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A Solution Approach for Multi-level Integer Indefinite Quadratic Programming Problems

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Abstract

A multi-level programming problem is a hierarchical optimization problem where the constraint region of the first level is implicitly determined by the other optimization problems. In this paper, a multi-level integer programming problem (MIPP) with bounded variables is considered in which the objective functions are indefinite quadratic and the feasible region is a convex polyhedron. An algorithm is developed for ranking and scanning the set of feasible solutions. These ranked solutions are used to solve (MIPP). The algorithm is explained with the help of examples.

Keywords: Indefinite quadratic programming problem, integer programming, multi-level programming problem, bounded variables, quasi-concave function.

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Introduction

A Multi-Level Programming Problem (MLPP) deals with decentralized planning problems with multiple decision makers in a multi-level or hierarchical organization where decisions have interacted with each other. In multi-level programming, the higher level decision makers make their decision in full view of the lower level decision makers. Each decision maker attempts to optimize its objective function and is affected by the actions of the other decision makers. The mathematical model of the K-level programming problem is as follows:

(MLPP) : $\max_{X_1} f_1(X) = c_{11}X_1 + ... + c_{1k}X_K$

$$\max_{X_2} f_2(X) = c_{21}X_1 + ... + c_{2k}X_K$$

$$. \quad . \quad . \quad . \quad .$$

$$\max_{X_K} f_K(X) = c_{K1}X_1 + ... + c_{KK}X_K$$

subject to

$$A_{i1}X_1 + A_{i2}X_2 + ... + A_{ik}X_k = b_i, \quad i = 1, 2, ..., m$$

$$X_1, X_2, ..., X_K \geq 0$$

The above problem has one decision maker at each level, n decision variables and m constraints. $X = (X_1, X_2, ..., X_k)$, $n = n_1 + n_2 + ... + n_k$, where the decision vector $X_K \in \mathbb{R}^{n_k}$, $k = 1, ..., K$ is under the control of k^{th} level decision maker and has n_k number of decision variables.

(MLPP) can be found in many decision making situations. Candler and Norton [3] presented a version of this problem in an economic policy context. Another application of (MLPP) can be seen in a market of consumers being served by a distribution centre as well as its competitor. Both the players must set a price level for its products so that their profits are maximized. However, the consumers are at liberty to buy from either of the players depending on the relative prices of this product. The customers' decision will be reached after taking into consideration economic criterion such as cost minimization.

Bard [2] formulated a normal non-linear programming problem by using the Kuhn-Tucker conditions for the problems of the third level and second level and proposes a cutting plane algorithm employing a vertex search procedure to solve a tri-level programming problem. P. Lasunon [8] in 2011 has proposed a new method for solving tri-level programming problem. T.I. Sultan et al.[16] in 2014 has given a decomposition algorithm for (TPP).

Faisca, Dua, Rustem, Saraiva and Pistikopoulas [14] in the year 2009 have discussed multi-parametric programming approach for multi-level hierarchical and decentralized optimization problems. Migdalas, Pardalos and Varbrand [12] in the year 1997 published a book on multi-level optimization, which is a series on Non-Convex optimization and its applications. Latest work on multilevel optimization can be found in [4, 5, 6, 10, 15].

Quadratic Programming represents a special class of non-linear programming in which the objective function is quadratic and the constraints are linear. R. Baker [1] in 2008 has given an interior point solution for multilevel quadratic programming problems.

Extensive work has been done on integer programming problems. These problems are of particular importance in business and industry where quite often, the fractional solutions are unrealistic because units are not divisible. Many cutting plane algorithms like Dantzig cut, Gomory cut, edge truncating cut etc. are used to solve such problems when decision variables are not bounded. Huang, Quing [7], MaZhong Fan [11] and Xu Chang [17] have developed various programming problems with bounded variables. Lev and Soyster [9] have developed an algorithm for integer programming with bounded variables and the upper bounds on the variables are small.

Indefinite Quadratic Integer Programming Problem with Bounded Variables

Consider the indefinite quadratic integer programming probem with bounded variables, defined as

(IQIPP): $\text{Max } Z(X) = Z_1(X).Z_2(X) = (C^T X + \alpha)(D^T X + \beta)$

subject to $X \in S$,

where $S = \{X \mid AX = b,\ \ell \le X \le u,\ X$ is an integer vector$\}$.

Here, S is non-empty and bounded, $X \in \mathbb{R}^n$ is a vector of variables; $b \in \mathbb{R}^m$; C, D $\in \mathbb{R}^n$; $\alpha, \beta \in \mathbb{R}$ and $A \in \mathbb{R}^{m\times n}$.

Both $Z_1(X)$ and $Z_2(X)$ are positive for all $X \in S$. Thus, the function Z(X) is both quasi-concave and quasi-convex on S. Hence, Z(X) is explicitly quasi-monotone on S. Therefore, the optimal solution to the problems (IQIPP) occurs at an extensive point of S.

Optimality Criterion for (IQIPP)

Theorem 1 [13]: Let X_B be a non-degenerate basic feasible to the problem (IQIPP) and let $\hat{X}_B$ be another basic feasible solution obtained by introducing a non-basic column vector a_j into the basis, for which $L_j = Z_1(z_j^2 - d_j) + Z_2(z_j^1 - c_j) - \theta_j(z_j^2 - d_j)(z_j^1 - c_j)$ is negative. Then, $\hat{X}_B$ is a basic feasible solution with an improved value of Z.

The optimality criterion for solving (IQIPP) with bounded variables is that $L_j \ge 0$ for upper bounded non-basic variables and $L_j \le 0$ for lower bounded non-basic variables.

Mathematical Formulation of the Problem (MIQPP)

The Multi-level Integer Indefinite Quadratic Programming Problem with bounded variables is formulated as,

(MIQPP): $\underset{X_1}{\text{Max}}\, Z_1(X_1, X_2, ..., X_p) = Z_{11}(X_1, X_2, ..., X_p).Z_{12}(X_1, X_2, ..., X_p)$

$\underset{X_2}{\text{Max}}\, Z_2(X_1, X_2, ..., X_p) = Z_{21}(X_1, X_2, ..., X_p).Z_{22}(X_1, X_2, ..., X_p),$ for a given X_1

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$\underset{X_p}{\text{Max}}\, Z_p(X_1, X_2, ..., X_p) = Z_{p1}(X_1, X_2, ..., X_p).Z_{p2}(X_1, X_2, ..., X_p)$ for a given $(X_1, X_2, ..., X_{p-1})$.

where $(X_1, X_2, ..., X_p) \in S^*$.

Here, $S^* = \{X \mid AX = b,\ L \le X \le U\}$ is non-empty and bounded.

Define, $S_1^* = \{X \mid AX = b;\ L \le X \le U, X$ is an integer vector$\}$.

Clearly, $S_1^* \subseteq S$. We are interested in finding the solution of the given problem in S_1^*.

Here, $Z_{i1}(X_1, X_2, ..., X_p) = C_{i1}X_1 + C_{i2}X_2 + ... + C_{ip}X_p + \alpha_i;\ i = 1, 2, ..., p$

$Z_{i2}(X_1, X_2, ..., X_p) = D_{i1}X_1 + D_{i2}X_2 + ... + D_{ip}X_p + \beta_i;\ i = 1, 2, ..., p$

$X_1 = (x_1^1, x_2^2, ..., x_{n_1}^1) \in \mathbb{R}^{n_1};\ X_2 = (x_2^1, x_2^2, ..., x_{n_2}^1) \in \mathbb{R}^{n_2};....$

$X_p = (x_1^p, x_2^p, ..., x_{n_p}^p) = \mathbb{R}^{n_p}$

Here, $C_{ir} = \sum_{j=1}^{n_r} c_{ij}^r x_j^r, \qquad i = 1, 2, ..., p$

$$D_{ir} = \sum_{j=1}^{n_r} d_{ij}^r x_j^r, \qquad i = 1, 2, ..., p$$

$$\alpha_i, \beta_i \in \mathbb{R}; \qquad i = 1, 2, ..., p$$

$$A = (A_1, A_2, ..., A_p), \quad A_i \in \mathbb{R}^{m \times n_i}; \quad i = 1, 2, ..., p; \quad b \in \mathbb{R}^{m \times 1}$$

$$L = (L_1, L_2, ..., L_p)^T, \text{ where } L_i \in \mathbb{R}^{n_i \times 1}; \qquad i = 1, 2, ..., p,$$

$$U = (U_1, U_2, ..., U_p)^T, \text{ where } U_i \in \mathbb{R}^{n_i \times 1}; \qquad i = 1, 2, ..., p .$$

The objective functions defined for each $Z_i(X)$; i = 1, 2, ...,p is the product of two positive valued affine functions, hence it is quasi-concave.

The polyhedron S^* defined by the constraint region of the problem (MIQPP) is assumed to be non-empty and compact.

Constraint region for the problem $Z_p(X)$ for some fixed value of $(X_1, X_2, ..., X_{p-1})$ is given by

$$S^* = (X_1, X_2, ..., X_{p-1}) = \{X_p \mid A_p X_p \le A_1 X_1 + A_2 X_2 + ... + A_{p-1} X_{p-1},\ L_p \le X_p \le U_p, \ X_p \text{ is an integer vector}\}$$

The rational reaction set for the follower's problem $Z_p(X)$, for fixed value of $(X_1, X_2, ..., X_{p-1})$ is given by

$$M(X_1, X_2, ...,X_{p-1}) = \{X_p \mid X_p \in \arg\max Z_p(X_1, X_2, ..., X_{p-1}, X_p); \ X_p \in S^*(X_1, X_2, ..., X_{p-1})\}.$$

The inducible region of (MIQPP) is given by

$$IR = \{(X_1, X_2, ..., X_p) \mid (X_1, X_2, ..., X_p) \in S_1^*,\ X_p \in M(X_1, X_2, ..., X_{p-1})\}$$

Technique to solve the problem (MIQPP) in S_1^*.

Firstly, consider the problem (MIQPP) in S^*. For i ≥ 1 and k ≥ 1, let B_k be the basis matrix corresponding to the basic feasible solution X_{B_K}. Suppose that the non-basic matrix is decomposed into N_k^1 and N_k^2, where,

$N_k^1 = \{j \mid a_j^k \notin B_k \text{ and } x_j^k \text{ is at its lower bound}\}$,

$N_k^2 = \{j \mid a_k^k \notin B_k \text{ and } x_j^k \text{ is at its upper bound}\}$.

$I_k = \{t \mid a_t \in B_k\}$.

Further, $A_{N_k^1} = \{a_j^k \in A \mid j \in N_k^1\}$, $A_{N_k^2} = \{a_j^k \in A \mid j \in N_k^2\}$.

Let $X_{N_k^1} = \{x_j^k \mid j \in N_k^1\}$ be a vector of non-basic variable at their lower bounds and $X_{N_k^2} = \{x_j^k \mid j \in N_k^2\}$ be a vector of non-basic variables at their upper bounds respectively.

For k ≥ 1, we have $B_k X_{B_k} + N_k^1 X_{N_k^1} + N_k^2 X_{N_k^2} = b$.

This implies $X_{B_k} + (B_k^{-1} N_k^1) X_{N_k^1} + (B_k^{-1} N_k^2) X_{N_k^2} = B_k^{-1} b$ (1)

This implies $X_{B_k} + \sum_{j \in N_k^1} y_{k_j} x_j^k + \sum_{j \in N_k^2} y_{k_j} x_j^k = B_k^{-1} b$. (2)

For i ≥ 1 and k ≥ 1, the objective function value corresponding to the basis B_k is given by

$$Z_{i1} = (C_{B_k})_i X_{B_k} + (C_{N_k^1})_i X_{N_k^1} + (C_{N_k^2})_i X_{N_k^2} + \alpha_i$$

$$= (C_{B_k})_i [B_k^{-1} b - (B_k^{-1} N_k^1) X_{N_k^1} - (B_k^{-1} N_k^2) X_{N_k^2}] + (C_{N_k^1})_i X_{N_k^1} + (C_{N_k^2})_i X_{N_k^2} + \alpha_i$$

$$= (C_{B_k})_i B_k^{-1} b + [(C_{N_k^1})_i - (C_{B_k})_i B_k^{-1} N_k^1] X_{N_k^1} + [(C_{N_k^2})_i - (C_{B_k})_i B_k^{-1} N_k^2)] X_{N_k^2} + \alpha_i$$

$$\therefore Z_{i1} = (C_{B_k})_i B_k^{-1} b - \sum_{j \in N_k^1} (z_j^{i1} - c_j)_k x_j^k - \sum_{j \in N_k^2} (z_j^{i1} - c_j)_k x_j^k + \alpha_i \qquad (3)$$

Similarly, we have

$$Z_{i2} = (D_{B_k})_i B_k^{-1} b - \sum_{j \in N_k^1} (z_j^{i1} - d_j)_k x_j^k - \sum_{j \in N_k^2} (z_j^{i1} - d_j)_k x_j^k + \beta_i \qquad (4)$$

Suppose that we have a current basic feasible solution, $X_{B_k}^0 = (x_{jk}^0)$, where $x_{jk}^0 = \ell_{jk}$, $j_k \in N_k^1$ and $x_{jk}^0 = u_{jk}$, $j_k \in N_k^2$.

Therefore, improved objective function values are given by

$$\left.\begin{aligned} Z_{i1}(X_{B_k}^0) &= (C_{B_k})_i B_k^{-1} b - \sum_{j \in N_k^1} (z_j^{i1} - c_j)_k l_{l_k} - \sum_{j \in N_k^2} (z_j^{i1} - c_j)_k u_{jk} + \alpha_i \\ Z_{i2}(X_{B_k}^0) &= (D_{B_k})_i B_k^{-1} b - \sum_{j \in N_k^2} (z_j^{i2} - d_j)_k \ell_{jk} - \sum_{j \in N_k^2} (z_j^{i2} - d_j)_k u_{jk} + \beta_i \end{aligned}\right] \qquad (5)$$

Also, $Z_i(X_{B_k}^0) = Z_{i1}(X_{B_k}^0).\, Z_{i2}(X_{B_k}^0)$, $i = 1, 2, \ldots., p.$

In order to find a new feasible solution, consider a non-basic variable x_{r_k} at its lower bound which undergoes a change ϕ_r^k. Using reference [13], the new solution is given by $\hat{X}_k = (\hat{x}_{jk})$, where

$$\left[\begin{aligned} &(\hat{x}_t)_k = (x_t^0)_k - y_{tr}^k \phi_r^k, && \forall t \in I_k \\ &(\hat{x}_r)_k = (\ell_r)_k + \phi_r^k && \\ &\hat{x}_{jk} = x_{jk}^0, && j_k \in N_k^1 \cup N_k^2 \setminus \{r\} \end{aligned}\right. \qquad (6)$$

The objective function value corresponding to a new feasible solution $\hat{X}_k$ is given by

$$Z_{i1}(\hat{X}_k) = C_{B_k}(B_k^{-1}b) - \sum_{j\in N_k^1\setminus\{r\}} (z_j^{i1} - c_j)_k \ell_{j_k} - (z_r^{i1} - c_r)_k(\ell_{rk} + \phi_r^k) - \sum_{j\in N_k^2} (z_j^{i1} - c_j)_k u_{jk} + \alpha_i$$

$$= C_{B_k}(B_k^{-1}b) - \left[\sum_{j\in N_k^1\setminus\{r\}} (z_j^{i1} - c_j)_k \ell_{j_k} + (z_r^{i1} - c_r)_k \ell_{rk}\right] - \phi_r^k (z_r^{i1} - c_r)_k - \sum_{j\in N_k^2} (z_j^{i1} - c_j)_k u_{jk} + \alpha_i$$

$$= \left[C_{B_k}(B_k^{-1}b) - \sum_{j\in N_k^1\setminus\{r\}} (z_j^{i1} - c_j)_k \ell_{j_k} - \sum_{j\in N_k^2} (z_j^{i1} - c_j)_k u_{jk} + \alpha_i\right] - \phi_r^k (z_r^{i1} - c_r)_k$$

$$= Z_{i1}(X^0) - \phi_r^k (z_r^{i1} - c_r)_k ..$$

$$\left.\begin{aligned} \therefore \qquad & Z_{i1}(\hat{X}_k) = Z_{i1}(X_k^0) - \phi_r^k (z_r^{i1} - c_r)_k \\ \text{similarly,} \qquad & Z_{i2}(\hat{X}_k) = Z_{i2}(X_k^0) - \phi_r^k (z_r^{i2} - d_r)_k \end{aligned}\right] \tag{7}$$

$$\therefore \qquad Z_i(\hat{X}_k) = Z_{i1}\hat{X}_k).Z_{i2}(\hat{X}_k)\,. \tag{8}$$

The new solution is a feasible extreme point, provided

$$\phi_r^k = \mathrm{Min}\left\{(u_r - \ell_r)_k,\ \left(\frac{x_{B_t}^k - \ell_{Bt}}{(y_{tj})_k}\middle|(y_{tj})_k > 0,\ t \in I_k\right)\left(\frac{u_{B_t}^k - x_{Bt}^k}{(y_{tj})_k}\middle|(y_{tj})_k < 0,\ t \in I_k\right)\right\}.$$

The following possibilities may arise depending on the value of ϕ_r^k:

(i) If $\phi_r^k = (u_r - \ell_r)_k$, then x_r^k attains its upper bound and remains non-basic. The change in the values of each basic variable $(\hat{x}_t)_k, t \in I_k$ and the objective functions $Z_{i1}(X_k)$ and $Z_{i2}(X_k)$ are given by the equations (6) and (7) respectively.

(ii) If $\phi_r^k = \left(\frac{x_{s_k} - \ell_{s_k}}{y_{rs_k}}\right)$, for some $s_k \in I_k$, then x_{r_k} becomes basic and x_{s_k} departs the basis and attains its lower bound. The change in the values of the basic variables $(\hat{x}_t)_k, t \in I_k$ and the objective functions $Z_{i1}(X_k)$ and $Z_{i2}(X_k)$ are given by the equations (6) and (7) respectively.

(iii) If $\phi_r^k = \left(\frac{u_{s_k} - x_{s_k}}{-(y_{rs_k})}\right)$, for some $s_k \in I_k$, then x_{r_k} becomes basic and x_{s_k} departs from the basis and attains its upper bound. The change in the values of the basic variables $(\hat{x}_t)_k, t \in I_k$ and the objective functions $Z_{i1}(X_k)$ and $Z_{i2}(X_k)$ are given by the equations (6) and (7) respectively.

The change in the value of the objective function $Z_i(X_k)$ $(i \geq 1)$ is given by

$$Z_i(\hat{X}_k) - Z_i(X_k^0) = [Z_{i1}(X_r^0) - \phi_r^k(z_r^1 - c_r)_k][Z_{i2}(X_k^0) - \phi_r^k(z_r^2 - d_r)_k] - Z_{i1}(X^0)Z_{i2}(X^0)$$

$$= -\phi_r^k[Z_{i1}(X_r^0)(z_r^{i2} - d_r)_k + Z_{i2}(X_k^0)(z_r^{i1} - c_r)_k - \phi_r^k(z_r^{i1} - c_r)_k.(z_r^{iz} - d_r)_k]$$

$$= -\phi_r^k(L_{ir})_k \tag{9}$$

where $(L_{ir})_k = Z_{i1}(X_k^0)(z_r^{i2} - d_r)_k + Z_{i2}(X_k^0)(z_r^{i1} - c_r)_k - \phi_{rk}(z_r^{i1} - c_r)_k(z_r^{i2} - d_r)_k$.

Similarly, if variable $x_{r_k} = u_{r_k}$ undergoes a change, then the new solution $\hat{X}_k = (\hat{x}_{jk})$ is defined as

$$\begin{cases} (\hat{x}_t)_k = (x_t^0)_k + y_{t_r}^k \phi_r^k, & \forall\, t \in I_k \\ (\hat{x}_r)_k = u_{rk} - \phi_r^k & \\ \hat{x}_{jk} = x_{jk}^0, & \forall\, j_k \in N_k^1 \cup N_k^2 \setminus \{r\} \end{cases} \tag{10}$$

The objective function value corresponding to a new integer feasible solution $\hat{X}_k$ is given by

$$Z_i(\hat{X}_k) = [Z_{i1}(X_k^0) + \phi_r^k (z_r^{i1} - c_r)_k][Z_{i2}(X_k^0) + \phi_r^k (z_r^{i2} - d_r)_k] \qquad (11)$$

The new solution is a feasible extreme point, provided

$$\phi_r^k = \text{Min}\left\{ (u_r - \ell_r)_k,\ \left(\frac{x_{B_t}^k - \ell_{Bt}}{-(y_{tj})_k} \middle| (y_{tj})_k < 0,\ t \in I_k \right), \left(\frac{u_{B_t}^k - x_{Bt}^*}{(y_{tj})_k} \middle| (y_{tj})_k > 0,\ \forall t \in I_k \right) \right\}$$

Thus, depending on the values of ϕ_r^k, the following possibilities may arise:

(i) If $\phi_r^k = (u_r - \ell_r)_k$, , then x_{r_k} attains its lower bound and remains non-basic. The change in the values of each basic variable $(\hat{x}_t)_k, t \in I_k$ and the objective function $Z_i(X_k)$ are given by the equations (10) and (11) respectively.

(ii) If $\phi_r^k = \dfrac{x_{s_k} - \ell_{s_k}}{-(y_{rs_k})}$, for some $s_k \in I_k$, then x_{r_k} becomes basic and x_{s_k} departs from the basis and attains its lower bound. The corresponding change in the values of the basic variables $(\hat{x}_t)_k, t \in I_k$ and the objective functions $Z_i(X_k)$ are given by the equations (10) and (11) respectively.

(iii) If $\phi_r^k = \dfrac{u_{s_k} - x_{s_k}}{y_{rs_k}}$, for some $s_k \in I_k$, then x_{r_k} becomes basic and x_{s_k} departs from the basis and attains its upper bound. The corresponding change in the values of the basic variables $(\hat{x}_t)_k, t \in I_k$ and the objective functions $Z_{i1}(X_k)$ is given by the equations (10) and (11) respectively.

The change in the value of the objective function $Z_i(X_k)$ $(i \geq 1)$ is given by

$$Z_i(\hat{X}_k) - Z_i(X_k^0) = \phi_r^k (L_{ir})_k \qquad (12)$$

where $(L_{ir})_k = Z_{i1}(X_k^0)(z_r^{i2} - d_r)_k + Z_{i2}(X_k^0)(z_r^{i1} - c_r)_k - \phi_r^k (z_r^{i1} - c_r)_k (z_r^{i2} - d_r)_k$.

Thus, we conclude that the non-basic variable x_{r_k} enters the basis which gives maximum improvement in the value of the objective function. We are interested in finding on optimal solution of the problem (MIQPP) in S_i .

Define, $J_1^k = \{j \mid j \in N_k^1 \text{ and } (L_{ij})_k = 0\}$.

$J_2^k = \{j \mid j \in N_k^2 \text{ and } (L_{ij})_k = 0\}$.

$T_1^k = \{j \mid j \in N_k^1 \text{ and } (L_{ij})_k \neq 0\}$.

$T_2^k = \{j \mid j \in N_k^2 \text{ and } (L_{ij})_k \neq 0\}$.

Any basic feasible solution to the problem $Z_i(X_k)$, $i \geq 1$, (i = 1, 2, ..., p) such that $(L_{ij})_k \leq 0 \; \forall \; j \in N_k^1$ and $(L_{ij})_k \geq 0 \quad \forall \; j \in N_k^2$ is a locally optimal solution. Since the objective function $Z_i(X_k)$ (i = 1, 2, ...,p) at each level is explicitly quasi-monotone and is maximised over a compact polyhedron, every locally optimal solution of $Z_i(X_k)$ $(i \geq 1)$ will also be a globally optimal solution.

An optimal integer feasible solution for $Z_i(X_k)$ $(i \geq 1)$ can be obtained by repeated application of cut in [13] in the simplex table. This yields optimal feasible solution for the problem in S_1^*.

Theorem 2: Let X_k $(k \geq 1)$ be an integer feasible solution of (MIQPP). Then, all integer feasible solutions of the problem (MIQPP) in S_1^* yielding value higher than $Z_i(X_k)$ $(i \geq 1)$ lies in the open half space,

$$\sum_{j \in T_1^k} (x_i - \ell_j) - \sum_{j \in T_2^k} (u_j - x_j) \leq 1 \qquad \text{(I)}$$

Proof: Let X_k, $k \geq 1$ be an integer feasible solution of (MIQPP). Let B_k be the basis matrix corresponding to X_{B_k} . We have,

$$AX_k = b$$

That is, $$B_K X_{B_k} + \sum_{j \in N_1^k} a_j x_j + \sum_{j \in N_2^k} a_j x_j = b \qquad (13)$$

Suppose that corresponding to the current optimal feasible solution, we have $x_{j_k} = \ell_{j_k}$, $j_k \in N_1^k$ and $x_{j_k} = u_{j_k}$, $j_k \in N_2^k$. Therefore, from (13), we get

$$B_K X_{B_k} + \sum_{j \in N_k^1} a_j^k \ell_{j_k} + \sum_{j \in N_k^2} a_j^k u_{j_k} = b \qquad (14)$$

For some $r \in T_1^k, k \geq 1,\ a_{r_k} = \sum_{t \in I_k} y_{t_r}^k b_r$, where $I_k = \{t \mid a_t \in B_k\}$.

Choose a scalar $\phi_r^k > 0$, equation (14) becomes

$$\sum_{t \in I_k} b_t x_{B_t}^k + \sum_{j \in N_1^k} a_{j_k} \ell_{j_k} + \sum_{j \in N_2^k} a_{j_k} u_{j_k} + \phi_r^k a_{r_k} - \phi_r^k a_{r_k} = b$$

That is, $$\sum_{t \in I_k} [x_{B_t}^k - \phi_r^k y_{t_r}^k] b_t + \sum_{j \in N_1^k \setminus \{r\}} a_{j_k} \ell_{j_k} + a_{r_k}(\ell_{r_k} + \phi_r^k) + \sum_{j \in N_k^k} a_{j_k} u_{j_k} = b \qquad (15)$$

Equation (15) gives a new feasible solution of (MIQPP) given by

$$X_k^1 = \begin{cases} x_{B_t}^1 = x_{B_t}^k - \phi_r^k y_{t_r}^k, & \forall t \in I_k \\ x_{r_k}^1 = \ell_{r_k} + \phi_r^k, & \text{for } r \in T_1^k \\ x_{j_k}^1 = \ell_{j_k}, & \forall j \in N_k^1 \setminus \{r\} \\ x_{j_k}^1 = u_{j_k}, & \forall j \in N_k^2 \end{cases}$$

Here, $x_{j_k}^1 = \ell_{j_k}\ \ \forall j \in N_k^1 \setminus \{r\}$ and $x_{j_k}^1 = u_{j_k},\ \forall j \in N_k^2$ are integers. Therefore, for X_k^1 to be an integer solution, it is required that ϕ_r^k should be a positive integer, so that $x_{r_k}^1 = \ell_{r_k} + \phi_r^k$, for $r \in T_1^k$ is also an integer. It is required that $\phi_r^k y_{t_r}^k$, $\forall t \in I_k$ is an integer, so that $x_{B_t}^1 = x_{B_t}^k - \phi_r^k y_{t_r}^k$, $\forall t \in I_k$ is an integer.

Besides this $x^1_{B_t}$ and $x^1_{r_k}$ should lie between the specified bounds, that is,

$$\ell_{B_t} \le x^1_{B_t} \le u_{B_t} \ \ \forall\, t \in I_k \ \text{ and } \ \ell_{r_k} \le x^1_{r_k} \le u_{r_k} \ \ \forall\, r \in T^k_1.$$

This implies $\ell_{r_k} \le \ell_{r_k} + \phi^k_r \le u_{rk}$, then is,

$$\phi^k_r \le u_{r_k} - \ell_{r_k} \quad \text{for } r \in T^k_1 \tag{16}$$

Again, we have $\ell_{B_t} \le x^1_{B_t} \le u_{B_t} \ \ \forall\, t \in I_k$, that is, $\ell_{B_t} \le x^k_{B_t} - \phi^k_r y^k_{t_r} \le u_{B_t} \ \ \forall\, t \in I_k$.

Three different cases arises depending on the value of $y^k_{t_r}$.

Case 1: If $y^k_{t_r} = 0$, then $\phi^k_r y^k_{t_r} = 0$.

This implies $\ell_{B_t} \le x^k_{B_t} \le u_{B_t} \ \ \forall\, t \in I_k$.

The condition is satisfied.

Case 2: If $y^k_{t_r} < 0$, then $(-\phi^k_r y^k_{t_r}) > 0$.

This implies $(x^k_{B_t} - \phi^k_r y^k_{t_r})$ is a positive integer which cannot exceed is upper bound, that is,

$$x^k_{B_t} - \phi^k_r y^k_{t_r} \le u_{B_t} \qquad \forall\, t \in I_k.$$

or $$\phi^k_r \le \frac{u_{B_t} - x^k_{B_t}}{y^k_{t_r}} \ \ \forall\, t \in I_k. \tag{17}$$

Case 3 : If $y^k_{t_r} > 0$, then $-(\phi^k_r y^k_{t_r}) < 0$.

This implies that $(x^k_{B_t} - \phi^k_r y^k_{t_r})$ is a positive integer, which cannot be less than its lower bound, that is,

$$\ell_{B_t} \le x^k_{B_t} - \phi^k_r y^k_{t_r} \qquad \forall\, t \in I_k$$

or $$\phi_r^k \le \frac{x_{B_t}^k - \ell_{B_t}}{y_{t_r}^k} \quad \forall\, t \in I_k. \tag{18}$$

Thus, from (16), (17) and (18), we get ϕ_r^k can assume any possible value given by

$$\phi_r^k = \text{Min}\left\{(u_{r_k} - \ell_{r_k}),\ \left(\frac{x_{B_t}^k - \ell_{B_t}}{y_{t_r}^k} : y_{t_r}^k > 0, t \in I_k\right), \left(\frac{u_{B_t} - x_{B_t}^k}{-y_{t_r}^k} : y_{t_r}^k < 0, t \in I_k\right)\right\}.$$

The objective function value corresponding to X_k is given by

$$Z_i(X_k) = Z_{i_1}(X_k).Z_{i_2}(X_k). \tag{19}$$

where

$$\left.\begin{array}{l} Z_{i_1}(X_k) = C_{B_k}(B_k^{-1}b) - \sum\limits_{j\in N_k^1} (z_j^{i_1} - c_j)_k \ell_{j_k} - \sum\limits_{j\in N_k^2} (z_j^{i_1} - c_j)_k u_{j_k} + \alpha_i \\ Z_{i_2}(X_k) = D_{B_k}(B_k^{-1}b) - \sum\limits_{j\in N_k^1} (z_j^{i_1} - d_j)_k \ell_{j_k} - \sum\limits_{j\in N_k^2} (z_j^{i_2} - d_j)_k u_{j_k} + \beta_i \end{array}\right] \tag{20}$$

The objective function value corresponding to a new integer feasible solution X_k^1 is given by

$$Z_i(X_k^1) = Z_{i_1}(X_k^1).Z_{i_2}(X_k^1). \tag{21}$$

Now,

$$Z_{i_1}(X_k^1) = C_{B_k}(B_k^{-1}b) - \sum_{j\in N_k^1\setminus\{r\}} (z_j^{i_1} - c_j)_k \ell_{j_k} - (z_r^{i1} - c_r)_k(\ell_{r_k} + \phi_r^k) - \sum_{j\in N_k^2} (z_j^{i_1} - c_j)_k u_{j_k} + \alpha_i$$

$$= (C_{B_k})(B_k^{-1}b) - \left[\sum_{j\in N_k^1\setminus\{r\}} (z_j^{i_1} - c_j)_k \ell_{j_k} + (z_r^{i1} - c_r)_k \ell_{r_k}\right] - (z_r^{i1} - c_r)_k \phi_r^k - \sum_{j\in N_k^2} (z_j^{i_1} - c_j)_k u_{j_k} + \alpha_i$$

$$= \left[C_{B_k}(B_k^{-1}b) - \sum_{j\in N_k^1} (z_j^{i_1} - c_j)_k \ell_{j_k} - \sum_{j\in N_k^2} (z_j^{i_1} - c_j)_k u_{j_k} + \alpha_i\right] - \phi_r^k (z_r^{i1} - c_r)_k$$

$\therefore \qquad Z_{i1}(X_k^1) = Z_{i1}(X_k) - \phi_r^k (z_r^{i1} - c_r)_k \qquad$ (Using (20))

Similarly, $\qquad Z_{i2}(X_{ik}) = Z_{i2}(X_k) - \phi_r^k (z_r^{i2} - d_r)_k$

Substituting these values in (21), we have

$$Z_i(X_k^1) = [Z_{i1}(X_k) - \phi_r^k (z_r^{i1} - c_r)_k].[Z_{i2}(X_k) - \phi_r^k (z_r^{i2} - d_r)_k] \qquad (22)$$

Subtracting (19) from (22), we have

$$Z_i(X_k^1) - Z_i(X_k) = [Z_{i1}(X_k) - \phi_r^k (z_r^{i1} - c_r)_k][Z_{i2}(X_k) - \phi_r^k (z_r^{i2} - d_r)_k] - Z_{i1}(X_k) Z_{i2}(X_k)$$

$$= -\phi_r^k [Z_{i1}(X_k)(z_r^{i2} - d_r)_k + Z_{i2}(X_k)\phi_r^k (z_r^{i1} - c_r)_k - \phi_r^k (z_r^{i1} - c_r)_k (z_r^{i2} - d_r)_k]$$

$$= -\phi_r^k (L_{ir})_k$$

Since $(L_{ir})_k \le 0$, $r \in T_1^k \quad \therefore Z_i(X_k^1) - Z_i(X_k) \ge 0.$

This implies $Z_i(X_k^1) \ge Z_i(X_k)$.

Thus, we get that X_k^1 is an integer feasible solution of the problem (MIQPP) with objective function value higher than the value corresponding to X_k.

We have $x_{j_k} = \ell_{jk} \ \ \forall\, j \in N_k^1 \setminus \{r\}$ and $x_{jk} = u_{jk} \ \ \forall j \in N_k^2$

$\therefore \qquad (x_{j_k} - \ell_{jk}) = 0 \ \ \forall\, j \in N_k^1 \setminus \{r\}$ and $(x_{jk} - u_{jk}) = 0, \ \ \forall j \in N_k^2$

$$\therefore \qquad \sum_{j \in N_k^1 \setminus \{r\}} (x_{jk} - \ell_{jk}) + \sum_{j \in N_k^2} (x_{jk} - u_{jk}) < 1$$

Hence, the integer feasible solution X_k^1 lies in the open half space

$$\sum_{j \in N_k^1 \setminus \{r\}} (x_{jk} - \ell_{jk}) + \sum_{j \in N_k^2} (x_{jk} - u_{jk}) < 1.$$

As ϕ_r^k assumes all possible integral values, we will obtain all integer feasible solutions with values higher than X_k, and all these integer solutions will lie in the open half space

$$\sum_{j\in N_k^1\setminus\{r\}} (x_{jk} - \ell_{jk}) + \sum_{j\in N_k^2} (x_{jk} - u_{jk}) < 1.$$

Definition 1: Edge: An edge E_r^k for some $\{r\} \in N_k^1$ incident at an integer feasible solution X_k is defined as

$$E_r^k : \begin{cases} x_t = x_{t_k} - \phi_r^k (y_{tr})_k, & t \in I_k \\ x_{t_k} = (\ell_{t_k}) + \phi_r^k, & \{r\} \in N_k^1 \\ x_{j_k} = \ell_{jk}, & j \in N_k^1 \setminus \{r\} \\ x_{j_k} = u_{jk}, & j \in N_k^2 \end{cases} \tag{23}$$

where

$$0 \le \phi_r^k \le \text{Min}\left\{ (u_{r_k} - \ell_{r_k}),\ \left(\frac{x_{B_t}^k - \ell_{B_t}^k}{(y_{t_j})_k} : (y_{t_j})_k > 0, t \in I_k \right), \left(\frac{u_{B_t}^k - x_{B_t}^k}{-(y_{t_j})_k} : (y_{t_j})_k < 0, t \in I_k \right), \right\} \tag{24}$$

Definition 2: An edge E_r^k, for some $\{r\} \in N_k^2$ incident at an integer feasible solution X_k is defined as

$$E_r^k : \begin{cases} x_t = x_{t_k} + \phi_r^k (y_{t_j})_k, & : \ t \in I_k \\ x_{rk} = u_{rk} - \phi_r^k \quad , & : \ \{r\} \in N_k^2 \\ x_{jk} = \ell_{jk} \quad , & : \ j \in N_k^1 \\ x_{jk} = u_{jk} \quad , & : \ j \in N_k^2 \setminus \{r\} \end{cases} \tag{25}$$

where

$$0 \le \phi_r^k \le \text{Min}\left\{ (u_{r_k} - \ell_{r_k}),\ \left(\frac{x_{B_t}^k - \ell_{B_t}^k}{-(y_{t_j})_k} : (y_{t_j})_k < 0, t \in I_k \right), \left(\frac{u_{B_t}^k - x_{B_t}^k}{(y_{t_j})_k} : (y_{t_j})_k > 0, t \in I_k \right) \right\} \tag{26}$$

Theorem 3 [13]: Edge Truncating cut: An integer feasible solution of (MIQPP) not lying on an edge $E_r^k, \{r\} \in T_k^1$ of the truncated region, through an integer point, say, X_k, lies in the closed half space

$$\sum_{j\in N_k^1\setminus\{r\}} (x_j - \ell_j) + \sum_{j\in N_k^2} (u_j - x_j) \geq 1 \tag{27}$$

Proposition 1: For i ≥ 1, k ≥ 1, all integer feasible solutions alternate to X_k, at each level depends on whether $\phi_r^k < 1$ or $\phi_r^k \geq 1$.

Proof: Consider the objective function $Z_i(X_k)$, i ≥ 1, at the i-th level. Let $X_k (k \geq 1)$ be its k-th best integer feasible solution.

Let A_j^k denote the set of integer feasible solutions alternate to X_k on an edge E_J^k. The alternate solution to X_k if it exists is obtained by moving along the edge E_j^k for some $j \in J_1^k \cup J_2^k$.

Suppose that for some $j \in J_1^k \cup J_2^k$, k ≥ 1, $\phi_r^k < 1$. Then, there are no eligible directions incident at the integer feasible solution X_k. Hence, there is no integer feasible solution on the edge E_j^k. This edge E_j^k is truncated by applying ETC. Let $\phi_r^k \geq 1$ for some $j \in J_1^k \cup J_2^k$. Since ϕ_r^k and $\phi_r^k y_{t_r}^k$ are integers for all t ∈ I_k, therefore, by moving on an edge E_j^k, an alternate solution to X_k is obtained. After obtaining all integer feasible solutions on the edge E_j^k, this edge is truncated using ETC.

Thus, an optimal feasible solution for $Z_i(X_k)$ (i ≥ 1, k ≥ 1) is obtained over the truncated region. It is either an integer feasible solution alternate to X_k or the next best integer solution X_{k+1} or a non-integer point. Therefore, by repeated application of ETC and the cut [13] whole feasible region for the integer solution at each level is scanned.

If after applying ETC's the solution at any level is infeasible, the problem (MIQPP) is infeasible. Thus, the process terminates.

Since the procedure for finding the integer solution moves from one extreme point to another which are finite in number, therefore, the procedure for finding the optimal solution to the problem (MIQPP) terminates in a fintie number of steps.

Algorithm for finding an optimal solution for Multi-level Integer Indefinite Quadratic Programming Problem with Bounded Variables

Consider the problem (MIQPP).

Step 1 : Set i = 1, k = 1 and r = 1.

Step 2 : Solve $Z_i(X_k)$. Let its optimal solution be $(X_k)_i^r$, where $(X_k)_i^r = (x_1^k, x_2^k, ..., x_{n_k}^k)$. If $(X_k)_i^r$ is an integer solution, go to step 3.

Otherwise, apply the cut [13] to find the integer solution for $Z_i(X_k)$

Step 3 : Solve $Z_{i+1}(X_k)$. Let its optimal integer solution be $(\tilde{X}_k)_{i+1}^r$, where $(\tilde{X}_k)_{i+1}^r = (\tilde{x}_1^k, \tilde{x}_2^k, ..., \tilde{x}_{n_k}^k)$.

If $(x_1^k, x_2^k, ..., x_{n_k}^k) = (\tilde{x}_1^k, \tilde{x}_2^k, ..., \tilde{x}_{n_k}^k)$, go to step 5, or to step 8.

Otherwise, set $J^k = (J_{1,r}^k)_i \cup (J_{2,r}^k)_i$. Go to step 4.

Step 4 : If $J^k = \phi$, introduce the cut (I) into the optimal table of $(X_k)_i^r$. Go to step 7. If $J^k \neq \phi$, choose $j \in J^k$ for which $\phi_j^k \geq 1$ and determine all the integer solutions along the edge E_j^k. Formulate the set $(A_j^k)_i^r$. Go to step 5.

If $\phi_j^k < 1$, for $j \in J^k$, choose any {j} and go to step 6.

Step 5 : Formulate the set $(A_j^k)_{i+1}^r$ If $(A_j^k)_i^r \cap (A_j^k)_{i+1}^r \neq \phi$, that is for some j, $(X_j^k)_i^r = (X_j^k)_{i+1}^r$, go to step 8. Otherwise, go to step 6.

If $(A_j^k)_i^r \cap (A_j^k)_{i+1}^r = \phi$, go to step 8.

Step 6 **:** Truncate the edge E_j^k by applying the cut

$$\sum_{j \in N_k^1} (x_j - \ell_j) + \sum_{j \in N_k^2} (u_j - x_j) \geq 1 \qquad \{j\} \in T_k^1$$

$$\text{or} \quad \sum_{j \in N_k^1} (x_j - \ell_j) + \sum_{j \in N_k^2 \setminus \{r\}} (u_j - x_j) \geq 1 \qquad \{j\} \in T_k^2 .$$

If the resulting problem is infeasible, go to step9. Otherwise, find an optimal feasible solution of this problem. Ser r = r + 1. Go to step 2.

Step 7 **:** If the problem so obtained is infeasible, go to step 9. Otherwise, set r = r + 1. Go to step 2.

Step 8 **:** Set i = i + 1. Go to step 2.

$(X_k)_i^r$ is an optimal solution for the problem (MIQPP).

Step 9 **:** (MIQPP) is infeasible.

Example : Consider the indefinite quadratic integer multi-level programming problem with bounded variables.

(TIQPP) : $\underset{x_1}{\text{Max}}\, Z_1(x_1, x_2, x_3, x_4) = (-x_1 + x_2 + 5)(x_1 + 2x_2 + 8)$

$$\underset{x_2, x_3}{\text{Max}}\, Z_2(x_1, x_2, x_3, x_4) = (x_1 + x_2 + 4)(x_2 - 2x_3 + x_4 + 5)$$

$$\underset{x_4}{\text{Max}}\, Z_3(x_1, x_2, x_3, x_4) = (x_1 - x_3 + 9)(2x_2 + 2x_4 + 9)$$

Subject to

$$\begin{aligned} 3x_1 - 2x_2 + x_4 &\leq 12 \\ x_1 + x_2 + x_3 + x_4 &\leq 14 \\ 2x_2 + 5x_3 &\leq 15 \end{aligned}$$

where $1 \leq x_1 \leq 5$, $0 \leq x_2 \leq 3$, $1 \leq x_3 \leq 3$, $0 \leq x_4 \leq 1$

x_1, x_2, x_3, x_4 are integers.

Solution: Consider the upper level problem w.r.t. the constraints

$$\underset{x_1}{\text{Max}}\ Z_1(x_1, x_2, x_3, x_4) = (-x_1 + x_2 + 5)(x_1 + 2x_2 + 8)$$

subject to

$$\begin{aligned} &3x_1 - 2x_2 + x_4 + x_5 &&= 12 \\ &x_1 + x_2 + x_3 + x_4 \quad + x_6 &&= 14 \\ &2x_2 + 5x_3 \qquad\qquad + x_7 &&= 15 \end{aligned}$$

where $1 \le x_1 \le 5$, $0 \le x_2 \le 3$, $1 \le x_3 \le 3$, $0 \le x_4 \le 1$

$0 \le x_5 \le \infty$, $0 \le x_6 \le \infty$, $0 \le x_7 \le \infty$.

At lower bound, we have $x_5 = 9$, $x_6 = 12$, $x_7 = 10$.

				ℓ	ℓ	ℓ	ℓ			
			$c_j \rightarrow$	1	–1	0	0	0	0	0
			$d_j \rightarrow$	1	2	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	x_5	9	3	–2	0	1	1	0	0
0	0	x_6	12	1	1	1	1	0	1	0
0	0	$x_{7=}$	10	0	2	5	0	0	0	1
$Z_{11} = -4$		$z_j^{11} - c_j \rightarrow$		–1	1	0	0	0	0	0
$Z_{12} = 9$		$z_j^{12} - d_j \rightarrow$		–1	–2	0	0	0	0	0
		$L_{1j} \rightarrow$		–8	27	0	0	0	0	0

Here, $\theta_1 = \min\left(\frac{9}{3}, \frac{12}{1}\right) = 3$, $\theta_2 = \min\left(\frac{12}{1}, \frac{10}{2}\right) = 5$.

Entering variable : x_2

Departing criterion : $\Delta_2 = \text{Min}\,(\gamma_1, \gamma_2, u_2 - \ell_2)$.

Here, $u_2 - \ell_2 = 3 - 0 = 3$.

$$\gamma_1 = \text{Min}\left(\frac{x_{B_t} - \ell_{B_t}}{y_{t_r}} : y_{t_r} > 0\right) = \text{Min}\left(\frac{12}{1}, \frac{10}{2}\right) = 5.$$

$$\gamma_2 = \text{Min}\left(\frac{u_{B_t} - x_{B_t}}{-y_{t_r}} : y_{t_r} < 0\right) = \text{Min}(\infty) = \infty..$$

$\therefore\ \Delta_2 = \text{Min}\ (5, \infty, 3) = 3.$

$$x_2 \to \ell_2 + \Delta_2 = 0 + 3 = 3..$$

Corresponding change in the value of x_i's is given by $X_B = b - y_2\Delta_2$

$$\begin{bmatrix} x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 10 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \\ 4 \end{bmatrix}.$$

The optimal table for the upper level problem $Z_1(X)$ is given by

				ℓ	u	ℓ	ℓ			
			$c_j \to$	1	–1	0	0	0	0	0
			$d_j \to$	1	2	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	x_5	15	3	–2	0	1	1	0	0
0	0	x_6	9	1	1	1	1	0	1	0
0	0	x_7	4	0	2	5	0	0	0	1
$Z_{11} = -7$		$z_j^{11} - c_j \to$		–1	1	0	0	0	0	0
$Z_{12} = 15$		$z_j^{12} - d_j \to$		–1	–2	0	0	0	0	0
		$L_{1j} \to$		–13	33	0	0	0	0	0

Table (1)

Here, $L_{1j} \le 0$ for lower bounded non-basic variables and $L_{1j} \ge 0$ for upper bounded non-basic variables.

$\therefore$ optimal solution for $Z_1(X)$ is $(X_1)_1^1 = (1,3,1,0)$.

Put $x_1^1 = 1$ in the lower level problem

$$\underset{x_2,x_3}{\text{Max}}\, Z_2(X) = (x_1 + x_2 + 4)(x_2 - 2x_3 + x_4 + 5)$$

subject to the constraints (28).

The problem reduces to

$$\underset{x_2,x_3}{\text{Max}}\, Z_2(X) = (x_2 + 5)(x_2 - 2x_3 + x_4 + 5)$$

subject to

$$\begin{aligned} -2x_2 + x_4 &\le 9 \\ x_2 + x_3 + x_4 &\le 13 \\ 2x_2 + 5x_3 &\le 15 \end{aligned} \qquad (29)$$

where $0 \le x_2 \le 3$, $1 \le x_3 \le 3$, $0 \le x_4 \le 1$, x_2, x_3, x_4 are integers.

Solving by the method, as explained above, the optimal table for $Z_2(X)$ is given by

				u	ℓ	u			
			$c_j \rightarrow$	–1	0	0	0	0	0
			$d_j \rightarrow$	1	–2	1	0	0	0
C_B	D_B	V_B	X_B	x_2	x_3	x_4	s_1	s_2	s_3
0	0	s_1	14	–2	0	1	1	0	0
0	0	s_2	8	1	1	1	0	1	0
0	0	s_3	4	2	5	0	0	0	1
$Z_{21} = -8$		$z_j^{21} - c_j \rightarrow$		1	0	0	0	0	0
$Z_{22} = 7$		$z_j^{22} - d_j \rightarrow$		–1	2	–1	0	0	0
		$L_{2j} \rightarrow$		17	–16	8	0	0	0

Table 2

The optimal solution for $Z_2(X)$ is $(X_1)_2^1 = (1,3,1,1)$. We have $(X_1)_1^1 \neq (X_1)_2^1$.

Consider $J^1 = (J_{1,1}^1)_1 \cup (J_{2,1}^1)$., where

$$(J_{1,1}^1)_1 = \{j : j \in N_k^1 : (L_{ij})_1 = 0\} = \{3, 4\}$$

$$(J_{2,1}^1)_2 = \{j : j \in N_k^2 : (L_{ij})_1 = 0\} = \phi$$

$\therefore \quad J^1 = \{3, 4\} \neq \phi.$

Therefore, an alternate feasible solution exists corresponding to $(X_1)_1^1$.

Take j = 3.

Using (24), we have $0 \le \phi_3^1 \le \text{Min}\left(2, \frac{9}{1}, \frac{4}{5}\right)$.

$$\therefore \quad 0 \le \phi_3^1 \le \frac{4}{5} < 1.$$

Since ϕ_3^1 has to be an integer, $\therefore$ no alternate integer solution exists on this edge, i.e., $(A_j^1)_1^r = \phi$.

Apply the cut (I) $\sum_{j \in N_k^1 \setminus \{r\}} (x_{j_k} - \ell_{j_k}) + \sum_{j \in N_k^2} (u_{j_k} - x_{j_k}) \ge 1$

$\Rightarrow \quad (x_1 - 1) + (x_4 - 0) + (3 - x_2) \ge 1$

or $\quad -x_1 + x_2 - x_4 \le 1.$

Introduce the cut in Table (1) and the solve as above, the optimal table is given by

				ℓ	u	ℓ					ℓ
			$c_j \rightarrow$	1	–1	0	0	0	0	0	0
			$d_j \rightarrow$	1	2	0	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	0	x_5	14	2	–1	0	0	1	0	0	1
0	0	x_6	8	0	2	1	0	0	1	0	1
0	0	x_7	4	0	2	5	0	0	0	1	0
0	0	x_4	1	–1	0	1	0	0	0	0	–1
$Z_{11} = -7$		$z_j^{11} - c_j \rightarrow$		–1	1	0	0	0	0	0	0
$Z_{12} = 15$		$z_j^{12} - d_j \rightarrow$		–1	2	0	0	0	0	0	0
		$L_{1j} \rightarrow$		–13	33	0	0	0	0	0	0

Table 3

We have $(X_1)_1^2 = (1,3,1,1)$.

Proceeding, we get corresponding to $(X_1)_1^2$, $(X_1)_2^2 = (1,3,1,1)$. and $(X_1)_3^1 = (1,3,1,1)$..

Now, take **j = 4.**

Using (24), we have $0 \le \phi_4^1 \le \min\left(1, \frac{15}{1}, \frac{9}{1}\right)$

$\therefore\ 0 \le \phi_4^1 \le 1$

Since ϕ_4^1 has to be an integer $\therefore$ ϕ_4^1 = 1.

Using (23), the solution so obtained is

$$(X_2)_1^2 = \begin{cases} x_5 = 14,\ x_6 = 8,\ x_7 = 4 \\ x_4 = 1 \\ x_1 = 1,\ x_3 = 1 \\ x_2 = 3 \end{cases}$$

That is, $(X_2)_1^2$ = (1, 3, 1, 11).

Put $x_2^1 = 1$ in $Z_2(X)$, the solution is $(X_2)_2^2$ = (1, 3, 1,1).

$\therefore$ we have $(X_2)_1^2 = (X_2)_2^2$.

Put $x_2^1 = 1$ and $x_2^2 = 3$ in $Z_3(X) = (x_1 - x_3 + 9)(2x_2 + 2x_4 + 9)$ subject to the constraints (28).

The optimal solution for $Z_3(X)$ is $(X_1)_3^2 = (1,3,1,1)$.

The observations for the above example have been summarized in Table 4.

$(X_k)_1^r$	$(X_k)_2^r$	$(X_k)_3^r$	$Z_1(X_k)$	$Z_2(X_k)$	$Z_3(X_k)$	(TIQPP)
$(X_1)_1^1 = (1,3,1,0)$	$(X_1)_2^1 = (1,3,1,1)$	$(X_1)_3^1 = (1,3,1,1)$	105	56	153	$(X_1)_1^1 = (1,3,1,0) \notin IR$
	$(X_2)_2^1 = (1,0,1,0)$	$(X_2)_3^1 = (1,0,1,1)$	105	15	99	
	$(X_3)_2^1 = (1,0,0,0)$	$(X_3)_3^1 = (1,0,0,1)$	105	25	110	
	$(X_4)_2^1 = (1,3,1,0)$	$(X_4)_2^1 = (1,3,1,1)$	105	48	153	
$(X_1)_1^2 = (1,3,1,1)$	$(X_1)_2^2 = (1,3,1,1)$	$(X_1)_3^1 = (1,3,1,1)$	105	56	153	$(X_1)_1^2 = (1,3,1,1) \in IR$
$(X_2)_1^3 = (5,3,1,0)$	$(X_2)_2^3 = (5,0,1,3)$	cannot proceed since $x_4 = 3$ is not possible.				$(X_2)_1^3 = (5,3,1,0) \notin IR$
$(X_3)_1^4 = (1,0,1,0)$	$(X_3)_2^4 = (1,3,1,1)$	$(X_3)_3^4 = (1,3,1,1)$	36	56	153	$(X_3)_1^4 = (1,0,1,0) \notin IR$
$(X_2)_1^2 = (1,3,1,1)$	$(X_2)_2^2 = (1,3,1,1)$	$(X_3)_3^2 = (1,3,1,1)$	105	56	153	$(X_2)_1^2 = (1,3,1,1) \in IR$

Table 4

From above table, we conclude that the optimal solution for the problem (TIQPP) is (1, 3, 1, 1).

Conclusions: The proposed algorithm scans the feasible region for the integral points. This is done using Gomory like cut and the edge truncating cut. The edge truncating cut removes the larger portion of the feasible region which contains no integer feasible solution. The algorithm scans the edges in such a manner that the edges once removed cannot reappear.

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