AMO - Advanced Modeling and Optimization, Volume 18, Number 1, 2016[1]

A Short Note on: Portfolio Weighting with Analytic Hierarchy Process (AHP)

Reza Habibi[2]
Iran Banking Institute, Central Bank of Iran

Abstract. The AHP weighting of a specified portfolio is studied. Weights of AHP are computed such that they correspond to Lagrange multiplier. Conclusions are proposed.

Keywords: AHP; Lagrange multiplier; Pair comparisons; Portfolio management

1 Introduction. Financial portfolios contain a collection of financial assets kept by an investor. The weight of each asset is the percentage composition of it holding in portfolio. There are different approaches to calculate the portfolio weights. For example, the dollar value approach and weighting by using of number of units of a given security are two basic types of weighting approaches. Also, the efficient frontier weights are derived using a quadratic programming and applying the Lagrange multiplier method. Let $r_i, i = 1,2,\dots,n$ be the return of n independent financial asset with mean μ_i and variance σ_i^2. The mean and variance of a portfolio with weight a_i assigned to i-th asset are $\sum_{i=1}^{n} a_i \mu_i$ and $\sum_{i=1}^{n} a_i^2 \sigma_i^2$, respectively. The following Lagrange function is minimized w.r.t a_i for a pre-determined threshold μ^*, where

$$\pounds = \sum_{i=1}^{n} a_i^2 \sigma_i^2 - \gamma\left(\sum_{i=1}^{n} a_i \mu_i - \mu^*\right),$$

which gives weights $a_i = k\frac{\mu_i}{\sigma_i^2}$ with $k^{-1} = \sum_{i=1}^{n} \frac{\mu_i}{\sigma_i^2}$. The mean and variance of portfolio using optimal weights are $k\sum_{i=1}^{n} \theta_i^2$ and $k^2\sum_{i=1}^{n} \theta_i^2$, respectively, where $\theta_i = \frac{\mu_i}{\sigma_i}$. Another method is to use the pair comparisons perspective of AHP (Saaty, 2001). Let b_{ij} denote the relative preference of i-th asset w.r.t j-th asset. Mehlawat (2013) developed a multi-criteria decision making framework for portfolio selection. The AHP is used to model the behavioral construct of suitability. Banihashemi and Sanei (2013) considered the asset allocation problems using a combined DEA/AHP method. The next section the AHP weighting is studied.

2 AHP weighting. To correspond the results of AHP and Lagrange weighting methods, it is necessary that $b_{ij} = \frac{a_i}{a_j}$. However, this assumption may be violated. Indeed, in practice,

[1] **AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

[2] **Reza Habibi (corresponding author) has a PhD in Statistics from Shiraz University. Recently, (after PhD program) he has been graduated in Financial Engineering from Amirkabir University. He was a free researcher in Department of Statistics of Central Bank of Iran (CBI). He is currently a lecturer in Iran Banking Institute of CBI. r_habibi@ibi.ac.ir**

determining the weights using AHP yields weights as

$$a_i^* = k\frac{\mu_i}{\sigma_i^2}\varepsilon_i.$$

The mean and variance of portfolio using AHP weights are $k\sum_{i=1}^n \theta_i^2\, \varepsilon_i$ and $k^2\sum_{i=1}^n \theta_i^2\, \varepsilon_i^2$. The proportions of AHP mean and variance to these value of quadratic programming are $\pi_1 = \sum_{i=1}^n w_i\, \varepsilon_i$ and $\pi_2 = \sum_{i=1}^n w_i\, \varepsilon_i^2$, where $w_i = \frac{\theta_i^2}{\sum_{i=1}^n \theta_i^2}$. Suppose that the acceptable upper

bound for π_1 be π, then solving again the Lagrange multiplier to minimize π_2 gives the $\varepsilon_i = \pi$. However, this is a multi-criteria decision making problem assuming $0 \le \varepsilon_i \le L_i$, $i = 1,2,\dots,n$. As follows the solution of this problem is surveyed. The problem is

$$\text{Problem: }\begin{cases} Min(\sum_{i=1}^n w_i\, \varepsilon_i, \sum_{i=1}^n w_i\, \varepsilon_i^2) \\ s.t.\ \ 0 \le \varepsilon_i \le L_i, i = 1,2,\dots,n \end{cases}$$

The weighting approach solution of this problem is

$$\text{Problem: }\begin{cases} MinZ = \sum_{i=1}^n w_i\, \varepsilon_i(\lambda + \varepsilon_i) \\ s.t.\ \ 0 \le \varepsilon_i \le L_i, i = 1,2,\dots,n \end{cases}$$

The Kuhn-Tucker conditions (Bazaraa *et al*., 1993) using shadow price $u_i \ge 0, i = 1,2,\dots,n$ are

$$\begin{cases} u_i(L_i - \varepsilon_i) = 0 \\ w_i(2\varepsilon_i + \lambda) - u_i L_i = 0. \end{cases}$$

Thus, if $u_i = 0$ then $\varepsilon_i = 0$ and if $u_i \ne 0$ then $\varepsilon_i = L_i$ and $\lambda = \bar{\lambda} = \frac{\sum_{i=1}^n \lambda_i}{n}$ where

$$\lambda_i = \begin{cases} 0 & \text{if} \quad u_i = 0, \\ (2 - \alpha_i)L_i & \text{if} \quad u_i \ne 0, \end{cases}$$

where $\alpha_i = \frac{u_i}{w_i}$. As w_i gets small then α_i gets large and λ_i is close to zero and therefore $\varepsilon_i = 0$ and for large w_i's the $\varepsilon_i = L_i$ is reasonable. As follows the AHP algorithm for portfolio weighting is given

Algorithm: AHP portfolio weighting

1. Compute $w_i = \frac{\theta_i^2}{\sum_{i=1}^n \theta_i^2}$ *for* $i = 1,2,\dots,n$, *where* $\theta_i = \frac{\mu_i}{\sigma_i}$.

2. Order from small to large w_i*'s and for small* w_i*'s let* $\varepsilon_i = 0$ *and for large* w_i*'s let* $\varepsilon_i = L_i$.

3. Select ε_i*'s such that the equation* $\lambda = \bar{\lambda}$ *is satisfied.*

Remark 1. Suppose that the consistency assumption is violated and $b_{ij} = \frac{a_i}{a_j}\varepsilon_{ij}$, where $a_i = k\frac{\mu_i}{\sigma_i^2}$ and $k^{-1} = \sum_{i=1}^n \frac{\mu_i}{\sigma_i^2}$. Saaty (2001) showed that the inconsistency ratio based on the largest eigenvalue (λ_{max}) of pair comparison matrix is given by

$$IR = \frac{\lambda_{max} - n}{n - 1} = \frac{\sum_{i=1}^n \sum_{j=1}^n (\varepsilon_{ij} - 1)}{n(n-1)}.$$

3 Conclusion. This short note studies the AHP portfolio weighting. The AHP portfolio weighting algorithm is given.

References

[1] Banihashemi, S. and Sanei, M. (2013). Portfolio optimization and asset allocations using DEA/AHP. *Indian J.Sci.Res*. **4**, 81-86.

[2] Bazaraa, M., Sherali, H. and Shetly, C. (1993). *Nonlinear programming theory and algorithms*. Wiley. New York.

[3] Mehlawat, K. M. (2013). Behavioral optimization models for multi-criteria portfolio selection. *Yugoslav Journal on Operations Research* **23**, 279-297.

[4] Saaty, T. L. *(2001). Fundamentals of Decision Making and Priority Theory*. RWS Publication. Pittsburgh. Pennsylvania.