Toward a new analytical approximation method for the system of integro-differential equations

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Abstract: In this paper, an analytical approximation method for solving the system of integro-differential equations has been proposed. The method is based on perturbation technique, polynomial series and Laplace transform. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution. It is shown that the results are in good agreement with the exact solutions for each problem.

***Keywords**:* system of integro-differential equations, polynomial series; perturbation technique; Laplace transform.

1. Introduction

The integro-differential equations play a key role in describing key scientific phenomena. These equations have gained a lot of interest in many application fields, such as biological, physical and engineering problems. Integro-differential equations are important, but they are hard to solve even numerically, so the progress on how to solve them is slow. Therefore, their numerical treatment is desired. Goswami et al. [1] used wavelet on bounded interval to solve the integral equations, Lakestani et al. [2] used spline wavelets to solve the integro-differentail equations, also Nevles et al. [3] used orthogonal wavelets to solve the integral equations, Chrysafinos [4] used wavelet-Galerkin method or integro-differential equations, Abbasa et al. [5] applied multiwavelet direct method for solving integro-differential equations. Furthermore other authors used different methods for solving integro-differential equations [6, 7]. Orthogonal functions and polynomials have been used by many authors for solving functional equations. The aim of this paper is to construct an approximate analytical solution of the integro-differential equations by using the Laplace transform, polynomial series and perturbation technique. The rest of this paper is organized as follows. In Section 2, a theoretical consideration is discussed. In Section 3, the method is employed for obtaining the exact solutions of the integro-differential equations . Finally, some conclusions are given in Section 4.

2. Analysis of the method

A system of non-linear integro-differential equations can be presented as

$$\begin{aligned}
f_1^{(i)}(x) &= H_1\left(x, f_1(x),\cdots,f_1^{(i-1)}(x), f_2(x),\cdots,f_2^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)\\
&\quad+\int_0^x K_1\left(x,t,f_1(t),\cdots,f_1^{(i-1)}(t),\cdots,f_n(t),\cdots,f_n^{(i-1)}(t)\right)dt,\\
f_2^{(i)}(x) &= H_2\left(x, f_1(x),\cdots,f_1^{(i-1)}(x), f_2(x),\cdots,f_2^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)\\
&\quad+\int_0^x K_1\left(x,t,f_1(t),\cdots,f_1^{(i-1)}(t),\cdots,f_n(t),\cdots,f_n^{(i-1)}(t)\right)dt, \qquad i=1,2,\dots l. \qquad (1)\\
&\vdots\\
f_n^{(i)}(x) &= H_n\left(x, f_1(x),\cdots,f_1^{(i-1)}(x), f_2(x),\cdots,f_2^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)\\
&\quad+\int_0^x K_n(x,t,f_1(t),\cdots,f_1^{(i-1)}(t),\cdots,f_n(t),\cdots,f_n^{(i-1)}(t)\,dt,
\end{aligned}$$

where $f_1, f_2, \dots, f_n$ are unknown functions should be determined.

Let us now consider the j^{th} equation of system (1) as

$$\begin{aligned}
f_j^{(i)}(x) &= H_j\left(x, f_1(x),\cdots,f_1^{(i-1)}(x), f_2(x),\cdots,f_2^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)\\
&\quad+\int_0^x K_j\left(x,t,f_1(t),\cdots,f_1^{(i-1)}(t),\cdots,f_n(t),\cdots,f_n^{(i-1)}(t)\right)dt, \qquad j=1,2,\dots n.
\end{aligned}$$

Moreover, suppose, H_j can be decomposed as,

$$\begin{aligned}
&H_j\left(x, f_1(x),\cdots,f_1^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)\\
&\quad= L_j\left(x, f_1(x),\cdots,f_1^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)\\
&\quad+ N_j\left(x, f_1(x),\cdots,f_1^{(i-1)}(x),\cdots,f_n(x),\cdots,f_n^{(i-1)}(x)\right)
\end{aligned}$$

where L_j is a linear operator, N_j is a nonlinear operator and $f_j^{(i)}(x)$ is a unknown analytical function.

In this section, we construct the solution of system of partial differential equations by extending the idea of [8] and [9] For solving system (1), we construct the following homotopy

$$\begin{aligned} & F_j^{(i)}(x) - F_{j,0}(x) + pF_{j,0}(x) \\ & \qquad - p(H_j\left(x, F_1(x), \ldots, F_1^{(i-1)}(x), F_2(x), \ldots, F_2^{(i-1)}(x), \ldots, F_n(x), \ldots, F_n^{(i-1)}(x)\right) \\ & \qquad + \int_0^x K_j\left(x, t, F_1(t), \ldots, F_1^{(i-1)}(t), \ldots, F_n(t), \ldots, F_n^{(i-1)}(t)\right)dt) = 0, \end{aligned} \tag{2}$$

where $p \in [0,1]$ is an embedding or homotopy parameter, $H(x;p) : \Omega \times [0,1] \to \mathbb{R}$ and $F_{j,0}(x), j = 1,2,\ldots n.$ are the initial approximation of solution of the problem in Eq. (2). Obviously, a monotonous change of parameter p from zero to one corresponds to a continuous change of the trivial problem $F_j^{(i)}(x) - F_{j,0}(x) = 0$ to the original problem. Next, we assume that the solution of equation $H(F_j, p)$ can be written as a power series in embedding parameter p as follows:

$$F_j(x) = F_{j,0}(x) + pF_{j,1}(x), j = 1,2,\ldots,n. \tag{3}$$

By applying the Laplace transform on both sides of Eq. (2), we have

$$\mathcal{L}\{F_j^{(i)}(x)\} = \mathcal{L}\left\{\begin{aligned} & f_{j,0}(x) - p(f_{j,0}(x) \\ & \qquad - H_j\left(x, F_1(x), \ldots, F_1^{(i-1)}(x), F_2(x), \ldots, F_2^{(i-1)}(x), \ldots, F_n(x), \ldots, F_n^{(i-1)}(x)\right) \\ & \qquad - \int_0^x K_j\left(x, t, F_1(t), \ldots, F_1^{(i-1)}(t), \ldots, F_n(t), \ldots, F_n^{(i-1)}(t)\right)dt) \end{aligned}\right\}$$

Now by using the differential property of the Laplace transform, we have:

$$\begin{aligned} & s^i \mathcal{L}\{F_j(x)\} - s^{i-1}F_j(0) - s^{i-2}F_j'(0) - \cdots - F_j^{(i-1)}(0) \\ & = \mathcal{L}\left\{\begin{aligned} & f_{j,0} - p(f_{j,0} \\ & \qquad - H_j\left(x, F_1(x), \ldots, F_1^{(i-1)}(x), F_2(x), \ldots, F_2^{(i-1)}(x), \ldots, F_n(x), \ldots, F_n^{(i-1)}(x)\right) \\ & \qquad - \int_0^x K_j\left(x, t, F_1(t), \ldots, F_1^{(i-1)}(t), \ldots, F_n(t), \ldots, F_n^{(i-1)}(t)\right)dt) \end{aligned}\right\}, \end{aligned} \tag{4}$$

Or

$$\mathcal{L}\{F_j(x)\} = \frac{1}{s^i}\{s^{i-1}F_j(0) + s^{i-2}F_j^{'}(0) + \cdots + F_j^{(i-1)}(0) + \mathcal{L}\left\{\begin{array}{l} f_{j,0} - p(f_{j,0} \\ \quad - H_j\left(x, F_1(x), \ldots, F_1^{(i-1)}(x), F_2(x), \ldots, F_2^{(i-1)}(x), \ldots, F_n(x), \ldots, F_n^{(i-1)}(x)\right) \\ \quad - \int_0^x K_j\left(x, t, F_1(t), \ldots, F_1^{(i-1)}(t), \ldots, F_n(t), \ldots, F_n^{(i-1)}(t)\right) dt) \end{array}\right\}, \tag{5}$$

By applying the inverse Laplace transform on both sides of Eq. (5), we have:

$$F_j(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^i}\left\{\begin{array}{l} s^{i-1}F_j(0) + s^{i-2}F_j^{'}(0) + \cdots + F_j^{(i-1)}(0) \\ \quad + \mathcal{L}\left\{\begin{array}{l} f_{j,0} - p(f_{j,0} \\ -H_j\left(x, F_1(x), \ldots, F_1^{(i-1)}(x), \ldots, F_n(x), \ldots, F_n^{(i-1)}(x)\right) \\ -\int_0^x K_j\left(x, t, F_1(t), \ldots, F_1^{(i-1)}(t), \ldots, F_n(t), \ldots, F_n^{(i-1)}(t)\right) dt) \end{array}\right\} \end{array}\right\}\right\}, \tag{6}$$

Suppose that the initial approximation of Eq. (6) has the following form

$$f_{j,0}(x) = \sum_{n=0}^{\infty} a_{j,n}P_n(x), j = 1, 2, \ldots, n, \tag{7}$$

where $a_{j,n}, n = 0, 1, 2, \ldots$ are unknown coefficients and $P_n(x), n = 0, 1, 2, \ldots$ are specific functions on the problem. By substituting Eq. (3) and Eq. (7) into Eq. (6), we get

$$F_{j,0}(x) + pF_{j,1}(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^i}\left\{\begin{array}{l} s^{i-1}F_j(0) + s^{i-2}F_j^{'}(0) + \cdots + F_j^{(i-1)}(0) \\ +\mathcal{L}\left\{\begin{array}{l} F_{j,0} - p(F_{j,0} - H_j\left(\begin{array}{l} x, F_{1,0} + pF_{1,1}, \ldots, F_{1,0}^{(i-1)} + pF_{1,1}^{(i-1)}, \ldots, \\ F_{n,0} + pF_{n,1}, \ldots, F_{n,0}^{(i-1)} + pF_{n,1}^{(i-1)} \end{array}\right) \\ \quad - \int_0^x K_j\left(\begin{array}{l} x, t, F_{1,0} + pF_{1,1}, \ldots, F_{1,0}^{(i-1)} + pF_{1,1}^{(i-1)}, \ldots, \\ F_{n,0} + pF_{n,1}, \ldots, F_{n,0}^{(i-1)} + pF_{n,1}^{(i-1)} \end{array}\right) dt) \end{array}\right\} \end{array}\right\}\right\}, \tag{8}$$

Equating the coefficients of like powers of p parameter, we get the following set of equations:

$$\text{coefficient of } p^0 \text{: } F_{j,0}(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^i}\left\{\begin{array}{l} s^{i-1}F_j(0) + s^{i-2}F_j^{'}(0) + \cdots + F_j^{(i-1)}(0) \\ +\mathcal{L}\left\{\sum_{n=0}^{\infty} a_{j,n}P_n\right\} \end{array}\right\}\right\},$$

$$\text{coefficient of } p^1\text{: } F_{j,1}(x) = \mathcal{L}^{-1}\left\{ -\frac{1}{s^i}\left(\mathcal{L}\left\{ \begin{array}{l} \sum_{n=0}^{\infty} a_{j,n}P_n + F_{j,1}(x) \\ -H_j\left(x, F_{1,0}, \ldots, F_{1,0}^{(i-1)}, \ldots, F_{n,0}, \ldots, F_{n,0}^{(i-1)}\right) \\ -\int_0^x K_j\left(x, t, F_{1,0}, \ldots, F_{1,0}^{(i-1)}, \ldots, F_{n,0}, \ldots, F_{n,0}^{(i-1)}\right) dt) \end{array} \right\} \right) \right\}.$$

Now, we solve these equations in such a way that $F_{j,1}(x) = 0$. Therefore, the approximate solution of Eq. (1) may be obtained as

$$f_j(x) = F_{j,0}(x) = \mathcal{L}^{-1}\left\{ \frac{1}{s^i}\left\{ s^{i-1}F_j(0) + s^{i-2}F_j'(0) + \cdots + F_j^{(i-1)}(0) + \mathcal{L}\left\{ \sum_{n=0}^{\infty} a_{j,n}P_n \right\} \right\} \right\}.$$

3. Examples

In this section, to illustrate the method and to show the ability of the method two examples regarding integro-differential system are presented.

Example 2. In this example we consider a nonlinear system of integro-differential equations with initial conditions, $f(0) = 0$ and $g(0) = 0$. The exact solutions are $f(x) = \sinh x$, $g(x) = \cosh x$.

$$\begin{aligned} f'(x) &= 1 - \frac{1}{2}g'^2(x) + \int_0^x ((x-t)f(t) + f(t)g(t))\,dt, \\ g'(x) &= 2x + \int_0^x ((x-t)f(t) - g^2(t) + f^2(t))\,dt, \end{aligned} \tag{9}$$

To solving the system (9) by new method we construct the following homotopy:

$$\begin{aligned} F'(x) &= f_0(x) - p\Big(f_0(x) - 1 + \frac{1}{2}g'^2(x) - \int_0^x ((x-t)F(t) + F(t)G(t))\,dt\Big), \\ G'(x) &= g_0(x) - p\Big(g_0(x) - 2x - \int_0^x ((x-t)F(t) - G^2(t) + F^2(t))\,dt\Big), \end{aligned} \tag{10}$$

By applying the Laplace transform to both sides of the Eq. (10) we obtain

$$\begin{aligned} s\mathcal{L}\{F(x)\} - F(0) &= \mathcal{L}\Big\{f_0(x) - p\Big(f_0(x) - 1 + \frac{1}{2}g'^2(x) - \int_0^x ((x-t)F(t) + F(t)G(t))\,dt\Big)\Big\} \\ s\mathcal{L}\{G(x)\} - G(0) &= \mathcal{L}\Big\{g_0(x) - p\Big(g_0(x) - 2x - \int_0^x ((x-t)F(t) - G^2(t) + F^2(t))\,dt\Big)\Big\} \end{aligned} \tag{11}$$

Assuming that, $f_0(x) = \sum_{n=0}^{\infty} a_n P_n(x)$, $g_0(x) = \sum_{n=0}^{\infty} b_n P_n(x)$, $P_i(x) = x^i$, $F(0) = f(0)$ and $G(0) = g(0)$ and applying the inverse Laplace transform on both sides of Eq. (11), we have:

$$F(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{f(0)+\mathcal{L}\left\{f_0(x)-p\begin{pmatrix}f_0(x)-1+\dfrac{1}{2}g'^2(x)\\ -\int_0^x((x-t)F(t)+F(t)G(t))\,dt\end{pmatrix}\right\}\right\}\right\}$$
$$G(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{g(0)+\mathcal{L}\left\{g_0(x)-p\begin{pmatrix}g_0(x)-2x\\ -\int_0^x((x-t)F(t)-G^2(t)+F^2(t))\,dt\end{pmatrix}\right\}\right\}\right\} \quad (12)$$

Suppose the solutions of system (10) have the following form:

$$F(x)=F_0(x)+pF_1(x),$$
$$G(x)=G_0(x)+pG_1(x). \quad (13)$$

where $F_i(x)$ and $G_i(x)$ are functions which should be determined. Substituting Eq. (13) into Eq. (12) and equating the coefficients of p with the same powers leads to

$$p^0:\begin{cases}F_0(x)=\mathcal{L}^{-1}\left\{\dfrac{1}{s}\left\{f(0)+\mathcal{L}\left\{\sum\limits_{n=0}^{\infty}a_nx^n\right\}\right\}\right\},\\ G_0(x)=\mathcal{L}^{-1}\left\{\dfrac{1}{s}\left\{g(0)+\mathcal{L}\left\{\sum\limits_{n=0}^{\infty}b_nx^n\right\}\right\}\right\},\end{cases}$$
$$p^1:\begin{cases}F_1(x)=\mathcal{L}^{-1}\left\{-\dfrac{1}{s}\left\{\mathcal{L}\left\{\sum\limits_{n=0}^{\infty}a_nx^n-1+\dfrac{1}{2}g'^2(x)-\int_0^x((x-t)F_0(t)+F_0(t)G_0(t))\,dt\right\}\right\}\right\},\\ G_1(x)=\mathcal{L}^{-1}\left\{-\dfrac{1}{s}\left\{\mathcal{L}\left\{\sum\limits_{n=0}^{\infty}b_nx^n-2x-\int_0^x((x-t)F(t)-G^2(t)+F^2(t))\,dt\right\}\right\}\right\}.\end{cases}$$

Now, if we set $F_1(x)=0,$ then

$$F_1(x)=\left(\frac{3}{2}-a_0-\frac{1}{2}a_0^2\right)x-\left(\frac{1}{2}a_1+\frac{1}{2}a_0a_1\right)x^2+\left(\frac{1}{6}-\frac{1}{3}a_2+\frac{1}{6}a_0-\frac{1}{6}a_1^2-\frac{1}{3}a_0a_2\right)x^3$$
$$+\left(\frac{1}{24}a_0-\frac{1}{4}a_3+\frac{1}{24}a_1-\frac{1}{4}a_1a_2+\frac{1}{24}a_0b_0-\frac{1}{4}a_0a_3\right)x^4+\cdots$$

and if we set $G_1(x)=0,$ then

$$G_1(x)=\left(\frac{1}{2}-\frac{1}{2}b_1\right)x^2-\left(\frac{1}{3}b_0+\frac{1}{3}b_2\right)x^3+\left(\frac{1}{12}a_0^2-\frac{1}{12}b_0^2+\frac{1}{24}a_0-\frac{1}{12}b_1-\frac{1}{4}b_3\right)x^4$$
$$+\left(\frac{1}{20}a_0a_1-\frac{1}{20}b_0b_1+\frac{1}{120}a_1-\frac{1}{30}b_2-\frac{1}{5}b_4\right)x^5+\cdots.$$

It can be easily shown that

$$a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}, a_3 = 0, a_4 = \frac{1}{24}, \ldots,$$
$$b_0 = 0, b_1 = 1, b_2 = 0, b_3 = \frac{1}{6}, b_4 = 0, \ldots.$$

Therefore, the exact solutions of the system of Eq. (9) can be expressed as follows:

$$\begin{aligned} f(x) &= a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3 + \frac{1}{4}a_3 x^4 + \frac{1}{5}a_4 x^5 + \cdots \\ &= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots = \sinh x, \\ g(x) &= 1 + b_0 x + \frac{1}{2}b_1 x^2 + \frac{1}{3}b_2 x^3 + \frac{1}{4}b_3 x^4 + \frac{1}{5}b_4 x^5 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots = \cosh x. \end{aligned}$$

Example 2. In this example we consider a nonlinear system of integro-differential equations with initial conditions, $f(0) = f''(0) = 0, f'(0) = 1$ and $g(0) = 1, g'(0) = 0, g''(0) = -1$. The exact solutions are $f(x) = \sin x$, $g(x) = \cos x$.

$$\begin{aligned} f'''(x) &= x - f'(x) - \int_0^x (f''^2(t) + g''^2(t))\,dt, \\ g'''(x) &= \sin x + \frac{1}{2}\sin^2 x + \int_0^x f''(t)\,g(t)\,dt \end{aligned} \tag{14}$$

To solving the system (14) by new method we construct the following homotopy:

$$\begin{aligned} F'''(x) &= f_0(x) - p\Big(f_0(x) - x + F'(x) + \int_0^x (F''^2(t) + G''^2(t))\,dt\Big), \\ G'''(x) &= g_0(x) - p\Big(g_0(x) - \sin x - \frac{1}{2}\sin^2 x - \int_0^x f''(t)\,g(t)\,dt\Big), \end{aligned} \tag{15}$$

By applying the Laplace transform to both sides of the Eq. (15) we obtain

$$\begin{aligned} s^3\mathcal{L}\{F(x)\} - s^2F(0) - sF'(0) - F''(0) &= \mathcal{L}\left\{f_0(x) - p\begin{pmatrix} f_0(x) - x + F'(x) \\ +\int_0^x (F''^2(t) + G''^2(t))\,dt \end{pmatrix}\right\}, \\ s^3\mathcal{L}\{G(x)\} - s^2G(0) - sG(0) - G''(0) &= \mathcal{L}\left\{g_0(x) - p\begin{pmatrix} g_0(x) - \sin x - \frac{1}{2}\sin^2 x \\ -\int_0^x f''(t)\,g(t)\,dt \end{pmatrix}\right\}, \end{aligned} \tag{16}$$

Assuming that, $f_0(x)=\sum_{n=0}^{\infty}a_nP_n(x),\quad g_0(x)=\sum_{n=0}^{\infty}b_nP_n(x),\ P_i(x)=x^i,\quad F(0)=f(0),$ $F'(0)=f'(0),\ F''(0)=f''(0)$ and $G(0)=g(0),G'(0)=g(0),G''(0)=g''(0)$ and applying the inverse Laplace transform on both sides of Eq. (16), we have:

$$F(x)=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\left\{\begin{array}{l}s^2f(0)+sf'(0)+f''(0)\\ +\mathcal{L}\left\{f_0(x)-p\Big(f_0(x)-x+F'(x)+\int_0^x(F''^2(t)+G''^2(t))\,dt\Big)\right\}\end{array}\right\}\right\},$$
$$G(x)=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\left\{\begin{array}{l}s^2g(0)+sg'(0)+g''(0)\\ +\mathcal{L}\left\{g_0(x)-p\Big(g_0(x)-\sin x-\frac{1}{2}\sin^2 x-\int_0^x F''(t)G(t)\,dt\Big)\right\}\end{array}\right\}\right\}. \tag{17}$$

Suppose the solutions of system (17) have the following form:

$$\begin{aligned}F(x)&=F_0(x)+pF_1(x),\\ G(x,t)&=G_0(x)+pG_1(x).\end{aligned} \tag{18}$$

where $F_i(x)$ and $G_i(x)$ are functions which should be determined.

Substituting Eq. (18) into Eq. (17) and equating the coefficients of p with the same powers leads to

$$p^0:\begin{cases}F_0(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{s^2f(0)+sf'(0)+f''(0)+\mathcal{L}\left\{\sum_{n=0}^{\infty}a_nx^n\right\}\right\}\right\},\\ G_0(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{s^2g(0)+sg'(0)+g''(0)+\mathcal{L}\left\{\sum_{n=0}^{\infty}b_nx^n\right\}\right\}\right\},\end{cases}$$
$$p^1:\begin{cases}F_1(x)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left\{\mathcal{L}\left\{\sum_{n=0}^{\infty}a_nx^n-x+F'(x)+\int_0^x(F''^2(t)+G''^2(t))\,dt\right\}\right\}\right\},\\ G_1(x)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left\{\mathcal{L}\left\{\sum_{n=0}^{\infty}b_nx^n-\sin x-\frac{1}{2}\sin^2 x-\int_0^x F''(t)G(t)\,dt\right\}\right\}\right\}.\end{cases}$$

Now, if we set $F_1(x)=0,$ and using the Taylor series of $\sin x$, then

$$F_1(x)=\Big(-\frac{1}{6}-\frac{a_0}{6}\Big)x^3-\frac{a_1}{24}x^4+\Big(-\frac{a_2}{60}-\frac{a_0}{120}+\frac{b_0}{60}\Big)x^5+\Big(\frac{b_1}{360}-\frac{{b_0}^2}{360}-\frac{{a_0}^2}{360}-\frac{a_1}{720}-\frac{a_3}{120}\Big)x^6$$
$$+\Big(\frac{b_2}{1260}-\frac{b_1b_0}{8460}-\frac{a_1a_0}{8460}-\frac{a_2}{2520}-\frac{a_4}{210}\Big)x^7\cdots=0$$

and if we set $G_1(x)=0,$ then

$$G_1(x) = -\frac{b_0}{6}x^3 + \left(\frac{1}{24} - \frac{b_1}{24}\right)x^4 + \left(-\frac{b_2}{60} + \frac{a_0}{120} + \frac{1}{120}\right)x^5 + \left(-\frac{b_3}{120} - \frac{1}{720} + \frac{a_1}{720}\right)x^6$$
$$+ \left(-\frac{a_0}{1680} + \frac{a_2}{2520} - \frac{1}{1260} - \frac{b_4}{210}\right)x^7 + \cdots = 0$$

It can be easily shown that

$$a_0 = -1, a_1 = 0, a_2 = \frac{1}{2}, a_3 = 0, a_4 = -\frac{1}{24}, \dots$$
$$b_0 = 0, b_1 = 1, b_2 = 0, b_3 = \frac{1}{6}, b_4 = 0, \dots.$$

Therefore, the exact solutions of the system of Eq. (14) can be expressed as follows:

$$f(x) = x + \frac{1}{6}a_0x^3 + \frac{1}{24}a_1x^4 + \frac{1}{60}a_2x^5 + \frac{1}{120}a_3x^6 + \frac{1}{210}a_4x^7 + \cdots$$
$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots = \sin x,$$
$$g(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}b_0x^3 + \frac{1}{24}b_1x^4 + \frac{1}{60}b_2x^5 + \frac{1}{120}b_3x^6 + \frac{1}{210}b_4x^7 + \cdots$$
$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 = \cos x.$$

5. Conclusion

In the present work, we proposed a combination of Laplace transform, polynomial series and homotopy technique to solve nonlinear system of integro-differential equations. The new method developed in the current paper was tested on two examples. The obtained results show that this approach can solve the problem effectively.

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