AMO - Advanced Modeling and Optimization, Volume 17, Number 2, 2015

An oddity property for two-person games

Christian Bidard*

Abstract

The parametric Lemke algorithm is used to show the existence of an odd number of solutions of a generalized bimatrix game in a certain domain. These solutions are classified into two types according to the relative sign of two determinants. The British economist David Ricardo made an implicit use of that algorithm at the beginning of the nineteenth century.

Keywords. Complementarity problems, Generalized bimatrix game, Oddity, Parametric Lemke algorithm, Ricardo

AMS classification. 01A55, 90-03, 90C33, 91A05

1 Introduction

The Lemke algorithm was first elaborated to find solutions of bimatrix games (Lemke [8]). One of its variants, the parametric Lemke algorithm, follows the deformations of a solution $(w(t), z(t))$ of a linear complementarity problem $LCP(q, M)$ when $q = q(t)$ varies with a parameter t. An important property is that the distribution of the zeroes in $w(t)$ and $z(t)$ remains the same by intervals. A critical point is reached when some positive component of $w(t)$ or $z(t)$ vanishes and, then, a switch in one basic variable is required. As long as a new basic variable can indeed be defined at critical points, a solution for $q = q(0)$ is transferred and allows us to obtain a solution for $q = q(1)$. We return to that algorithm for a two-person game which is a nonlinear extension of bimatrix games (Sections 2 and 3). We show the connection between its working and the existence of an odd number of solutions in a certain domain, with a classification

*EconomiX, University Paris Ouest, 200 avenue de la République, F-92001 Nanterre. Tel: +(33) 1 40 97 59 47. E-mail: christian.bidard@u-paris10.fr

of the solutions into two types according to the relative sign of two determinants (Section 4). That (generic) oddity result was left unnoticed even for bimatrix games. In the historical Section 5, we show that the algorithm was first applied by the economist David Ricardo at the beginning of the nineteenth century and we examine the economic consequences of the presence of antitone moves.

2 Existence

The two-person games G (or, later, $G(d)$) we consider are complementarity problems which, either directly or as a result of an optimization criterion, are of the type

$$\begin{aligned} f(x) &\geq c \quad [y] && (1)\\ g(y) &\geq d \quad [x] && (2)\\ x &\geq 0, y \geq 0 && (3) \end{aligned}$$

where x is an $n \times 1$ vector in an open set $\Omega_f \subset R^n$, with $R^n_+ \subset \Omega_f$, $f : \Omega_f \to R^l$ a continuous function, c an $l \times 1$ vector, y an $l \times 1$ vector in an open set Ω_g, with $R^l_+ \subset \Omega_g$, $g : \Omega_g \to R^n$ a continuous function, and $d \in R^n$ is an $n \times 1$ vector (notation ≥ 0 means nonnegativity, > 0 semipositivity, $>> 0$ positivity). The bimatrix game is obtained when f and g are respectively represented by $l \times n$ and $n \times l$ matrices A and B. Theorem 1 below generalizes a well known existence result for bimatrix games, the extension of assumption $A + B^T \geq 0$ being

$$\forall x \geq 0 \;\; \forall y \geq 0 \;\; y^T f(x) + x^T g(y) \geq 0 \tag{4}$$

Its proof makes use of the following version of the Gale-Nikaido-Debreu lemma (Gale [7], Nikaido [9], Debreu [5]):

Lemma 1 *(GND Lemma). Let S be a compact convex set in R^m and $s : p \in S \to s(p) \in R^m$ be a continuous function satisfying the Walras identity $p^T s(p) = 0$. There exists a vector $p_0 \in S$ such that $p^T s(p_0) \geq 0$ for any $p \in S$.*

Proof. Let C be a compact convex set in R^m containing the image set $s(S)$ and $\varphi : C \to S$ be the upper hemi-continuous correspondence defined by

$$\varphi(x) = \left\{ p; p \in S,\, p^T x = \min_{\pi \in S} \pi^T x \right\}$$

By the Kakutani fixed point theorem, the correspondence $(p, x) \to (\varphi(x), s(p))$ from $S \times C$ into itself admits a fixed point $(p_0, s_0 = s(p_0))$, and p_0 has the required property. ■

If S is the unit simplex, the conclusion is that p_0 is a solution of the complementarity problem $s(p) \geq 0$, $p \geq 0$, $p^T s(p) = 0$. When the Walras identity does not hold, the idea of the following proof (see also Bidard [1, 2]) is to force it by considering the function $\overline{s}(p,t) = (ts(t^{-1}p), -ps(t^{-1}p))$.

Definition 1 *A continuous function f defined on R^n_+ is infra-homogeneous if $f(\lambda x) \leq \lambda f(x)$ for any $\lambda \geq 1$.*

Theorem 1 *Let $f : R^n_+ \to R^l$ and $g : R^l_+ \to R^n$ be continuous functions satisfying condition (4). Let f be infra-homogeneous and such that*

$$\{x > 0, f(x) \geq 0\} \Rightarrow x^T d < 0 \tag{5}$$

If:
(i) either $c << 0$
(ii) or g is infra-homogeneous and

$$\{y > 0, g(y) \geq 0\} \Rightarrow y^T c < 0 \tag{6}$$

the game (1)-(2)-(3) admits a solution.

Proof. For $\varepsilon \geq 0$, let the simplex S_ε in $R^n_+ \times R^l_+ \times R_+$ be defined by

$$S_\varepsilon = \left\{ (x,y,t); x \geq 0, y \geq 0, t \geq \varepsilon, \sum_i x_i + \sum_j y_j + t = 1 \right\}$$

For any $\varepsilon > 0$, the continuous function $\overline{s} : S_\varepsilon \to R^n \times R^l \times R$ defined by

$$\overline{s}(x,y,t) = (tg(t^{-1}y) - td, tf(t^{-1}x) - tc, x^T d + y^T c - x^T g(t^{-1}y) - y^T f(t^{-1}x)) \tag{7}$$

satisfies the Walras identity $(x,y,t)^T \overline{s}(x,y,t) = 0$ on S_ε. According to the GND lemma, there exists a point $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in S_\varepsilon$ such that inequality

$$\forall (x,y,t) \in S_\varepsilon \quad (x,y,t)^T \overline{s}(x_\varepsilon, y_\varepsilon, t_\varepsilon) \geq 0 \tag{8}$$

holds. Taking into account assumption (4) and the infra-homogeneity of f, we obtain in particular that

$$\forall (0,y,t) \in S_\varepsilon \;\; y^T(f(x_\varepsilon) - t_\varepsilon c) + t(x_\varepsilon^T d + y_\varepsilon^T c) \geq 0 \tag{9}$$

Consider a cluster point $(x_0, y_0, t_0) \in S_0$ of $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ when ε tends to zero, and assume first that $t_0 = 0$, hence $(x_0, y_0) > 0$. Property (9) can be extended by continuity to

$\varepsilon = 0$, therefore $f(x_0) \geq 0$ and $x_0^T d + y_0^T c \geq 0$. Hypothesis (5) excludes $y_0 = 0$, therefore $y_0 > 0$. We distinguish two cases:

(i) If $c << 0$, the scalar $y_0^T c$ is negative and a contradiction is obtained with (5).

(ii) If g is infra-homogeneous and hypothesis (6) holds, a symmetrical reasoning shows that $g(y_0) \geq 0$ and $x_0 > 0$, and a contradiction is obtained between inequality $x_0^T d + y_0^T c \geq 0$, (5) and (6).

In both cases, $t_0 \neq 0$ and property (8) can be extended by continuity to $\varepsilon = 0$. This shows that $\overline{s}(x_0, y_0, t_0) \geq 0$, therefore $(t_0^{-1} x_0, t_0^{-1} y_0)$ is a solution of the game. ■

In the remaining of the paper, we study the case corresponding to hypothesis

$$c << 0 \tag{10}$$

and assume that f is linear and represented by matrix A. The constructive proof to which we proceed introduces an asymmetric treatment of inequalities (1) and (2) and replaces inequality (2) by an equality. Vector d will then be interpreted as a parameter of a two-person game $G(d)$. Note first that, when d is negative, the solution of $G(d)$ is unique:

Lemma 2 *Under hypotheses (4) and (10), the unique solution of $G(d)$ for $d << 0$ is $x = 0$, $y = 0$.*

Proof. For a solution (x, y), complementarity implies that $y^T c + x^T d = y^T f(x) + x^T g(y) \geq 0$, therefore $x = 0$ and $y = 0$. Conversely, $(x = 0,\ y = 0)$ is indeed a solution since inequality (4) implies $f(0) \geq 0 >> c$ and $g(0) \geq 0 >> d$. ■

Let the functions $\overline{f} : \Omega_f \to R^l \times R^n$ and $\overline{g} : \Omega_g \times R^n \to R^n$ be respectively defined by

$$\begin{aligned} \overline{f}(x) &= \begin{pmatrix} f(x) \\ x \end{pmatrix} && (11) \\ \overline{g}(y, z) &= g(y) - z && (12) \end{aligned}$$

and $\overline{c}$ be the $(l+n) \times 1$ vector c once completed by n zeroes. By setting $\overline{y}^T = (y^T, z^T)$, we have $\overline{y}^T \overline{f}(x) + x^T \overline{g}(\overline{y}) = y^T f(x) + x^T g(y)$, assumption (4) implies

$$\forall x \geq 0 \ \ \forall \overline{y} \geq 0 \ \ \overline{y}^T \overline{f}(x) + x^T \overline{g}(\overline{y}) \geq 0 \tag{13}$$

A solution (x, y) of the two-person game $G(d)$ generates a solution $(x, \overline{y})$ of the system $\overline{G}(d)$

$$\begin{aligned} \overline{f}(x) &\geq \overline{c} \quad [\overline{y}] && (14) \\ \overline{g}(\overline{y}) &= d && (15) \\ \overline{y} &\geq 0 && (16) \end{aligned}$$

where $\overline{y}$ is the $(l+n)\times 1$ vector obtained by stacking vectors y and $g(y)-d$. Conversely, a solution $(x,\overline{y})$ of $\overline{G}(d)$ generates a solution (x,y) of $G(d)$, where vector y consists of the first n components of $\overline{y}$.

An n-set $K\subset\{1,...,l+n\}$ is said to sustain a solution of $G(d)$ if it is the support of a nonnegative vector $\overline{y}$ (i.e., the set of its nonzero components) in a solution of $\overline{G}(d)$. Let R_K^{l+n} is the set of the $(l+n)\times 1$ vectors with support K. Given a subset $S\subset\{1,...,l+n\}$ with cardinal s, $\overline{f}_S:\Omega_f\to R^s$ is the projection of $\overline{f}$ on R^S, i.e. it is the function defined by its components $\overline{f}_i$ for $i\in S$. Function $\overline{g}_K$ is the restriction of function $\overline{g}$ to vectors with support K. $\left[J\overline{f}_K\right]$ (respectively $[J\overline{g}_K]$) is the Jacobian matrix of $\overline{f}_K$, $J\overline{f}_K$ (respectively $J\overline{g}_K$) its determinant (f and g are now assumed to be of class C^1). When f is linear and represented by an $l\times n$ matrix A, $\overline{f}$ is represented by $(l+n)\times n$ matrix $\overline{A}$ obtained by stacking A and the identity matrix I_n, and $\left[J\overline{f}_K\right]=\overline{A}_K$ is the $n\times n$ sub-matrix of $\overline{A}$ made of the rows corresponding to the set K.

3 The parametric Lemke algorithm

We consider the generic case, with no specific algebraic relationship between the data $(\overline{A},\overline{g},c,d)$. Flukes apart, the complementarity relationship in inequality (14) requires that the number n of components of x is at least equal to the number of positive components of $\overline{y}$, while equality (15) requires that the number of positive components of $\overline{y}$ is at least equal to n. The number of positive components of $\overline{y}$ is therefore generically equal to n (if $\overline{g}(0)=0$, as in a bimatrix game, the point $d=0$ is an exception, since $\overline{y}=0$ works) and the number of strict inequalities in (14) is generically equal to l. If $\overline{A}_K$ and $\overline{g}_K$ are injective, each arbitrary n-set K sustains at most one solution of (15) and of (14), therefore the number of solutions is finite: finding a solution amounts basically to identifying the n-set K of the positive components of vector $\overline{y}$. For that purpose, we make use of the parametric Lemke algorithm (Cottle *et al.*[3]), where vector d is a parameter. The algorithm follows the deformations of a solution $(x(d),\overline{y}(d))$ of $\overline{G}(d)$ when d varies, with a specific attention given to the support of $\overline{y}(d)$. An important property is that the support remains the same by intervals. A critical point is reached when some positive component of $\overline{y}(d)$ vanishes and, then, a switch in one basic variable is required. As long as a new basic variable can indeed be defined, a solution obtained for $d_0=d(0)$ allows us to obtain a solution for another vector d. The following Lemma reinterprets the above assumption (5) as a condition for the working of the algorithm.

Lemma 3 *Assume (4) and (10). Let d belong to the set $\mathcal{D}$*

$$\mathcal{D} = \left\{ d; \exists \widehat{z} \geq 0 \quad d << -A^T z \right\} \tag{17}$$

and let $\overline{y}$ be a solution of $\overline{G}(d)$. If $\det \overline{A}_K \neq 0$ and $J\overline{g}_K(\overline{y}) \neq 0$, the parametric Lemke algorithm works in a neighborhood of d, except by fluke.

Proof. Let $(x_0, \overline{y}_0)$ be a solution of (14)-(15)-(16) at d_0, and assume first that $\overline{y}_0$ has n positive components (set K). Since $\overline{g}_K$ is a local diffeomorphism, a slight change in d is met by an adaptation of the positive components of $\overline{y} = \overline{y}(d)$, with no change in x_0. Assume now that some positive component k of $\overline{y}(d) = \overline{g}_K^{-1}(d)$ vanishes at point $d_1 \in \mathcal{D}$ for $\overline{y} = \overline{y}_1$ ('critical point'). The component k which vanishes is unique, flukes apart. For $L = K \setminus \{k\}$, let $\overline{A}_L$ be the corresponding rows of $\overline{A}$. $\overline{A}_L$ has dimension $(n-1) \times n$ and the set of solutions to equation $\overline{A}_L x - \overline{c}_L = 0$ is $x = x_0 + \lambda x'$ where λ is an arbitrary scalar and x' a nonzero vector in the kernel of $\overline{A}_L$. Choose x' such that $\overline{A}_k x' > 0$ (otherwise, replace x' by $-x'$), so that the kth inequality in (14) with $\overline{f} = \overline{A}$ is met for any nonnegative scalar λ. Consider the complementary subset K' of K, for which the strict inequality $\overline{A}_{K'} x_0 >> \overline{c}_{K'}$ holds except by fluke, and assume for a moment that inequality

$$\overline{A}_{K'} x' \geq 0 \tag{18}$$

also holds. Then, by definition of the nonzero vector x',we have $\overline{A}x' \geq 0$, hence $Ax' \geq 0$ and $x' \geq 0$. Since x' is nonzero, we obtain from (17) that $d_1^T x' < -\widehat{z}^T Ax' \leq 0$, and from $\overline{g}_L(\overline{y}_1) = d_1$ that $\overline{g}_L(\overline{y}_1)^T x' < 0$. Let x'' be the semipositive vector x' completed by l zeroes. According to property (13) applied to $x = x''$ and $y = \overline{y}_1$, we have $\overline{y}_1^T \overline{A} x'' \geq -\overline{g}(\overline{y}_1)^T x'' = -\overline{g}_L(\overline{y}_1)^T x' > 0$, therefore $\overline{y}_1^T \overline{A}_L x' > 0$. A contradiction being obtained with the definition of x', inequality (18) does not hold, i.e. there exists at least one row m outside K such that $\overline{A}_m x' < 0$. Consider the smallest positive value of λ for which one of the strict inequalities $\overline{A}_{K'} x_0 >> \overline{c}_{K'}$ is turned into an equality when x_0 is replaced by $x_1 = x_0 + \lambda x'$ (flukes apart, that inequality is uniquely determined). Then $(x_1, \overline{y}_1)$ is another solution of $G(d_1)$. When the positive components of $\overline{y}$ remain close to those of $\overline{y}_1$, but its mth component $\overline{y}_m$ becomes positive (half-neighborhood of $\overline{y}_1$,with support $M = L \cup \{m\}$), $(x_1, \overline{y})$ sustains a solution of $G(d)$ for d varying in a half-neighborhood of d_1. ■

We now examine whether the half-neigborhoods associated with each of the n-sets K and M coincide or are complementary. These n-sets have the $(n-1)$-set L in common.

Lemma 4 *The determinants of the $n \times n$ matrices $\overline{A}_K$ and $\overline{A}_M$ have opposite signs.*

Proof. $\overline{A}_K$ and $\overline{A}_M$ have the submatrix $\overline{A}_L$ in common, which is such that $\overline{A}_L x' = 0$. They only differ by one row, respectively $\overline{A}_k$ or $\overline{A}_m$ and, by construction, $\overline{A}_k$ and $\overline{A}_m$ are such that $\overline{A}_k x' = a > 0$ and $\overline{A}_m x' = b < 0$, therefore $(b\overline{A}_k - a\overline{A}_m)x' = 0$. There follows that $(b\overline{A}_K - a\overline{A}_M)x' = 0$, hence $\det(b\overline{A}_K - a\overline{A}_M) = 0$ and the result. ∎

Lemma 5 *Let d be close to a critical point $d_1 = \overline{g}(\overline{y}_1)$. The components of $\overline{g}_K^{-1}(d)$ (respectively, $\overline{g}_M^{-1}(d)$) other than the kth (respectively, the mth) are positive. The components $(\overline{g}_K^{-1}(d))_k$ and $(\overline{g}_M^{-1}(d))_m$ have the relative sign as $J\overline{g}_K(\overline{y}_1)$ and $J\overline{g}_M(\overline{y}_1)$.*

Proof. By continuity, the components of $\overline{y} = \overline{g}_K^{-1}(d)$ are close to those of $\overline{g}_K^{-1}(d_1)$, therefore they are close to zero for the kth component and positive for the others. For $i \in K$, let b_i be the gradient of $\overline{g}_i$ at $\overline{y}_1$. The algebraic equality $d - d_1 = \sum_{i \in K} (\overline{y} - \overline{y}_1)_i b_i$ holds up to a first order approximation, from which there follows that the scalar $\det(b_i, i \in L; d - d_1)$ has the same sign as $(\overline{y} - \overline{y}_1)_k \det(b_i, i \in L; b_k) = (\overline{g}_K^{-1}(d))_k J\overline{g}_K(\overline{y}_1)$. The same scalar has also the sign of $(\overline{g}_M^{-1}(d))_m J\overline{g}_M(\overline{y}_1)$. The result follows. ∎

Definition 2 *Let K be an n-set sustaining a solution of $\overline{G}(d)$. That solution is called white if $\det(-\overline{A}_K)$ and $J\overline{g}_K(\overline{y})$ have the same sign, black if they have opposite signs.*

For instance, the unique solution obtained for $c << 0$ and $d << 0$ corresponds to the subset $K = \{l+1, ..., l+n\}$ for which $\overline{A}_K = I_n$ and $\overline{g}_K = -I_n$, therefore that solution is white.

Consider a path $d = d(t)$ in the set $\mathcal{D}$, a solution $(x_0, \overline{y}(t))$ of $G(d(t))$ for $t \in S = [t_1 - \eta, t_1[$ for which the support of $\overline{y}(t)$ is the set K, with $\overline{y}_k(d(t))$ vanishing at $d(t_1) = d_1$. By Lemma 3, there exists an n-set M obtained from K by replacing k by an adequately chosen m, such that M is the support of another solution $(x_1, \overline{y}')$ of $G(d(t))$ for t varying in one of the semi-intervals S or $T =]t_1, t_1 + \eta]$ (the semi-interval for which $\overline{y}'_m(d(t))$ is positive). Both values $\overline{y}_k(t) = \overline{y}_k(d(t))$ and $\overline{y}'_m(t) = \overline{y}'_m(d(t))$ change their sign at $t = t_1$ when $d(t)$ crosses the hypersurface corresponding to $\overline{y}_k = \overline{y}'_m = 0$, therefore $\overline{y}_k(t)$ and $\overline{y}'_m(t)$ have either always opposite signs or always the same sign on $]t_1 - \eta, t_1 + \eta, [$. Assume first they have opposite signs. For the first solution with support K, $\overline{y}_k(t)$ is positive when t is smaller than t_1, therefore $\overline{y}'_m(t)$ is negative on S and positive on T. This means that M sustains a solution when t continues to move in the same direction after t_1. By Lemma 5, this case occurs when the Jacobians $J\overline{g}_K(\overline{y}_1)$ and $J\overline{g}_M(\overline{y}_1)$ have opposite signs or, taking into account Lemma 4 and Definition 2, when the successive sets K and M have the same color. On the contrary, if $\overline{y}_k(t)$ and $\overline{y}'_m(t)$ have the same sign, M sustains another solution on the same half-interval as K, i.e. when $d = d(t)$ makes a U-turn on the path (antitone move). That change of direction occurs when K and M have opposite colors.

4 Main result

Consider the parametric problem $\overline{G}(d)$ where $d = d(t)$ moves on an oriented curve in $\mathcal{D}$ which does not cross itself, starting from $d_0 << 0$, going to a given point d, and coming back to $d_1 << 0$. Starting from the unique solution for d_0, that solution is transferred along the path. Even in the presence of antitone moves, the important property of the algorithm is that the same solution is not found twice (the law of succession of the sets K, M, ..., being uniquely defined along the path and reversible, a contradiction would be obtained by considering the first set which appears twice in that sequence), therefore the algorithm starting at d_0 goes first to d, then reaches d_1. (The algorithm starting from d_1 follows the reverse path.) Each time the algorithm reaches d, a new solution of $\overline{G}(d)$ is obtained. Taking into account the connection between the change of color and the change of direction on the path, we obtain that the solutions thus defined, which we call the accessible solutions (i.e., they are reached by following the given path $d(t)$), satisfy the following existence and oddity result:

Definition 3 *Functions $f = A$ and g are called regular if, for any n-set $K \subset \{1, ..., l+n\}$, $\overline{A}_K$ and $\overline{g}_K$ are diffeomorphisms.*

Theorem 2 *Let A and g be regular and satisfy condition (4). Let $c << 0$. For d in the set $\mathcal{D}$ defined by (17), and flukes apart, the number of white solutions of the two-person game $G(d)$ exceeds by one the number of black solutions.*

Proof. Let us start from a given solution at d, which is sustained by a set K. By Lemma 3, it can be transferred along the path. If either d_0 or d_1 is reached ultimately, the solution belongs to the unique sequence of accessible solutions, for which the oddity result holds. If not, the sequence starting from K makes a loop and returns to K, hence to d. Considering the corresponding move on the path $d(t)$, that loop sustains as many white as black solutions at d. The non accessible solutions are therefore partitioned into subsets containing as many solutions of each color. Hence the result when $n \geq 2$. The case $(n = 1, d > 0)$ is treated apart as any curve from $d_0 < 0$ to $d > 0$ then to $d_1 < 0$ makes a to-and-fro movement and crosses itself. Then the inequalities (1) are written $a_1 x \geq c, ..., a_l x \geq c$ and the only constraint which matters is the one corresponding to the smallest a_i, therefore we may assume $l = 1$. The system is reduced to the scalar inequalities

$$\begin{aligned} ax &\geq c \quad [y] \\ g(y) &\geq d \quad [x] \\ x &\geq 0, y \geq 0 \end{aligned}$$

with $c < 0$, $d > 0$, $ay + g(y) \geq 0$ for any $y \geq 0$ (hypothesis (4)) and $a < 0$ (otherwise, no scalar $d > 0$ belongs to $\mathcal{D}$). If $g(0) < d$, the set of solutions is $S = \{(x, y); x = c/a, g(y) = d)\}$; if $g(0) > d$, it is $(x = 0, y = 0) \cup S$. (The limit case $g(0) = d$ corresponds to a degeneracy.) Since the color of a solution in S depends on the sign of $g'(y)$ when $g(y) = d$, the result holds in both cases. ■

One may wonder if the proof of Lemma 3 and the oddity result can be extended to the case when f is nonlinear (then, one must distinguish between the local behavior of f associated with its Jacobian matrix and its global behavior). The proof of the Lemma makes use of the following property: consider the curve $(C) = \left\{x; , \overline{f}_L(x) = \overline{c}_L\right\}$ which we also parameterize as $x = x(\lambda)$ with $x(0) = x_0$, and choose a direction on that curve such that $\overline{f}_k(x(\lambda))$ is locally increasing when λ becomes positive. Then, for any i, function $\overline{f}_i$ varies monotonously on the oriented curve and, if it is decreasing, its decreases to $-\infty$. That property holds when f is linear (then (C) is a straight line) but its extension to the nonlinear case seems rather artificial.

It follows from Theorem 2 that global uniqueness is obtained if and only if any solution is white (Erreygers [6]). A reason why the oddity property has not been stated previously for bimatrix games is that it is not apparent. In the example (inspired from the land problem below)

$$A = \begin{bmatrix} -4 & 1 \\ -5 & 1 \end{bmatrix}, B = \begin{bmatrix} 7 & 6 \\ -1 & -1 \end{bmatrix}, c = \begin{pmatrix} -4 \\ -6 \end{pmatrix}, d = \begin{pmatrix} 63 \\ -10 \end{pmatrix}$$

the hypotheses $A + B^T \geq 0$ and $c << 0$ are met, but the bimatrix game admits two solutions: $(x_1 = 1, x_2 = 0, y_1 = 9, y_2 = 0)$ and $(x_1 = 2, x_2 = 4, y_1 = 3, y_2 = 7)$. The reason why the oddity result does not apply here that vector d is located outside the set $\mathcal{D}$ on which the working of the parametric Lemke algorithm is guaranteed.

5 A pioneer

In this historical Section, we draw attention on the connection between the work of the British economist David Ricardo (1772-1823) who, in the *Principles* (Ricardo

[10]), elaborated a theory of long-term prices and rents, and the parametric Lemke algorithm. We follow here a modern formalization of Ricardo's theory mainly due to Sraffa [12]. The data are the set of methods of production, the rate of profit r per year ($r \geq 0$), the vector δ ($\delta \geq 0$) of demand for consumption and the vector $\overline{\Lambda}$ of scarcity constraints on lands. Let there be g goods and h qualities of lands. The ith method of production is represented by a vector $u_i \in R_+^g$ of material inputs, a vector $\lambda_i \in R_+^h$ of land inputs and a quantity $-c_i \in R_{++}$ of labor ($c << 0$), and the product obtained one year after investment is represented by a vector $v_i \in R_+^g$. Nonnegative combinations of the methods are allowed. The unknowns are the price vector (the wage being set equal to one) $p \in R_+^g$, the vector $\rho \in R_+^h$ of rents per acre of lands and the vector $y \in R_+^m$ representing the activity levels of the methods. A long-term equilibrium is a solution of the system

$$-V^T p + (1+r)U^T p + \Lambda^T \rho \quad \geq \quad c \quad [y] \tag{19}$$

$$-\Lambda y \quad \geq \quad -\overline{\Lambda} \quad [\rho] \tag{20}$$

$$(V-U)y \quad \geq \quad \delta \quad [p] \tag{21}$$

Condition (19) states that all operated methods yield the ruling rate of profit, while non-operated methods are less profitable. Condition (20) expresses the scarcity constraints on lands, and the complementarity relationship means that non fully cultivated lands yield a zero rent (competition between landowners). Condition (21) means that the demand δ for consumption is met by the net product of the year, the overproduced commodities being zero priced. By introducing vector x obtained by stacking p and ρ, the problem is transformed into a bimatrix game with $A = \left[(1+r)U^T - V^T, \Lambda^T\right]$ and $B = [V-U, -\Lambda]$ for which condition $A + B^T \geq 0$ is met. The parallel was pointed out by Salvadori [11], who applied the existence result to the economic problem.

Ricardo was interested in the effect of an increase of demand for corn due to the increase in the number of workers. He stressed that, starting from an equilibrium for the present level of demand, it suffices to increase the activity levels on partially cultivated lands, with no effect on prices and rents as long as no new scarcity constraint on lands is met. When such a constraint is met, the price of corn rises up to the point where some new agricultural method, which was not operated before because it was too expensive when corn was cheap, becomes profitable. That law of succession of methods defines a new equilibrium with higher prices and higher rents. Ricardo's ideas, even if not formalized (at that time, economists did not write equations), are clearly the same as those sustaining the parametric Lemke algorithm.

Ricardo considered two ways, which he considered as basically equivalent, to extend the production of corn when demand increases. One consists in extending cultivation on a new land (extensive cultivation), the other in changing a method already in use on some land and adopting a more productive method on the same land (intensive cultivation). He did not see, however, that in the second case the incoming method designated by the law of succession of methods, which is selected on the basis of a profitability criterion, may in fact be less productive than the method it replaces and may not meet the evolution of demand. In other words, Ricardo missed the possibility of an antitone move: from a formal point of view, the 'Ricardian dynamics' are the parametric Lemke algorithm with no antitone move. Contemporary economists have pointed at the possible multiplicity of equilibria but missed the link between the multiplicity phenomenon and the failure of the Ricardian dynamics on the path leading from low levels of demand (Lemma 2 considers the extreme case of a negative demand) to high levels (Bidard [3]).

6 Conclusion

Oddity results, with a distinction between two types of solutions, are not infrequent in mathematics. In this paper we draw a bridge between that property and the working of the parametric Lemke algorithm. It is likely that, beyond two-person games, the strategy of proof applies to other types of complementarity problems.

7 Acknowledgements

With acknowledgements to Emmanuel Hyon for comments and suggestions.

References

[1] Bidard, C. (2004), *Prices, Reproduction, Scarcity*, Cambridge University Press, Cambridge

[2] Bidard, C. (2011), Complementarity problems and general equilibrium, *Advanced Modeling and Optimization*, 13, 153-161

[3] Bidard, C. (2014),The Ricardian rent theory: an overview, Centro Sraffa working paper

[4] Cottle, R.W., Pang, J-S., Stone, R.E. (1992), *The Linear Complementarity Problem*, Academic Press, New York

[5] Debreu, G. (1956), Market equilibrium, *Proceedings of the National Academy of Sciences of the USA*, 38, 886-93

[6] Erreygers, G. (1995), On the uniqueness of cost-minimizing techniques, *The Manchester School*, 63, 145-166

[7] Gale, D. (1955), The law of supply and demand, *Mathematica Scandinavia*, 3, 155-169

[8] Lemke, C.E. (1965), Bimatrix equilibrium points and mathematical programming, *Management Science*, 11, 681-689

[9] Nikaido, H. (1956), On the classical multilateral exchange problem, *Metroeconomica*, 8, 135-145

[10] Ricardo, D. (1951-1973), *On the Principles of Political Economy and Taxation* (1817). Vol. I in *The Works and Correspondence of David Ricardo*, P. Sraffa (ed.), Cambridge University Press, Cambridge

[11] Salvadori, N. (1986), Land and choice of techniques within the Sraffa framework, *Australian Economic Papers*, 25, 94-105

[12] Sraffa, P. (1960), *Production of Commodities by Means of Commodities*, Cambridge University Press, Cambridge