AMO – Advanced Modeling and Optimization, Volume 17, Number 2, 2015

An Alternative Economic Equilibrium Model with Different Implementation Mechanisms

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Abstract

We consider a general economic equilibrium model with divisible commodities and price functions based on the material balance condition. We show that mechanisms for attaining its solution points are closely related to information exchange schemes attributed to the model within its basic information framework. In particular, it contains features from both perfect and imperfect competitive models and its equilibrium state can be attained within a completely decentralized transaction mechanism. Therefore, we obtain a rather flexible modeling framework for describing various complex economic systems. At the same time, the model admits an equivalent variational inequality formulation, hence there exists a great collection of results for its investigation and solution. We discuss its relationships with some other economic equilibrium models.

Key words: Economic equilibrium model; divisible commodities; price functions; basic information framework; implementation mechanism; information exchange scheme; bilateral transactions; auction model; oligopolistic equilibria.

1 Introduction

The concept of equilibrium plays a central role in various natural and socio-economical sciences. For example, in terms of the classical Newtonian mechanics the equilibrium state for a system means that the impact of all the forces on this system equals zero and this state can be maintained for an indefinitely long period. This formulation enables us to write the corresponding mathematical model, and the solutions of the corresponding problem can be used for evaluation of the behavior of the system. For instance, we can correct the deviation between the current state of the system and the equilibrium state. In the above example, the mathematical model is clearly a system of equations, which admits a solution in an explicit (closed) form. In the presence of certain binding constraints, one can utilize proper generalized equilibrium concepts; see e.g. [Baiocchi and Capelo, 1984], [Konnov, 2007a]. However, further complications in this direction can appear in the case of various systems with active elements having

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*AMO – Advanced Modeling and Optimization. ISSN: 1841-4311**

their own interests and sets of feasible actions, which are typical for socio-economical sciences, where suitable formulations of equilibrium models might give us non-trivial conclusions on the behavior of very complicated systems.

We know that such mathematical models of systems with active elements are most developed in Economics, where perfectly and imperfectly competitive equilibrium models are classical ones and paid most attention; see e.g. [Arrow and Hahn, 1971], [Friedman, 1977], [Okuguchi and Szidarovszky, 1990] and references therein. They reflect different decentralized equilibration mechanisms and information frameworks.

We recall that investigation of imperfectly competitive (or oligopolistic) models dates back to the book by A. Cournot [Cournot, 1838]. In these models, one side of market (say, consumers) is presented by a price function, whereas actions of each agent from the other side (producer) can change the state of the whole system so that utility functions of all the agents depend on these actions. Therefore, the agents utilize information about interests of the others for their rational decisions and the model is usually formulated as the Nash equilibrium problem of a non-cooperative game [Nash, 1951]. Investigation of perfectly competitive models dates back to the book by L. Walras [Walras, 1874]. These models usually describe a market of a great number of similar buyers and traders so that actions of any separate agent can not impact the state of the whole system and any agent does not utilize the information about the behavior of the others. We should recall also the auctions, which are known from the ancient times; see e.g. [Weber, 1985], [Milgrom, 2004]. It is well-known that auction models appear very useful in describing contemporary energy markets and telecommunication systems; see e.g. [Anderson and Philpott, 2002], [Beraldi et al., 2004], [Courcoubetis and Weber, 2003], [Bitsaki et al., 2006]. Observe that the auction models involve the upper level; participants report their offer/bid prices and capacities to a manager for the exchange decision within some auction rules. The auction mechanism however works under minimal information requirements about the behavior of the participants.

A new approach to modeling auction markets with infinitely divisible goods and price functions was proposed in [Konnov, 2006], [Konnov, 2007a], where the equivalence result with a variational inequality problem was established. Its further development and applications are described in [Konnov, 2007b], [Konnov, 2009], [Allevi et al., 2012], [Konnov, 2015a], [Konnov, 2015b]. Being based on these models, we now suggest a general flexible class of economic equilibrium models with divisible commodities and price functions, which is subordinated to the material balance condition. It admits various schemes of information flows among participants (economic agents) and provides adequate implementation mechanisms for each case. In particular, it contains features from both perfect and imperfect competitive models. At the same time, it can be formulated as a variational inequality problem, hence one can utilize the well-developed theory and methods of variational inequalities (see e.g. [Konnov, 2007a] and references therein) for investigation and solution finding in this economic equilibrium model.

2 A single commodity equilibrium model

For the simplicity of exposition, we begin our considerations from a simple equilibrium market model of a homogeneous commodity, which was suggested in [Konnov, 2006], [Konnov, 2007a] for description of a simple two-sided auction market.

The model involves a finite number of traders and buyers of this commodity, their index sets will be denoted by I and J, respectively. For each $i \in I$, the i-th trader has his/her capacity segment $[0, \alpha_i]$ and a price function g_i, so that his/her offer volume x_i belongs to $[0, \alpha_i]$. Similarly, for each $j \in J$, the j-th buyer has his/her capacity segment $[0, \beta_j]$ and a price function h_j, so that his/her bid volume y_j belongs to $[0, \beta_j]$. Then we can define the feasible set of offer/bid volumes

$$D = \left\{ (x, y) \;\middle|\; \sum_{i \in I} x_i = \sum_{j \in J} y_j;\; 0 \leq x_i \leq \alpha_i, i \in I, 0 \leq y_j \leq \beta_j, j \in J \right\}, \tag{1}$$

where $x = (x_i)_{i \in I}, y = (y_j)_{j \in J}$. We suppose that the prices may in principle depend on offer/bid volumes of all the commodities, i.e. $g_i = g_i(x, y)$ and $h_j = h_j(x, y)$. We say that a pair $(\bar{x}, \bar{y}) \in D$ constitutes an *equilibrium point* if $(\bar{x}, \bar{y}) \in D$ and there exists a number $\bar{\lambda}$ such that

$$\begin{array}{ll} g_i(\bar{x}, \bar{y}) \left\{ \begin{array}{ll} \geq \bar{\lambda} & \text{if } \bar{x}_i = 0, \\ = \bar{\lambda} & \text{if } \bar{x}_i \in (0, \alpha_i), \\ \leq \bar{\lambda} & \text{if } \bar{x}_i = \alpha_i, \end{array} \right. & h_j(\bar{x}, \bar{y}) \left\{ \begin{array}{ll} \leq \bar{\lambda} & \text{if } \bar{y}_j = 0, \\ = \bar{\lambda} & \text{if } \bar{y}_j \in (0, \beta_j), \\ \geq \bar{\lambda} & \text{if } \bar{y}_j = \beta_j, \end{array} \right. \\ \text{for } i \in I; & \text{for } j \in J. \end{array} \tag{2}$$

Observe that the number $\bar{\lambda}$ can be treated as a market clearing price, which equilibrates the market and yields also the offer/bid volumes for all the participants. In fact, the minimal offer (bid) volumes correspond to traders (buyers) whose prices are greater (less) than $\bar{\lambda}$, and the maximal offer (bid) volumes correspond to traders (buyers) whose prices are less (greater) than $\bar{\lambda}$. The prices of other participants are equal to $\bar{\lambda}$ and their volumes may be arbitrary within their capacity bounds, but should be subordinated to the balance equation.

In [Konnov, 2006] (see also [Konnov, 2007a], [Konnov, 2007b]), the following basic relation between the equilibrium problem (1)–(2) and a variational inequality (VI, for short) was established.

Proposition 2.1 *(a) If $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies (2) and $(\bar{x}, \bar{y}) \in D$, then $(\bar{x}, \bar{y})$ solves VI: Find $(\bar{x}, \bar{y}) \in D$ such that*

$$\sum_{i \in I} g_i(\bar{x}, \bar{y})(x_i - \bar{x}_i) - \sum_{j \in J} h_j(\bar{x}, \bar{y})(y_j - \bar{y}_j) \geq 0 \quad \forall (x, y) \in D. \tag{3}$$

(b) If a pair $(\bar{x}, \bar{y}) \in D$ solves VI (3), then there exists $\bar{\lambda}$ such that $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies (2).

Therefore, we can apply various results from the theory of VIs or more general equilibrium problems (see, e.g., [Facchinei and Pang, 2003], [Konnov, 2007a], [Konnov, 2013a]) for its investigation and solution.

For instance, if the set D in (1) is nonempty and bounded, the functions g_i and h_j are continuous for all $i \in I$ and $j \in J$, then VI (3) has a solution. In the unbounded case, a suitable coercivity condition can be utilized; see e.g. [Konnov, 2015a].

However, we feel that the model is essentially incomplete without the *basic information framework*, which includes possible contents and directions of information flows among participants, scenarios of information exchange, information uncertainty, etc. This enables us to investigate the main problem, which is to *find an adequate and transparent implementation mechanism for attaining the equilibrium point defined above.* The implementation mechanism is clearly attributed to a suitable information exchange scheme, which falls into the the basic information framework. We observe that several different information exchange schemes can be implemented within one model and one basic informational framework. This assertion will be illustrated in the next sections.

Remark 2.1 *Clearly, the question on existence of a suitable implementation mechanism arises not only for economic models. Investigation of any complex system with active elements having their own interests and sets of feasible actions can not be complete without description of its basic information framework(s) associated to possible behavior and implementation scenarios. At the same time, creation of a formal theory of information frameworks is not a goal of this paper. We hope that it will be now sufficient to briefly describe several essential features in each case.*

3 Auction and centralized regulation mechanisms

Within the *standard auction mechanism*, existing traders and buyers submit their offers and bids (prices and capacities) to an auction manager within a fixed time period (session). After closing the session, the manager determines the cutting price $\bar{\lambda}$ in conformity with rule (2) and tells it to the participants, which also implies all the actual commodity volumes. Therefore, it can be applied even in the case where each participant has no knowledge about features of the others, although this knowledge may appear useful in determining participant's price functions, which contain their desired particular offer/bid relations. Observe that it is not essential for the manager whether the submitted prices and capacities correspond to real abilities of the participants (say, reflect their expenses/utilities) or not. Clearly, the auction manager can take an arbitrary suitable method for finding the equilibrium solution, although further information exchange between the manager and participants is not permitted. In particular, he/she can select an iterative method for solving VI (3). In general, these methods require certain monotonicity assumptions for convergence; see e.g. [Konnov, 2007a], [Konnov, 2007b]. Additional calculation methods appear in the potential case where prices are partial derivatives of some differentiable function f, i.e.

$$g_i(x,y) = \frac{\partial f(x,y)}{\partial x_i},\ i \in I;\ \text{and}\ h_j(x,y) = -\frac{\partial f(x,y)}{\partial y_j},\ j \in J.$$

Then, VI (3) is rewritten as follows:

$$\langle \nabla f(\bar{x},\bar{y}), (x,y) - (\bar{x},\bar{y})\rangle \geq 0 \quad \forall (x,y) \in D; \tag{4}$$

and it yields the optimality condition for the optimization problem:

$$\min_{(x,y)\in D} \to f(x,y). \tag{5}$$

This is the case if the price functions are separable, i.e. $g_i(x,y) = g_i(x_i)$ for each $i \in I$ and $h_j(x,y) = h_j(y_j)$ for each $j \in J$. Then,

$$f(x,y) = \sum_{i\in I} \mu_i(x_i) - \sum_{j\in J} \eta_j(y_j) \tag{6}$$

where

$$\mu_i(x_i) = \int_0^{x_i} g_i(\tau)d\tau,\ i \in I;\ \text{and}\ \eta_j(y_j) = \int_0^{y_j} h_j(\tau)d\tau,\ j \in J. \tag{7}$$

Clearly, μ_i and η_j are treated as utility functions of the participants (traders and buyers).

In the integrable (potential) case, a great number of iterative optimization methods can be applied for calculation of a solution of problem (4) via approximating that of problem (5); see e.g. [Konnov, 2013a]. It is well-known that (5) implies (4); the reverse property is true if for instance the function f is convex. That is, the equilibrium states do not provide the optimal value of the system goal in (5) in general.

Furthermore, these iterative methods can be also treated as dynamic processes of sequential auction decisions. To illustrate this assertion, we take only two well-known methods in the separable case.

The **Frank-Wolfe or conditional gradient method** represents sequential solutions of the corresponding auction problems with fixed prices. At the k-th iteration (session), $k = 0, 1, \ldots$, the participants report their offer/bid prices calculated at some current volumes $(x^k, y^k) \in D$, and the manager solve the auction optimization problem:

$$\begin{array}{l} \min \to \left\{ \sum\limits_{i\in I} g_i(x_i^k)x_i - \sum\limits_{j\in J} h_j(y_j^k)y_j \right\} \\ \text{subject to } (x,y) \in D; \end{array} \tag{8}$$

cf. (4); which gives vectors $(\tilde{x}^k, \tilde{y}^k)$. Then the participants correct their offer/bid volumes as follows:

$$\begin{aligned} x^{k+1} &= \theta_k \tilde{x}^k + (1-\theta_k)x^k, \\ y^{k+1} &= \theta_k \tilde{y}^k + (1-\theta_k)y^k, \ \theta_k \in (0,1); \end{aligned}$$

and so on.

The stepsize parameter θ_k can be chosen in conformity with the rule:

$$\lim_{k\to\infty} \theta_k = 0, \ \sum_{k=0}^{\infty} \theta_k = \infty; \tag{9}$$

which provides convergence under rather mild conditions, including however the boundedness of D; see [Yachimovich, 1980]. We observe that the linear programming problem (8) can be solved in a finite number of iterations by a simple arrangement type procedure and that the participants may not use any information about each other. This means that the manager may report auction decisions separately and in parallel. Nevertheless, this process of repeated sequential auctions with fixed prices and correction rule (9) still approximates a solution of the initial problem (3).

The well-known **gradient projection method** can be written within the above iterative scheme. The main difference is that the participants report affine (linearized) price functions instead of the fixed offer/bid prices.

Namely, at the k-th iteration, $k = 0, 1, \dots$, the participants have some current volumes $(x^k, y^k) \in D$ and report the price functions $\tilde{g}_i(x_i) = g_i(x_i^k) + \theta_k^{-1}(x_i - x_i^k)$ for $i \in I$ and $\tilde{h}_j(y_j) = h_j(y_j^k) - \theta_k^{-1}(y_j - y_j^k)$ for $j \in J$, where $\theta_k > 0$ is a stepsize parameter. Then the manager solves the auction problem: Find $(x^{k+1}, y^{k+1}) \in D$ such that

$$\begin{aligned} &\sum_{i\in I}(g_i(x_i^k) + \theta_k^{-1}(x_i^{k+1} - x_i^k))(x_i - x_i^{k+1}) \\ &- \sum_{j\in J}(h_j(y_j^k) - \theta_k^{-1}(y_j^{k+1} - y_j^k))(y_j - y_j^{k+1}) \geq 0 \\ &\forall (x,y) \in D; \end{aligned} \tag{10}$$

cf. (4); which thus yields the next volume vectors, and so on.

Unlike (8), problem (10) has always a unique solution, and the process with rule (9) converges to a solution of (3) under even weaker conditions; see e.g. [Konnov, 2007a], [Konnov, 2013a].

Therefore, the equilibrium concept in (2) can be implemented within various auction mechanisms so that it was before associated with auction models; see [Konnov, 2007a], [Konnov, 2006], [Konnov, 2007b], [Konnov, 2009], [Allevi et al., 2012], [Konnov, 2015a], [Konnov, 2015b].

In comparison with the above auction mechanisms, the *centralized planning scheme* is also two-level, however, the upper level unit may require peculiarities of technology and financial abilities of lower level participants in order to formulate a system optimization problem such as (5), solve this problem with a suitable iterative method, and then send so derived allocation decisions to the lower level. At the same time, in case of some lack of necessary information or control tools for its influence, the upper level unit may apply some other control mechanisms, which do not require for the participants to report all the detailed information and seem more flexible. In particular, the upper level unit can take a suitable decomposition scheme; see e.g. [Lasdon, 1970], [Minoux, 1989]. In order to illustrate this mechanism, we recall the so-called **price decomposition**. In this method, problem (5)–(6) is replaced by its dual:

$$\max_{\lambda} \rightarrow \quad \psi(\lambda), \tag{11}$$

where

$$\begin{aligned} \psi(\lambda) \quad = \quad & \min_{(x,y)\in X\times Y} \left\{ \left(\sum_{i\in I} \mu_i(x_i) - \sum_{j\in J} \eta_j(y_j) \right) - \lambda \left(\sum_{i\in I} x_i - \sum_{j\in J} y_j \right) \right\}, \\ & X = \prod_{i\in I} [0, \alpha_i], \quad Y = \prod_{j\in J} [0, \beta_j]. \end{aligned} \tag{12}$$

It can be solved with a suitable one-dimensional search method, say, golden section. Given an approximation λ_k, calculation of the value of $\psi(\lambda_k)$ in (12) and its gradient decomposes into a set of one-dimensional problems:

$$\min_{x_i\in[0,\alpha_i]} \rightarrow \ (\mu_i(x_i) - \lambda_k x_i)\,, \ i \in I, \tag{13}$$

$$\max_{y_j\in[0,\beta_j]} \rightarrow \ (\eta_j(y_j) - \lambda_k y_j)\,, \ j \in J. \tag{14}$$

If these problems have unique solutions x_i^k, $i \in I$, and y_j^k, $j \in J$, then

$$\psi'(\lambda_k) = -\left(\sum_{i\in I} x_i^k - \sum_{j\in J} y_j^k \right),$$

and we can even set $\lambda_{k+1} = \lambda_k + \theta_k \psi'(\lambda_k)$ with some $\theta_k > 0$, thus following the Uzawa method; see [Arrow et al., 1958], Ch. 10. These methods for problem (11) also have rather natural interpretation. The upper level unit corrects sequentially the current price λ_k by collecting the private volume decisions x_i^k, $i \in I$, and y_j^k, $j \in J$, and using the balance relation. The price is reported to the lower level participants, whereas they find their response volumes via independent solutions of partial problems (13)–(14) with this given price. Note that again the participants do not use any information about each other.

We observe that in this case the regulator can thus provide convergence to a solution of problem (11), but is not able to identify the problem under solution if the functions μ_i and η_j are unknown to him/her. This means that the price regulation mechanism admits the difference between the preferences of upper and lower level units; see also

[Konnov, 2013b]. Therefore, if the regulator has full and precise information about the genuine utility functions of the lower level units, then the direct solution of problem (5) gives the best result, however, if they can report false data to cover their real purposes, the result of this centralized planning scheme may be far from the optimal one, even in comparison with the above price decomposition scheme, where the influence of the regulator is restricted essentially, but each lower level unit solves separately its partial problem of form (13)–(14) without reporting his/her utility function at all. This example clearly shows that the basic informational framework is very significant for creation of models of this complex system.

4 An alternative equilibrium treatment and its decentralized regulation mechanism

The regulation mechanisms of the previous section, which were destined for attainment of the equilibrium state determined by relations (1)–(2), require the creation of an upper control level. Therefore, it would be worthwhile to find a decentralized mechanism, which could provide the same result within some exchange transactions among the economic agents, i.e. without any upper level regulation. In this section, we will show that the question has positive answer, but this requires another treatment of the market equilibrium concept.

We restrict now our attention for simplicity only to bilateral transactions and say that such a transaction is *useful* for a pair of participants (economic agents) if either

(i) this is a (buyer/trader) pair, their price difference is positive, and they are able to increase their current exchange volumes,

or

(ii) a position of a participant in some transaction is not favorable, but he/she is able to decrease the exchange volume within a feasible transaction, which keeps the capacity and balance conditions.

We say that the market system described in Section 2 attains its *equilibrium state* $(\bar{x}, \bar{y})$ if the following conditions hold:

1. *Participant's activity levels satisfy their individual capacity restrictions*, i.e.

$$0 \leq \bar{x}_i \leq \alpha_i, i \in I, \ 0 \leq \bar{y}_j \leq \beta_j, j \in J.$$

2. *The balance equality holds*, i.e.

$$\sum_{i \in I} \bar{x}_i = \sum_{j \in J} \bar{y}_j.$$

3. *Any useful bilateral transaction is not possible.*

So, any feasible transaction must provide conditions 1–2. Hence, if condition 3 does not hold, economic agents can continue the transaction process, which can be treated as a mechanism for attaining the equilibrium state. It follows that we have to show that the relations in (1)–(2) in fact mean the impossibility of further profitable bilateral transactions.

We first give an equivalent version of conditions (1)–(2) by a specialization of Proposition 2.2 in [Konnov, 2015b].

Proposition 4.1 *(a) If conditions (2) hold for* $(\bar{x}, \bar{y}, \bar{\lambda})$ *and* $(\bar{x}, \bar{y}) \in D$*, then*

$$\forall l \in I, k \in J: \quad g_l(\bar{x}, \bar{y}) - h_k(\bar{x}, \bar{y}) \begin{cases} > 0 & \Longrightarrow \quad \bar{x}_l = 0 \ \text{or} \ \bar{y}_k = 0, \\ < 0 & \Longrightarrow \quad \bar{x}_l = \alpha_l \ \text{or} \ \bar{y}_k = \beta_k; \end{cases} \tag{15}$$

$$\forall k, l \in I, k \neq l: \quad g_l(\bar{x}, \bar{y}) - g_k(\bar{x}, \bar{y}) > 0 \Longrightarrow \quad \bar{x}_l = 0 \ \text{or} \ \bar{x}_k = \alpha_k; \tag{16}$$

$$\forall k, l \in J, k \neq l: \quad h_l(\bar{x}, \bar{y}) - h_k(\bar{x}, \bar{y}) < 0 \Longrightarrow \quad \bar{y}_l = 0 \ \text{or} \ \bar{y}_k = \beta_k. \tag{17}$$

(b) If a pair $(\bar{x}, \bar{y}) \in D$ *satisfies (15)–(17), then there exists* $\bar{\lambda}$ *such that conditions (2) hold.*

Proof. Let conditions (2) hold and $(\bar{x}, \bar{y}) \in D$. Then:

(i) If $l \in I$, $k \in J$, $\bar{x}_l > 0$ and $\bar{y}_k > 0$ with (2) imply $g_l(\bar{x}, \bar{y}) \leq \bar{\lambda} \leq h_k(\bar{x}, \bar{y})$, a contradiction to $g_l(\bar{x}, \bar{y}) > h_k(\bar{x}, \bar{y})$.

(ii) If $l \in I$, $k \in J$, $\bar{x}_l < \alpha_l$ and $\bar{y}_k < \beta_k$ with (2) imply $g_l(\bar{x}, \bar{y}) \geq \bar{\lambda} \geq h_k(\bar{x}, \bar{y})$, a contradiction to $g_l(\bar{x}, \bar{y}) < h_k(\bar{x}, \bar{y})$.

(iii) If $k, l \in I$, $k \neq l$, $\bar{x}_l > 0$ and $\bar{x}_k < \alpha_k$ with the first relation in (2) imply $g_l(\bar{x}, \bar{y}) \leq \bar{\lambda} \leq g_k(\bar{x}, \bar{y})$, a contradiction to $g_l(\bar{x}, \bar{y}) > g_k(\bar{x}, \bar{y})$.

(iv) If $k, l \in J$, $k \neq l$, $\bar{y}_l > 0$ and $\bar{y}_k < \beta_k$ with the second relation in (2) imply $h_l(\bar{x}, \bar{y}) \geq \bar{\lambda} \geq h_k(\bar{x}, \bar{y})$, a contradiction to $h_l(\bar{x}, \bar{y}) < h_k(\bar{x}, \bar{y})$.

Therefore, assertion (a) is true.

Conversely, let a pair $(\bar{x}, \bar{y}) \in D$ satisfy (15)–(17). We rewrite (15)–(17) equivalently as follows:

$$\forall l \in I, k \in J: \qquad \bar{x}_l \in (0, \alpha_l], \bar{y}_k \in (0, \beta_k] \Longrightarrow g_l(\bar{x}, \bar{y}) \leq h_k(\bar{x}, \bar{y}), \tag{18}$$

$$\forall l \in I, k \in J: \qquad \bar{x}_l \in [0, \alpha_l), \bar{y}_k \in [0, \beta_k) \Longrightarrow g_l(\bar{x}, \bar{y}) \geq h_k(\bar{x}, \bar{y}), \tag{19}$$

$$\forall k, l \in I, k \neq l: \qquad \bar{x}_l \in (0, \alpha_l], \bar{x}_k \in [0, \alpha_k) \Longrightarrow g_l(\bar{x}, \bar{y}) \leq g_k(\bar{x}, \bar{y}), \tag{20}$$

$$\forall k, l \in J, k \neq l: \qquad \bar{y}_l \in (0, \beta_l], \bar{y}_k \in [0, \beta_k) \Longrightarrow h_l(\bar{x}, \bar{y}) \geq h_k(\bar{x}, \bar{y}). \tag{21}$$

Define the index sets: $I_- = \{i \in I \mid \bar{x}_i = 0\}$, $I_0 = \{i \in I \mid \bar{x}_i \in (0, \alpha_i)\}$, $I_+ = \{i \in I \mid \bar{x}_i = \alpha_i\}$, and $J_- = \{j \in J \mid \bar{y}_j = 0\}$, $J_0 = \{j \in J \mid \bar{y}_j \in (0, \beta_j)\}$, $J_+ = \{j \in J \mid \bar{y}_j = \beta_j\}$. Then the following cases are possible.

Case 1: $I_0 \neq \varnothing$. Set $\bar{\lambda} = g_t(\bar{x}, \bar{y})$ for some $t \in I_0$, then $\bar{\lambda} = g_i(\bar{x}, \bar{y})$ for any $i \in I_0$ due to (20). Next, taking $k \in I_-$ in (20) gives $g_k(\bar{x}, \bar{y}) \geq \bar{\lambda}$, whereas taking $l \in I_+$ in (20) gives $g_l(\bar{x}, \bar{y}) \leq \bar{\lambda}$ as well. These relations yield the first relation in (2). Besides, (18)–(19) then give the second relation in (2).

Case 2: $J_0 \neq \varnothing$. It is argued similarly. Set $\bar{\lambda} = h_t(\bar{x}, \bar{y})$ for some $t \in J_0$, then $\bar{\lambda} = h_j(\bar{x}, \bar{y})$ for any $j \in J_0$ due to (21). Next, taking $k \in J_-$ in (21) gives $h_k(\bar{x}, \bar{y}) \leq \bar{\lambda}$, whereas taking $l \in J_+$ in (21) gives $h_l(\bar{x}, \bar{y}) \geq \bar{\lambda}$ as well. These relations yield the second relation in (2). Besides, (18)–(19) then give the first relation in (2).

Case 3: $I_0 = \varnothing$ and $J_0 = \varnothing$. Set $\tau_1' = \max_{i \in I_+} g_i(\bar{x}, \bar{y})$ and $\tau_2' = \min_{i \in I_-} g_i(\bar{x}, \bar{y})$, then (20) gives $\tau_1' \leq \tau_2'$. Similarly, set $\tau_1'' = \max_{j \in J_-} h_j(\bar{x}, \bar{y})$ and $\tau_2'' = \min_{j \in J_+} h_j(\bar{x}, \bar{y})$, then (21) gives $\tau_1'' \leq \tau_2''$. Besides, (18) and (19) yield $\tau_1' \leq \tau_2''$ and $\tau_1'' \leq \tau_2'$, respectively. Hence, there exists a number

$$\bar{\lambda} \in [\tau_1', \tau_2'] \cap [\tau_1'', \tau_2''],$$

and all the conditions in (2) hold. Therefore, assertion (b) is also true. □

If the conditions in (15)–(17) are not satisfied, the participants can make some profitable bilateral transactions or change their current volumes toward such a transaction, thus creating a **decentralized (self-regulation) transaction mechanism**. We suppose that economic agents can strike preliminary (or virtual) bargains but fix them only after attaining an equilibrium state. Let $(x, y) \in D$ be some pair of current preliminary transaction volumes. Then the following cases may occur:

(i) There exist $l \in I, k \in J$ such that $x_l < \alpha_l$, $y_k < \beta_k$, and $g_l(x, y) < h_k(x, y)$. Trader l and buyer k can make a profitable bilateral transaction and increase their volumes, i.e. set $x_l := x_l + \varepsilon$ and $y_k := y_k + \varepsilon$ for some $\varepsilon > 0$.

(ii) There exist $l \in I, k \in J$ such that $x_l > 0$, $y_k > 0$, and $g_l(x, y) > h_k(x, y)$. This situation is not favorable for both trader l and buyer k. They can make a bilateral transaction and decrease their volumes, i.e. set $x_l := x_l - \varepsilon$ and $y_k := y_k - \varepsilon$ for some $\varepsilon > 0$.

(iii) There exist $k, l \in I, k \neq l$ such that $x_l > 0$, $x_k < \alpha_k$, and $g_l(x, y) > g_k(x, y)$. This situation is not favorable for trader l. The traders can make a bilateral transaction which reduces the volume of trader l but maintains the balance, namely, set $x_l := x_l - \varepsilon$ and $x_k := x_k + \varepsilon$ for some $\varepsilon > 0$.

(iv) There exist $k, l \in J, k \neq l$, such that $y_l > 0$, $y_k < \beta_k$, and $h_l(x, y) < h_k(x, y)$. This situation is not favorable for buyer l. The buyers can make a bilateral transaction which reduces the volume of trader l but maintains the balance, namely, set $y_l := y_l - \varepsilon$ and $y_k := y_k + \varepsilon$ for some $\varepsilon > 0$.

Thus, the participants can make simultaneous changes of their volumes in order to maintain the material balance and to provide a profitable transaction (case (i)) or to move to a more favorable position for such transactions (cases (ii)–(iv)) at the market. At the same time, all these transactions become impossible if the equilibrium conditions in (2), or equivalently, in (15)–(17), are fulfilled. We can suppose that these changes yield an equilibrium state after a finite number of steps. In general, this question needs further investigations, however, the positive answer can be substantiated in the case of absence of upper capacity bounds and one-dimensional price functions for participants; see [Konnov, 2014]. For this reason, we will consider this case in more detail.

5 A simplified exchange model

So, we return to the model described in Section 2, but suppose now that $\alpha_i = \beta_j = +\infty$ for all $i \in I$ and $j \in J$ and that the price function of each participant depends only on his/her volume, i.e. $g_i = g_i(x_i)$ for $i \in I$ and $h_j = h_j(y_j)$ for $j \in J$. This situation may arise in the case when each participant does not have the information (or finds it useless) about the behavior and features of the others, which yields the integrability of the price functions; see (5)–(7).

Then we have the feasible set

$$\widetilde{D} = \left\{ (x,y) \,\middle|\, \sum_{i \in I} x_i = \sum_{j \in J} y_j;\ x_i \geq 0, i \in I,\ y_j \geq 0, j \in J \right\} \tag{22}$$

and the equilibrium conditions

$$\begin{array}{ll} g_i(\bar{x}_i) \left\{ \begin{array}{ll} \geq \bar{\lambda} & \text{if } \bar{x}_i = 0, \\ = \bar{\lambda} & \text{if } \bar{x}_i > 0, \end{array} \right. & h_j(\bar{y}_j) \left\{ \begin{array}{ll} \leq \bar{\lambda} & \text{if } \bar{y}_j = 0, \\ = \bar{\lambda} & \text{if } \bar{y}_j > 0, \end{array} \right. \\ \text{for } i \in I; & \text{for } j \in J; \end{array} \tag{23}$$

instead of (1)–(2), respectively. Further, Proposition 2.1 is rewritten as follows.

Proposition 5.1 *(a) If $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies (23) and $(\bar{x}, \bar{y}) \in \widetilde{D}$, then $(\bar{x}, \bar{y})$ solves VI: Find $(\bar{x}, \bar{y}) \in \widetilde{D}$ such that*

$$\sum_{i \in I} g_i(\bar{x}_i)(x_i - \bar{x}_i) - \sum_{j \in J} h_j(\bar{y}_j)(y_j - \bar{y}_j) \geq 0 \quad \forall (x,y) \in \widetilde{D}. \tag{24}$$

(b) If a pair $(\bar{x}, \bar{y}) \in \widetilde{D}$ solves VI (24), then there exists $\bar{\lambda}$ such that $(\bar{x}, \bar{y}, \bar{\lambda})$ satisfies (23).

We now give an equivalent version of conditions (22)–(23) as a specialization of Proposition 5.1 in [Konnov, 2015b].

Proposition 5.2 *A pair $(\bar{x}, \bar{y}) \in \widetilde{D}$ satisfies (23) if and only if the following conditions hold:*

$$\forall i \in I, j \in J: \quad g_i(\bar{x}_i) - h_j(\bar{y}_j) \left\{ \begin{array}{lll} > 0 & \Longrightarrow & \bar{x}_i = 0 \text{ or } \bar{y}_j = 0, \\ \geq 0 & \Longleftrightarrow & (\bar{x}, \bar{y}) \in \widetilde{D}. \end{array} \right. \tag{25}$$

Proof. Let conditions (23) hold and $(\bar{x},\bar{y}) \in \widetilde{D}$. Then $g_i(\bar{x}_i) \geq h_j(\bar{y}_j)$ for all $i \in I$, $j \in J$. Besides, if $\bar{x}_i > 0$ and $\bar{y}_j > 0$, then (23) implies $g_i(\bar{x}_i) = h_j(\bar{y}_j)$. Hence, (23) implies (25). Let now a pair $(\bar{x},\bar{y}) \in \widetilde{D}$ satisfy (25). Set

$$\alpha' = \min_{i\in I} g_i(\bar{x}_i), \alpha'' = \max_{j\in J} h_j(\bar{y}_j),$$

then $\alpha' \geq \alpha''$. If $\bar{x}_i = 0$ for all $i \in I$, then $\bar{y}_j = 0$ for all $j \in J$ and conversely. Then taking any $\alpha \in [\alpha'',\alpha']$ yields (23). Otherwise, there exists at least one pair of indices $k \in I$, $l \in J$ such that $\bar{x}_k > 0$ and $\bar{y}_l > 0$. Then setting $\alpha = \alpha' = \alpha''$ again yields (23). □

Comparing conditions (15)–(17) and (25), we conclude that the **self-regulation transaction mechanism** can involve now only two kinds of feasible bilateral transactions (instead of four in Section 4):

(i) There exist $l \in I, k \in J$ such that $g_l(x_l) < h_k(y_k)$. Trader l and buyer k can make a profitable bilateral transaction and increase their volumes, i.e. set $x_l := x_l + \varepsilon$ and $y_k := y_k + \varepsilon$ for some $\varepsilon > 0$.

(ii) There exist $l \in I, k \in J$ such that $x_l > 0$, $y_k > 0$, and $g_l(x_l) > h_k(y_k)$. This situation is not favorable for both trader l and buyer k. They can make a bilateral transaction and decrease their volumes, i.e. set $x_l := x_l - \varepsilon$ and $y_k := y_k - \varepsilon$ for some $\varepsilon > 0$.

Any transactions between two traders or two buyers become not necessary. Being based on these transactions and following [Konnov, 2014], we can create the following adaptive exchange procedure.

Adaptive bi-coordinate algorithm.
Initialization: Choose a pair $(\tilde{x}^0,\tilde{y}^0) \in \widetilde{D}$ and sequences $\{\delta_l\} \searrow 0$, $\{\tau_l\} \searrow 0$. Set $l := 1$.
Basic cycle:
Step 0: Set $k := 0$, $x^0 := \tilde{x}^{l-1}$, $y^0 := \tilde{y}^{l-1}$.
Step 1: Set $x^{k+1} := x^k$, $y^{k+1} := y^k$. If there is a pair of indices $i \in I$, $j \in J$ such that

$$g_i(x_i^k) - h_j(y_j^k) \leq -\delta_l,$$

set $x_i^{k+1} := x_i^k + \tau_l$, $y_j^{k+1} := y_j^k + \tau_l$, $k := k+1$ and go to Step 1.
Step 2: If there is a pair of indices $i \in I$, $j \in J$ such that $x_i^k \geq \tau_l$, $y_j^k \geq \tau_l$, and

$$g_i(x_i^k) - h_j(y_j^k) \geq \delta_l,$$

set $x_i^{k+1} := x_i^k - \tau_l$, $y_j^{k+1} := y_j^k - \tau_l$, $k := k+1$ and go to Step 1. Otherwise stop the basic cycle.
Restart: Set $\tilde{x}^l := x^k$, $\tilde{y}^l := y^k$, $l := l+1$ and begin the basic cycle.

At each iteration of the basic cycle, a selected (trader/buyer) pair makes a transaction if the modulus of their current price difference is large enough, i.e. δ_l serves for a decision threshold, and τ_l serves for a sufficient stepsize. If the (buyer/trader) price difference is positive, i.e. $h_j(y_j^k) - g_i(x_i^k) \geq \delta_l$, they increase simultaneously their volumes at Step 1. If this price difference is negative, i.e. $g_i(x_i^k) - h_j(y_j^k) \geq \delta_l$, they decrease simultaneously their volumes at Step 2. That is, the participants make comparisons of bilateral price differences that can result in changing their volumes. Besides, termination of the basic cycle means that the current threshold values is too big and need certain reduction. This reduction reflects behavior features of the participants, who intend to continue the transaction process. Afterwards, the basic cycle begins with a lower threshold (restart) and so on. We will show that this algorithm, which seems close to certain implementation of the well known A. Smith "invisible hand" process (see e.g. [Arrow and Hahn, 1971]), yields a sequence of volumes converging to an equilibrium point, but not to an optimal point in general.

We observe that problem (24) (or equivalently, (23) or (25)) yields the optimality condition for the optimization problem:

$$\min_{(x,y)\in\widetilde{D}} \to f(x,y) = \sum_{i\in I} \mu_i(x_i) - \sum_{j\in J} \eta_j(y_j); \tag{26}$$

where the functions μ_i and η_j are defined in (7); cf. (5)–(6). That is, (26) implies (24), the reverse implication is true if all the functions μ_i and $-\eta_j$ are convex.

The above algorithm can be substantiated under certain additional assumptions. First we suppose that all the price price functions f_i and h_j are Lipschitz continuous. Next, since the set $\widetilde{D}$ in (22) may be unbounded, we suppose that the function f is coercive on a set $\widetilde{D}$, i.e. $\{f(x^k, y^k)\} \to +\infty$ for any sequence $\{(x^k, y^k)\} \subset \widetilde{D}$, $\{\|(x^k, y^k)\|\} \to \infty$.

Then, using the results from [Konnov, 2014], we obtain the following convergence property.

Theorem 5.1 *Under the assumptions indicated it holds that:*

(a) for each l, the number of changes of index k in the basic cycle is finite;

(b) the sequence $\{(\tilde{x}^l, \tilde{y}^l)\}$ has limit points, all these limit points are solutions of VI (24).

6 Relationships with some other basic equilibrium models

We begin our comparisons from the well known perfectly and imperfectly competitive economic equilibrium models.

We recall that any *Walrasian equilibrium model* describes a market of a great number of similar buyers and traders so that actions of any separate agent can not impact the state of the whole system and any agent does not utilize information about the behavior of the others. That is, each agent accepts the general price values and then determines his/her personal demand or supply values. These values create the general market excess demand value that has certain impact on the prices. Namely, a positive (negative) excess demand forces the price to increase (decrease), thus defining the so-called tâtonnement process; see e.g. [Arrow and Hahn, 1971]. Hence, the model essentially exploits (in fact, postulates) the assumption that there exists a general (market) price value for each commodity, which can be recognized at any moment by any separate agent. Next, the equilibrium conditions in the Walrasian model determine complementarity type relationships between the price and market excess demand values (see e.g. [Arrow and Hahn, 1971]), whereas the equilibrium conditions in Sections 2 and 4 postulate the balance of supply and demand (i.e. zero market excess demand) at any moment and certain equilibrium conditions for prices. That is, they have reverse relations for prices and quantities. Besides, the implementation mechanisms suggested in Sections 4 and 5 seem different from the Walrasian tâtonnement process. In addition we observe that within the information framework of the model of Sections 2 and 4, each participant may in principle utilize information about the behavior of the others, unlike the Walrasian model.

We recall that any *imperfectly competitive (or oligopolistic) model* describes usually a market of a homogeneous commodity with uneven roles of market sides. In fact, one side of market (say, consumers) is presented by a price function, whereas actions of each agent from the other side (traders or producers) can change the state of the whole system so that utility functions of all these agents depend on their actions. Hence, the agents are forced to utilize information about interests and abilities of the others for their rational decisions and the model is usually formulated as the Nash equilibrium problem of a non-cooperative game; see e.g. [Friedman, 1977], [Okuguchi and Szidarovszky, 1990]. Therefore, this concept involves the balance condition implicitly via the price function of the subordinated side. We can conclude that the model from Section 2 foresees more active role of agents of both the sides and their individual actions. At the same time, the equilibrium concept from Section 2 can be used even if the participant's behavior information is very limited and not reported to others or restricted by simple bilateral transactions.

We observe that the usual strategic non-cooperative game framework assumes that players have a sufficient information about the interests and abilities of others as well as the dependence of their utility functions from other strategies in order to make rational decisions, although the personal choice remains still unknown. We recall that the well known fictitious play algorithm, which can be viewed as a dynamic game modeling process (see [Brown, 1951], [Belen'kii and Volkonskii 1974]), consists in simultaneous choices of best responses by all the players on the current state with proper shifts to the previous decisions. This also implies that each player receives the information about the current decisions of all the others. These assumptions seem more restrictive in comparison with the bilateral transaction mechanism.

We should also mention a class of *imperfectly competitive models with* so-called *conjectural variations* [Bowley, 1924], [Dixon and Somma, 2003], where each active agent inserts additional terms in his/her utility (payoff) function, which reflect possible reactions of others on his/her decisions. Clearly, agents' price functions in the presented model may also involve variables related to other agents, but their roles are different in general. In the imperfectly competitive models, conjectural variation terms describe some presentations of an agent about possible actions of others, i.e. it is supposed that the agent is able to create such an adequate model, which clearly requires rather detailed information about the whole system. In the model from Section 2 each agent only indicates his/here desirable relations for prices and volumes and can utilize them in his/her actions, but again the presence of these terms in the price functions can be provided by very limited information.

Remark 6.1 *In addition we should notice that agents' price functions may reflect various strategies rather than only the marginal strategy determined by the profit maximization, which is usually attributed to the above perfectly and imperfectly competitive economic equilibrium models. For instance, if an agent has no sufficient information about the others, the uncertainty does not make the marginal strategy very reasonable. It then seems more suitable for this agent to provide some satisfactory income level that admits a broader feasible set of possible actions and is closer to the previous volume decisions. The satisfactory approach to rationality is also well known; see [Simon, 1978].*

In order to illustrate some applications of the presented model we give two additional examples.

Example 6.1 (Transportation mode choice) Suppose for simplicity that there is an alternative for each citizen to either use public transport or take a private car as a transportation mode in a city, so that the price (dis-utility) in each case depends on the distribution of shares x_1 and x_2 of all the citizens between these modes.

First we consider the case when all the citizens must make this choice. Then we have the feasible set

$$X = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 + x_2 = 1, x_1 \geq 0,\ x_2 \geq 0 \right\}.$$

Let $a_i(x_i)$ be the price for the i-th mode if the share of citizens who chose it is equal to x_i, $i = 1, 2$. Since the population may be too large, any separate citizen will have rather small impact on the whole situation, so that it is natural to suppose that each citizen will not take into account information about actions of other separate citizens. Therefore, the presented model becomes more suitable than the game-theoretic ones; cf. [Moulin, 1981]. The equilibrium point $\bar{x} \in X$ is defined by the conditions:

$$a_i(\bar{x}_i) \left\{ \begin{array}{ll} \geq \bar{\lambda} & \text{if } \bar{x}_i = 0, \\ = \bar{\lambda} & \text{if } \bar{x}_i > 0; \end{array} \right. \text{ for } i = 1, 2;$$

for some $\bar{\lambda}$; cf. (2). This model corresponds to a one-side resource allocation scheme, which is a particular case of the two-side market; see e.g. [Konnov, 2014], [Konnov, 2015a].

Let us consider now the case when the citizens have their price function $h(y)$, where y denotes the share of citizens who agree to make the mode choice at this price level. Then we have the feasible set

$$W = \left\{(x,y) \in \mathbb{R}^3 \,\middle|\, x_1 + x_2 = y,\ x_1 \geq 0,\ x_2 \geq 0,\ 0 \leq y \leq 1\right\}.$$

Again, since the population may be too large, the presented model becomes more suitable, but now we have the two-side market model. The equilibrium pair $(\bar{x},\bar{y}) \in W$ is defined by the conditions:

$$a_i(\bar{x}_i)\left\{\begin{array}{ll} \geq \bar{\lambda} & \text{if } \bar{x}_i = 0, \\ = \bar{\lambda} & \text{if } \bar{x}_i > 0; \end{array}\right. \text{ for } i = 1,2; \quad h(\bar{y})\left\{\begin{array}{ll} \leq \bar{\lambda} & \text{if } \bar{y} = 0, \\ = \bar{\lambda} & \text{if } \bar{y} \in (0,1), \\ \geq \bar{\lambda} & \text{if } \bar{y} = 1, \end{array}\right.$$

for some $\bar{\lambda}$; cf. (2). Due to Proposition 2.1, we can rewrite both these problems as variational inequalities. In fact, the two-side equilibrium conditions with price functions are equivalent to VI: Find a pair $(\bar{x},\bar{y}) \in W$ such that

$$a_1(\bar{x}_1)(x_1 - \bar{x}_1) + a_2(\bar{x}_2)(x_2 - \bar{x}_2) - h(\bar{y})(y - \bar{y}) \geq 0 \quad \forall (x,y) \in W.$$

Example 6.2 (Oligopolistic market problem) The model involves n firms supplying a homogeneous product whose price p depends on its quantity σ, i.e. $p = p(\sigma)$ is the inverse demand function of consumers. Next, the value $c_i(x_i)$ represents the i-th firm total cost of supplying x_i units of the product. As usual, each output level x_i is supposed to be nonnegative and in principle bounded from above, i.e. $x_i \in X_i = [0, \alpha_i]$ with $\alpha_i \leq +\infty$ for $i \in I = \{1, ..., n\}$; see e.g. [Okuguchi and Szidarovszky, 1990]. This means that we have a noncooperative game of n players where each i-th player has the strategy set X_i and the payoff function $f_i(x) = x_i p_i\left(\sum_{j\in I} x_j\right) - c_i(x_i)$, where $x = (x_1, \ldots, x_n)^\top$. We recall that a vector $x^* \in X = \prod_{i\in I} X_i$, is said to be a *Nash-Cournot equilibrium point* if

$$f_i(x^*_{-i}, v_i) \leq f_i(x^*) \quad \forall v_i \in X_i, i \in I.$$

However, we can obtain the same equilibrium solution within the presented model if we suppose in addition that the functions p and c_i are differentiable and that each function f_i is concave in x_i; see [Konnov, 2015c]. In fact, let us consider the market with n traders (firms) and a unique buyer (consumer), having the price function $h(y) = p(y)$. Each i-th trader thus chooses some offer value x_i in his/her capacity segment $[0, \alpha_i]$ and has the price function $g_i(x,y) = c'_i(x_i) - x_i p'(y)$ for $i \in I$. Then we have the feasible set

$$W = \left\{(x,y) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i\in I} x_i = y;\ x_i \in [0, \alpha_i], i \in I,\ y \geq 0\right\}.$$

The equilibrium pair $(\bar{x},\bar{y}) \in W$ is defined by the conditions:

$$g_i(\bar{x},\bar{y}) \left\{ \begin{array}{ll} \geq \bar{\lambda} & \text{if } \bar{x}_i = 0, \\ = \bar{\lambda} & \text{if } \bar{x}_i \in (0,\alpha_i), \\ \leq \bar{\lambda} & \text{if } \bar{x}_i = \alpha_i, \end{array} \right. \text{ for } i \in I; \quad h(\bar{y}) \left\{ \begin{array}{ll} \leq \bar{\lambda} & \text{if } \bar{y} = 0, \\ = \bar{\lambda} & \text{if } \bar{y} > 0, \end{array} \right.$$

for some $\bar{\lambda}$; cf. (2). Due to Proposition 2.1, this problem is equivalent to VI: Find a pair $(\bar{x},\bar{y}) \in W$ such that

$$\sum_{i\in I} g_i(\bar{x},\bar{y})(x_i - \bar{x}_i) - h(\bar{y})(y - \bar{y}) \geq 0 \quad \forall (x,y) \in W.$$

In this model the unique buyer representing all the customers has just their price function $p(y)$, whereas the price function $g_i(x,y)$ of the i-th trader involves the marginal cost $c_i'(x_i)$ at each output level and the extrapolated value of the desired marginal profit $-x_i p'(y)$. Since the inverse demand $p(y)$ is usually non-increasing, the second term must be non-negative.

If all the prices g_i and h_j are fixed, conditions (2) or VI (3) yield the linear programming problem

$$\begin{array}{l} \min \to \quad \left\{ \sum\limits_{i\in I} g_i x_i - \sum\limits_{j\in J} h_j y_j \right\} \\ \text{subject to } (x,y) \in D, \end{array}$$

where D is defined in (1). However, the previous examples show that this problem is not then sufficient to adequately describe the agent and market behavior. Appearance of just price functions rather than fixed values can be invoked not only by agents' actions and intentions, but by a complexity level of the market system. It was shown in [Konnov, 2015a], [Konnov, 2015b] that general network flow and spatial price equilibrium problems, and resource allocation problems in wireless communication networks also fall into format (1)–(2), they containing price functions due to the complex nature of the systems. Hence, these models admit a decentralized regulation mechanism based on bilateral transactions, however, the problem with fixed prices can be treated as a very particular case of the general model.

7 A vector multi-commodity equilibrium problem

We now describe a multi-commodity vector extension of the equilibrium model given in Section 2, which follows that in [Konnov, 2015b]. Again denote by I and J the finite index sets of traders and buyers at the market. For each s-th commodity, $s = 1,\dots,n$, each i-th trader chooses some volume x_{is} in his/her capacity segment $[0,\alpha_{is}]$. Similarly, for each $j \in J$, the j-th buyer chooses some volume y_{js} in his/her capacity segment $[0, \beta_{js}]$. For brevity, we set $x_{(s)} = (x_{is})_{i \in I}$, $y_{(s)} = (y_{js})_{j \in J}$,

$$X_{(s)} = \prod_{i \in I} [0, \alpha_{is}] \text{ and } Y_{(s)} = \prod_{j \in J} [0, \beta_{js}].$$

We can thus define the feasible set

$$W = W_1 \times \cdots \times W_n,$$

where

$$W_s = \left\{ w_{(s)} = (x_{(s)}, y_{(s)}) \in X_{(s)} \times Y_{(s)} \;\middle|\; \sum_{i \in I} x_{is} = \sum_{j \in J} y_{js} \right\}$$

for $s = 1, \dots, n$. Similarly, we define

$$w = (w_{(1)}, \dots, w_{(n)}).$$

We note that the usual price can be treated as a dis-utility (negative validity) for consumers and that it seems worthwhile for the dis-utility of a separate commodity to be reflected by several different factors, hence it is worthwhile to take vector-valued validity functions instead of scalar price ones. Namely, we will suppose that, for each s-th commodity, each i-th trader has a validity mapping $G_{is} : W \to \mathbb{R}^{m_s}$ and each j-th buyer has a validity mapping $H_{js} : W \to \mathbb{R}^{m_s}$. In other words, we suppose that each commodity is equivalent to a separate class with several (m_s) continuously changing properties, reflecting its (negative) validity. Hence, the price is only one of such properties, which are essential for exchange market actions.

Further, we suppose that there is some ordering $\succ_s$ in each space $\mathbb{R}^{m_s}$ induced by some convex cone P_s, i.e.

$$a \succ_s b \iff a - b \in P_s.$$

We say that a vector $\bar{w} \in W$ is a *vector equilibrium point* if it satisfies the following conditions:

$$\forall l \in I, k \in J : \quad G_{ls}(\bar{z}) - H_{ks}(\bar{z}) \begin{cases} \succ_s \mathbf{0} \implies & \bar{x}_{ls} = 0 \text{ or } \bar{y}_{ks} = 0, \\ \prec_s \mathbf{0} \implies & \bar{x}_{ls} = \alpha_{ls} \text{ or } \bar{y}_{ks} = \beta_{ks}; \end{cases} \tag{27}$$

$$\forall k, l \in I, k \neq l : \quad G_{ls}(\bar{z}) - G_{ks}(\bar{z}) \succ_s \mathbf{0} \implies \quad \bar{x}_{ls} = 0 \text{ or } \bar{x}_{ks} = \alpha_{ks}; \tag{28}$$

$$\forall k, l \in J, k \neq l : \quad H_{ls}(\bar{z}) - H_{ks}(\bar{z}) \prec_s \mathbf{0} \implies \quad \bar{y}_{ls} = 0 \text{ or } \bar{y}_{ks} = \beta_{ks}; \tag{29}$$

for $s = 1, \dots, n$; cf. (15)–(17).

In order to formulate a suitable variational inequality as in the scalar case, we need a unique space $\mathbb{R}^m$ of validity properties and a unique ordering cone P that induces a proper ordering $\succ$. For instance, we can collect all the cones together, then

$$m = \sum_{i=1}^{n} m_s.$$

Next, if we replace $\succ_s$ with $\succ$ in (27)–(29), these relations are implied by the following vector inequality: Find a point $\bar{w} \in W$ such that

$$\sum_{s=1}^{n} \left[\sum_{i \in I} (x_{is} - \bar{x}_{is}) G_{is}(\bar{z}) - \sum_{j \in J} (y_{js} - \bar{y}_{js}) H_{js}(\bar{z}) \right] \not\succ \mathbf{0} \quad \forall w \in W;$$

although the reverse implication does not hold in general; see [Konnov, 2015b, Theorem 3.1]. These results show that the self-regulation transaction mechanism described in Section 4 (see (i)–(iv)) is applicable here, so that the above general vector model corresponds to the proposed market equilibrium concept.

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