1} - y''_n$

(d) the accuracy of the approximation y_{n+4} requires the solution of implicit equation (2.11) rewritten as

$$F(y_{n+4}) = 0$$

This can be achieved by the adoption of quasi Newton iteration scheme

$$[y_{n+4}^{mt1} - y_{n+4}^{m}] - \mathbb{G}[y_{n+4}^{m}]/(I - \frac{h^2}{12}\xi)$$

$$\xi = \frac{\partial_{\mathbb{G}}}{\partial_{y_{n+4}}}(y_{n+4}^{m}), m = 0, 1, 2$$

The convergence condition is that

$$\theta = \frac{|y_{n+4}^{(m+1)} - y_{n+4}^{(m)}|}{|y_{n+4}^{(m)} - y_{n+4}^{(m-1)}|} \leq Tolerance.$$

3 Basic properties of the method.

In order to ascertain the accuracy and suitability of the method (2.11), analysis of its basic properties such as consistency, order of accuracy and error constant, symmetry, convergence and zero-stability are undertaken.

3.1 Order of accuracy and error constant

The local truncation error (2.7) when $k = 4$ is written as

$$T_{n+4} = C_0 y_n + C_1 h y_n^1 + C_2 h^2 y_n^{(2)} + C_3 h^3 y_n^{(3)} + \cdots + C_8 h^8 y_n^{(8)} + C_9 h^9 y_n^{(9)} + O(h^{10})$$

Using the values of $\alpha_{j's}$ and $\beta_{j's}$ earlier obtained in (2), we have

$$\begin{aligned}
C_0 &= -1+4-6+4-1=0\\
C_1 &= 4-12+12-4=0\\
C_2 &= \frac{1}{2}(4-24-36)+\frac{1}{12}(1+8-18+8+1)=0\\
C_3 &= \frac{1}{6}(4-48+108-64)+\frac{1}{12}(8-36+24+4)=0\\
C_4 &= \frac{1}{24}(4-96+324+8-72+72+16-256)=0\\
C_5 &= \frac{1}{120}(4-192+972-1024)+\frac{1}{72}(8-144+216+64)=0\\
C_6 &= \frac{1}{720}(4-384+2916+8-4096)+\frac{1}{288}(8-288+648+256)=0\\
C_7 &= \frac{1}{5040}(4-768+8748-16384)+\frac{1}{1440}(8-576+1944+1024)=0\\
C_8 &= \frac{1}{40320}(4-1536+26244-65536)+\frac{1}{8640}(8-1152+5832+4096)=-\frac{1}{240}
\end{aligned}$$

Thus, $C_8 \neq 0$, which by [19] implies that $C_0 = C_1 = C_2 = C_3 = C_4 =, \ldots, = C_7 = 0$, but $C_8 = C_{p+2} \neq 0$

Hence, method (2.11) is of order $P = 6$ with principal error constant

$$C_{p+2} = -\frac{1}{240}$$

3.2 Symmetry of the method

Definition 1. A linear multistep method (2.11) is symmetric ([19], [12], [**?**]) if the parameters $\alpha_{j's}$ and $\beta_{j's}$ satisfy the following conditions:

$$\begin{aligned} \alpha_j &= \alpha_{k-j}, \beta_j = \beta_{k-j}, j = 0(1)k \\ \alpha_j &= -\alpha_{k-j}, \beta_j = -\beta_{k-j}, j = 0(1)k \end{aligned}$$

for even and odd step numbers respectively.
The method (2.11) is symmetric, which means that

$$\alpha_j = \alpha_{k-j}, \beta_j = \beta_{k-j}, j = 0(1)k$$

$$\begin{aligned} \alpha_0 &= \alpha_4 = 1 \qquad \beta_0 = \beta_4 = 1 \\ \alpha_1 &= \alpha_3 = -4 \qquad \beta_1 = \beta_3 = 8 \\ \alpha_2 &= \alpha_2 = 6 \qquad \beta_2 = \beta_2 = -18 \end{aligned}$$

It is easily proved then that both the order of the method and the step number k are even numbers [19],[28]. Hence, method (2.11) is symmetric.

3.3 Consistency

The method (2.11) is consistent, since

(i) it has order $P \geq 1$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\rho(r) = \rho'(r) = 0, r = 1$

(iv) $\rho''(r) = 2!\delta(r), r = 1$

3.4 Zero stability

Definitions:

(i) A linear multistep method for a given initial value problem is said to be zero stable if no root of the first characteristic polynomial $\rho(r)$ has modulus greater than one and if every root with modulus one is simple. That is

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j = 0$$

from (2.11),

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$$

implies that $(r-1)^4 = 0$,
therefore, method (2.11) is zero stable, since the roots of $\rho(r)$ all lie in the unit disk, and those that lie on the unit circle have multiplicity one.

(ii) A numerical solution to the class of system (1.1) is stable if the difference between the numerical solution and the theoretical solution can be made as small as possible, that is, if there exist two positive numbers ℓ_n and C such that

$$\|y_n - y(t_n)\| \leq C\|\ell_n\|$$

We obtain the region of absolute stability (RAS) for the method (2.11) by setting

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 = e^i\theta, \tag{3.1}$$

this region is contained in (0,1] that is $0 < h \leq 1$,

Definition. A linear multistep or predictor-corrector method is said to be absolutely or relatively stable in a region $\Re$ of the complex plane if, for all $\hbar \epsilon \Re$, all roots of the stability polynomial $\pi(r, \hbar)$ associated with the method satisfy

$$|r_s| < 1$$

, for $s = 1, 2, \ldots, k$

3.5 Convergence

Definition A linear multistep method that is consistent and zero stable is convergent ([2], [12], [20]).

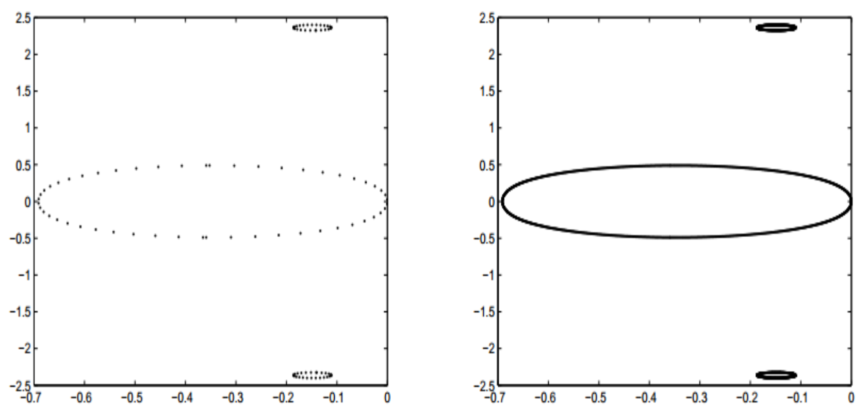

Figure 1: The stability regions of method (2.11) at different steps

4 Numerical experiments.

In this section, efficiency and applicability of our new method is demonstrated on some initial-value problems having as solution the combination of cosine and sine functions. The first two examples are taken from the previous work of Simos [25] and combined effort of Vadeen Berghe et. al [27]. The last example is taken from the independent work work of Badmus and Yahaya [9] which was later revised by Kayode [17] for the purpose of comparison.

4.1 Example 1.

We consider the nonlinear undamped Duffin's equation,

$$y'' + y + y^3 = B\cos(\omega - x)' \tag{4.1}$$

where $B = 0.002$ and $\omega = 1.01$. The exact solution is given by

$$y(x) = \sum_{i=0} 3A_{2i+1}\cos[(2i+1)\omega x], \tag{4.2}$$

where $A_1 = 0.200179477536$, $A_3 = 0.24696143\text{x}10^{-3}$, $A_5 = 0.304016\text{x}10^{-6}$ and $A_7 = 0.374\text{x}10^{-9}$. Equation (4.1) has been solved with various step-sizes, $h = 2^{-n}$ $n \geq o$, $0 \leq x \leq 1$. In Table 1, comparison of the end-point global errors in approximation obtained by using method of Simos [25] (denoted as Method [a]), Vanden Barghe et al. [27] (denoted as Method [b]) and the new implicit method (denoted as Method [e])

Table 1: Solution to problem 4.1

h	Exact	[E] Computed	Method [A]	Method [B]	Method [E]
1	0.2004263232	0.1981153035	1.10e-03	1.70e-03	2.31e-03
0.5	0.2004266269	0.2003937937	5.42e-05	1.88e-04	3.28e-05
0.25	0.2004267028	0.2004259025	1.86e-06	1.37e-05	8.00e-07
0.125	0.2004267217	0.2004266854	6.19e-08	8.70e-07	3.64e-08
0.0625	0.2004267265	0.2004267248	2.40e-09	5.41e-08	1.73e-09

4.2 Example 2.

$$u'' + u = 0.001\cos(x), \quad u(0) = 1, \quad u'(0) = 0, \tag{4.3}$$

with exact solution

$$u(x) = \cos(x) + 0.0005x\sin(x). \tag{4.4}$$

Table 2 displays the result of equation (4.3) that has been solve with several step-sizes,

Table 2: Solution to problem (4.3)

h	Exact	[E] Computed	Method [A]	Method [B]	Method [E]
1	0.5407230414	0.5399419065	1.40e-02	1.20e-03	7.81e-04
0.5	0.8777024183	0.8776930516	8.52e-04	7.54e-05	9.37e-06
0.25	0.9689433472	0.9689427425	5.30e-05	4.74e-06	6.05e-07
0.125	0.9922054594	0.9922054162	3.31e-06	2.96e-07	4.32e-08
0.0625	0.9980494626	0.9980494597	2.07e-07	1.86e-08	2.88e-09

$h = 2^{-2}\ n$ is positive number in the interval $0 \leq x \leq 1$, comparison of the end-point global errors in approximation obtained by using method of Simos [25] (denoted as Method [a]), Vanden Barghe et al. [27] (denoted as Method [b]) and the new implicit method (denoted as Method [e])

4.3 Example 3.

$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0, y(0) = 1, y'(0) = 1, x > 0, h = \frac{0.1}{32}, \tag{4.5}$$

whose theoretical solution is

$$y(x) = \frac{5}{3x} - \frac{2}{3x^4}. \tag{4.6}$$

In Table 3, we present the comparison of the global errors in approximations of the new method (indicated as Method [E])at some selected step-size given in the first column by

Table 3: Solution to problem (4.5)

x	Exact	[E] Computed	Method [C]	Method [D]	Method [E]
0.025	1.022049164	1.022049012	2.21e-04	7.74e-06	1.52e-07
0.015625	1.014447543	1.014447461	1.56e-04	5.93e-06	8.18e-08
0.0125	1.011741018	1.011740982	1.35e-04	3.68e-06	3.63e-08
0.00625	1.006057503	1.006057499	7.50e-04	1.86e-07	4.09e-09
0.003125	1.003076526	1.003076525	3.84e-05	1.10e-07	1.40-09

using both the block method [9] (which is denoted as Method [C]) and the zero-stable method [17](denoted as Method [D]).

5 Conclusions

A new method of order six for direct integration of linear, nonlinear, special and general second order initial value problems has been developed. The method is found to be consistent and zero-stable, these two conditions are the major ingredients for a linear multistep method to be convergent. It is clear from the results obtain in Tables 1-3 that the new method enjoys a significant level of accuracy on comparison with the existing methods that solved the same set of problems.

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