# LP modeling for the time optimal control problem with an application

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**ABSTRACT**

The time optimal control problem is to guide control system from a given state to a specified target. By extending measure theoretical approach for the classical optimal control problem to this case, the problem of finding the time optimal control is reduced to one consisting of minimizing a linear form over a set of positive measures. The resulting problem can be approximated by a finite dimensional linear programming (LP) problem. The nearly optimal control is constructed from the solution of the final LP problem. To find the lower bound of the optimal time a search algorithm is proposed. Numerical results are also given for several test examples to demonstrate the applicability and the efficiency of the proposed scheme. A demonstrative example illustrates the effectiveness of the method.

**Keywords and phrases:** Moving missile, fixed target, time optimal control, functional space, measure space, linear programming, search algorithm.

## 1. Introduction

Minimum-time control problems are an important class of problems in the study of optimal control systems that arise frequently in practical applications. For this reason, they have been extensively studied in the literature by both mathematicians and engineers alike. Some relevant references are [1]-[23]. These types of problems are usually complicated problems and there is no analytic method in general to solve them. For some analytic methods to solve special cases with various conditions we refer to [5] where time optimal for a class of second-order non-linear control systems is given. In [6] the analytic method of solving minimum-time

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control problems for linear systems is derived based on Pontryagin maximum principle. In contrast with analytic method, there are many numerical methods to solve such problems. For example, switching time computations method [7], switching time variation method [8, 9], switching time optimization [10], and Newtons method [11]. In [12] a numerical method is also proposed for minimum time control to a moving target for a linear time-varying system utilizing discretized form of the system's equations.

In order to solve the time optimal control model, we have extended a measure theory–based approach. Developed by Rubio [24], measure theory is an effective method for solving classical optimal control problems. In this method, to each admissible control-state, a linear continuous functional is first associated. Correspondence between continuous positive linear functionals and positive Borel measures leads to an optimization problem in measure space. The transformed problem in measure space is an appropriate formulation of the optimal control problem since it is a LP problem in measure space. The solution of this LP problem is then approximated by the solution of a finite-dimensional LP problem which is attractive for consistent numerical computations. The sub-optimal control–trajectory will be found from the solution of the corresponding LP problem. In this connection one may refer to [21], [25]-[34].

## 2. Problem statement

We consider a time optimal control problem in the following form

$$\text{minimize} \quad \int_0^T dt \tag{1}$$

subject to

$$\dot{x} = \mathcal{G}(t, x, u), \quad \text{a.e.} \quad t \in (0, T), \tag{2}$$

with initial condition

$$x(0) = x_0, \tag{3}$$

and final condition

$$x(T) = x_f. \tag{4}$$

Throughout this paper, we assume $J = [0, T]$ is the time interval of control. $T$ is unknown but we assume that there exist an upper bound $T_2$ and a temporary lower bound $T_1$ for $T$.

**Definition 2.1** *We shall say that the pair $p = (x(\cdot), u(\cdot))$ is admissible if the following conditions hold:*

*1) The state function $x(\cdot) : J \to \mathbb{R}^n$ of the system (2) is absolutely continuous and takes its values in the bounded connected set $A \subset \mathbb{R}^n$.*

*2) The control function $u(\cdot) : J \to \mathbb{R}^m$ of the system (2) is a measurable function on $J$ and takes its values in the given bounded set $U \subset \mathbb{R}^m$.*

*3) The pair $p$ satisfies the system (2)–(4).*

It is assumed that the set of all admissible pairs is nonempty and it is denoted by $\mathcal{P}$. Let $p$ be an admissible pair, $B$ be an open ball in $\mathbb{R}^{n+1}$ including $J \times A$ and $C^1(B)$ be the space of all real-valued continuously differentiable functions on $B$. Let $\phi \in C^1(B)$ and define functions $\phi^{\mathcal{G}} \in C(\Omega)$ as follows:

$$\phi^{\mathcal{G}} := \phi_x(t,x).\mathcal{G}(t,x,u) + \phi_t(t,x), \tag{5}$$

for each $(t,x,u) \in \Omega$, where $\Omega = J \times A \times U$. Since $p = (x(\cdot), u(\cdot))$ is an admissible pair, thus

$$\int_J \phi^{\mathcal{G}}(t,x,u)dt = \int_J [\phi_x(t,x).\mathcal{G}(t,x,u) + \phi_t(t,x)]dt \tag{6}$$

$$= \int_J \frac{d}{dt}\phi(t,x(t))dt = \phi(T,x(T)) - \phi(0,x(0)) := \Delta\phi. \tag{7}$$

Let $D(J^o)$ be the space of infinitely differentiable all real-valued functions with compact support in $J^o$, where $J^o$ is interior of $J$. For each $\psi \in D(J^o)$ define

$$\psi_j(t,x,u) := x_j\psi'(t) + \mathcal{G}_j(t,x,u)\psi(t), \quad j = 1,2,...,n,$$

where $x_j(t)$ is the jth component of $x(t)$. Then if $p = (x(\cdot), u(\cdot))$ be an admissible pair, we have

$$\begin{aligned}
&\int_J \psi_j(t,x(t),u(t))dt = \int_J [x_j(t)\psi'(t) + \mathcal{G}_j(t,x,u)\psi(t)]dt \\
&= \int_J x_j(t)\psi'(t)dt + \int_J \mathcal{G}_j(t,x,u)\psi(t)dt \\
&= x_j(t)\psi(t) \mid_0^T - \int_J [\dot{x}_j(t) - \mathcal{G}_j(t,x,u)]\psi(t)dt = 0.
\end{aligned}$$

Since the pair $p = (x(\cdot), u(\cdot))$ is admissible and $\psi(0) = \psi(T) = 0$; the above relations for $j = 1,2,...,n$ are reduced to

$$\int_J \psi^u(t,x(t),u(t))dt = 0, \tag{8}$$

where $\psi^u = (\psi_1, ..., \psi_n)^T$. Also, by choosing the functions which are dependent only on time and for each $p \in \mathcal{P}$, the following equalities can be also obtained as

$$\textstyle\int_J f(t,x(t),u(t))dt = a_f, \quad f \in C_1(\Omega), \tag{9}$$

where $C_1(\Omega)$ is the space of all functions in $C(\Omega)$ that are dependent only on time and $a_f$ is the integral of $f$ over $J$.

# 3. Metamorphosis

In the following, the problem of minimization $\int_0^T dt$ over $\mathcal{P}$ is transferred, into another, nonclassical problem which appears to have some better properties from computational stand point.

For each $p \in \mathcal{P}$, we consider the following continuous positive linear functional $\Lambda_p \in C^*(\Omega)$:

$$\Lambda_p(F) := \int_J F(t, x(t), u(t))dt, \qquad \forall F \in C(\Omega). \tag{10}$$

The mapping $p \to \Lambda_p$ is an injection [24], so this is an embedding from $\mathcal{P}$ to $C^*(\Omega)$. So problem of minimizing (1) over the constrains (2)-(4) is enlarged to the problem of minimizing

$$\Lambda(1) \tag{11}$$

subject to

$$\Lambda(\phi^{\mathcal{G}}) = \Delta\phi, \quad \phi \in C^1(B), \tag{12}$$

$$\Lambda(\psi^u) = 0, \quad \psi \in D(J^o), \tag{13}$$

$$\Lambda(f) = a_f, \quad f \in C_1(\Omega), \tag{14}$$

$$\Lambda \in C^*(\Omega). \tag{15}$$

Let $M^+(\Omega)$ denotes the space of all positive Borel measures on $\Omega$. By the Riesz representation theorem [35], there is an one-to-one correspondence between $\Lambda \in C^*(\Omega)$ and $\mu \in M^+(\Omega)$ as

$$\Lambda(F) = \mu(F) = \int_\Omega F d\mu, \qquad \forall F \in C(\Omega). \tag{16}$$

So one may change the problem (11) subject to (12)-(15) in functional space to the following equivalent optimization problem in measure space:

$$\text{minimize } \mu(1) \tag{17}$$

subject to

$$\mu(\phi^{\mathcal{G}}) = \Delta\phi, \quad \phi \in C^1(B), \tag{18}$$

$$\mu(\psi^u) = 0, \quad \psi \in D(J^o), \tag{19}$$

$$\mu(f) = a_f, \quad f \in C_1(\Omega), \tag{20}$$

$$\mu \in M^+(\Omega). \tag{21}$$

Define the set of all positive Borel measures satisfying (18)-(21) as $Q$, and *topologize* the space $M^+(\Omega)$ by the *weak*$^*$*-topology.* Consider the functional $\mathcal{I} : Q \longrightarrow \mathbb{R}$ defined by

$$\mathcal{I}(\mu) = \mu(1). \tag{22}$$

Now the measure theoretical problem (17)-(21) may be interpreted as the problem of minimizing $\mathcal{I}$ over $Q$. Now it is necessary to verify the existence of a solution for this problem.

**Theorem 3.1** *(i) The set $Q$ consisting of all measures satisfying in (18)-(21) is compact in $M^+(\Omega)$.*

*(ii) The functional $\mathcal{I} : Q \to \mathbb{R}$, defined in (22) is a linear continuous functional on the set $Q$ with weak*-topology.*

*(iii) The measure-theoretic problem, which consists in finding the minimum of the functional $\mathcal{I}$ in (22) over the set $Q$ of $M^+(\Omega)$, attains its minimum, say $\mu^*$, in the set $Q$.*

**Proof.** The proof is similar to Theorems 6.1 and 6.2 of [33]. □

**Remark 3.2** *Two main advantages of considering this measure theoretic form of the problem are:*

*(i) The existence of an optimal measure in the sets $Q$ that satisfies (18)-(21) can be studied in a straightforward manner without having to impose conditions such as convexity which may be artificial.*

*(ii) The functionals in (17)-(20) are linear although the main problem may be nonlinear.*

# 4. Approximation of the optimal measure

The minimizing problem (17)-(21) is an infinite-dimensional LP problem and we are mainly interested in approximating it. It is possible to approximate the solution of the problem (17)-(21) by the solution of a finite dimensional LP of sufficiently large dimension.

First we consider the minimization of (17) not over the set $Q$ but over a subset of it defined by requiring that only a finite number of constraints (17)-(21) be satisfied. Consider the equalities of (17)-(21). Let the sets $\{\phi_i, i \in I\!N\}$ and $\{\psi_h, h \in I\!N\}$ are the sets of total functions respectively in $C^1(B)$ and $D(J^o)$. This means that the set of linear combinations of these functions is dense in the $C^1(B)$ and $D(J^o)$, respectively. Now we can prove:

**Proposition 4.1** *Let $Q(M_1, M_2)$ be a subset of $M^+(\Omega)$ consisting of all measures which satisfy*

$$\begin{cases} \mu(\phi_i^{\mathcal{G}}) = \Delta\phi_i, & i = 1, 2, \ldots, M_1, \\ \mu(\psi^u) = 0, & h = 1, 2, \ldots, M_2. \end{cases} \tag{23}$$

*As $M_1$ and $M_2$ tend to infinity, $\eta(M_1, M_2) = \inf_{Q(M_1,M_2)} \mu(1)$ tends to $\eta = \inf_Q \mu(1)$.*

**Proof.** The proof is similar to Proposition 2 of [32]. □

The first stage of the approximation is completed successfully. As the second stage, from Theorem $A.5$ of [24], we can characterize a measure, say $\mu^*$, in the set $Q(M_1, M_2)$ at which the function $\mu \to \mu(1) = T$ attains its minimum. It follows from a result of Rosenbloom [36], that:

**Proposition 4.2** *The measure $\mu^*$ in the set $Q(M_1, M_2)$ at which the function $\mu \to \mu(1)$ attains its minimum has the following form*

$$\mu^* = \sum_{k=1}^{M_1+M_2} \kappa_k^* \delta_\Omega(\iota_k^*), \tag{24}$$

*where $\iota_k^* \in \Omega$ and the coefficients $\kappa_k^*, \geq 0$, $k = 1, 2, \cdots, M_1 + M_2$.*

Here $\delta_\Omega(\iota)$ is the unitary atomic measure characterized by

$$\delta_\Omega(\iota) = \begin{cases} 1 & \text{if } \iota \in \Omega \\ 0 & \text{otherwise,} \end{cases}$$

which also implies that

$$\delta_\Omega(\iota_k^*)(F) = F(\iota_k^*), \qquad \forall F \in C(\Omega).$$

The above representation of $\mu^*$ as a combination of unitary atomic measure changes the strange problem of finding a measure to a problem of finding $\{(\kappa_k^*, \iota_k^*) \ : \ k = 1, 2, ..., M_1 + M_2\}$ with linear constraints. If we could reduce the problem to one in which $\iota_1^*, \iota_2^*, ..., \iota_{M_1+M_2}^*$ are fixed and unknowns are the non-negative coefficients $\kappa_1^*, \kappa_2^*, ..., \kappa_{M_1+M_2}^*$, then we have a finite dimensional LP problem. This is the second stage of approximation.

**Proposition 4.3** *Let $\sigma$ be a countable dense subsets of $\Omega$. Given $\epsilon > 0$, a measures $\bar{\mu} \in M^+(\Omega)$ can be found such that*

$$\begin{aligned} &|(\mu^* - \bar{\mu})(1)| \leq \epsilon, \\ &|(\mu^* - \bar{\mu})(\phi_i^{\mathcal{G}})| \leq \epsilon, \ i = 1, 2, \ldots, M_1, \\ &|(\mu^* - \bar{\mu})(\psi_h^u)| \leq \epsilon, \ h = 1, 2, \ldots, M_2, \end{aligned}$$

*where the measure $\bar{\mu}$ has the following form*

$$\bar{\mu} = \sum_{k=1}^{M_1+M_2} \kappa_k^* \delta_\Omega(\iota_k), \tag{25}$$

*and the coefficients $\kappa_k^*$ are the same as in the optimal measures (24) and $\iota_k \in \sigma$.*

**Proof.** See the proof of Proposition III.3 in [24]. □

Thus the infinite-dimensional LP problem of minimizing (17) with constraints defined by (18)-(21) can be approximated by the following LP, where $\iota_1, \iota_2, ..., \iota_{M_1+M_2}$ are fixed in a countable dense subset of $\sigma$. Finally, the above results enable us to approximate the problem via the finite dimensional LP problem:

$$\text{Minimize} \quad \sum_{l=1}^{N} \kappa_l \tag{26}$$

subject to

$$\sum_{l=1}^{N} \kappa_l \phi_i^{\mathcal{G}}(\iota_l) = \Delta\phi_i, \quad i = 1, 2, \ldots, M_1, \tag{27}$$

$$\sum_{l=1}^{N} \kappa_l \psi_h^{u}(\iota_l) = 0, \quad h = 1, 2, ..., M_2, \tag{28}$$

$$\sum_{l=1}^{N} \kappa_l f_s(\iota_l) = a_{f_s}, \quad s = 1, 2, \ldots, L_1, \tag{29}$$

$$\kappa_l \geq 0, \quad l = 1, 2, ..., N. \tag{30}$$

In order to solve the LP problem (26)–(30), $\Omega$ is first partitioned into $N > M_1 + M_2$ subregions $\Omega_1, \Omega_2, ..., \Omega_N$, where $\Omega = \bigcup_{l=1}^{N} \Omega_l$, and $\iota_l$ is chosen in $\Omega_l$. From [21], assume that $J$ is divided to $L_1$ portion and $A \times U$ to $L_2$ portion, that is $N = L_1 L_2$. As the end part of $J$ is unknown, we divide $[0, T_1]$ into $L_1 - 1$ portion and $[T_1, T]$ is the rest partition.

We shall consider how one can choose total functions in the constraints (27)–(29). First we consider functions $\varphi_i$'s $\in C^1(B)$ of the form

$$x^q, \quad q = 0, 1, 2, \cdots.$$

Trivially the linear combinations of these functions are uniformly dense in the space $C^1(B)$, and we choose only $M_1$ of them. For the functions $\psi_h$ in (28), $\psi$ is chosen as the following forms:

$$\psi(t) = \begin{cases} \sin(\frac{2\pi r t}{T_1}), & t \in [0, T_1] \\ 0, & t \in [T_1, T] \end{cases}$$

$$\psi(t) = \begin{cases} 1 - \cos(\frac{2\pi r t}{T_1}), & t \in [0, T_1] \\ 0, & t \in [T_1, T] \end{cases}$$

where a finite number of positive integers $r$ is chosen. Finally, it is necessary to choose $L_1$ number of functions $f_s$ in (29) as follows:

$$f_s(t) := \begin{cases} 1, & t \in J_s \\ 0, & \text{otherwise}, \end{cases}$$

where

$$J_s = (\frac{(s-1)T_1}{L_1 - 1}, \frac{sT_1}{L_1 - 1}), \qquad s = 1, 2, ..., L_1 - 1,$$

$$J_{L_1} = [T_1, T].$$

From [21], we see that $a_{f_s}$ in (29) is achieved by

$$a_{f_s} = \begin{cases} \frac{T_1}{L_1 - 1}, & s = 1, 2, ..., L_1 - 1 \\ T - T_1, & s = L_1 \end{cases}$$

and then

$$\sum_{l=1}^{N} \kappa_l = T.$$

Solving the LP problem (26)-(30), we find the optimal values of decision variables $\{\kappa_1, \kappa_2, ..., \kappa_N\}$. In section 5 we discuss the construction of optimal control function.

## 5. Calculating the approximated optimal pair $(u(.), T)$

In this section, a combined algorithm is derived to find the best lower bound for optimal time $T$. Then, a piecewise constant control function from the solution of the LP problem (26)-(30) related to $T$ is constructed.

To solve the problem of choosing the lower bound $T_1$ in LP (26)-(30), we use a search algorithm to find the best choice for this lower bound which is proposed in [21]. The consequence of this algorithm is that the function evaluation $T_1 \rightarrow T(T_1)$ is done.

**Algorithm 5.1**

First let $I = [T_1, T_2]$ where $T_1 = 0$ and $T_2$ is an upper bound for $T$. Choose a penalty $M >> T_2$.

*Step 1*: Let $T^1 = T_1 + 0.382(T_2 - T_1)$ and $T^2 = T_1 + 0.618(T_2 - T_1)$ and solve the corresponding LP to find $T(T^1)$ and $T(T^2)$. The penalty $M$ is assigned to $T(T^1)$ or $T(T^2)$ if there exists no feasible solution for the corresponding LP problem.

*Step 2*: If $T(T^1) > T(T^2)$; then set $T_1 = T^1$ and $T_2 = T_2$; else if $T(T^1) < T(T^2)$ set $T_1 = T_1$ and $T_2 = T^2$.

*Step 3*: If the length of the interval $I = [T_1, T_2]$ is small enough, then stop with $\frac{T_1+T_2}{2}$ as the minimum value for $T_1$; else go to Step 1.

Now we explain construction of a nearly optimal control from the LP solution. From [24], a piecewise-constant optimal control function can be constructed by considering

$$t_k = \sum_{l \leq k} \kappa_l,$$

such that

$$u(t) \approx u_k, \quad t \in I_k = [t_{k-1}, t_k), \tag{31}$$

where $[0, T] = \bigcup_{k=1}^{M_1+M_2+L_1} I_k$. It is clear that the optimal control $u(.)$ in (31) can be written as

$$u(t) = \sum_{k=1}^{M_1+M_2+L_1} u_k \chi_{I_k}(t),$$

where $\chi_{I_k}$ is the characteristic function of the set $I_k$.

We need only to construct the control function $u(.)$, sine $x(.)$ is then simply the corresponding solution of differential equation (2), with $x(0) = x_0$ and $x(T) = x_f$, which can be estimated numerically.

# 6. Application in air and space sciences

As an applicable example in air and space sciences, we concentrate on developing an appropriate dynamic model for a moving missile to hit fixed and moving targets. In this model, the initial acceleration and the missile's angle are considered as the control inputs. Next, the performance measures consisting of the minimization of the missile flight time under diverse motion scenarios are considered. This performance measure is used to formulate the time optimal control problem, analytically. Some aspects of such a system have been studied previously by some researchers in [1]-[4]. As a minimum-time problem, this problem has many economical and military applications.

**6.1 Mathematical modeling of the problem**

Consider the Mass $M$ (sitting along the $x_2(t)$) which is acted upon by a thrust force, $F = M\alpha(t)$, as shown in Figure 1. It is desired that $M$ would hit a target which is sitting along the $x_1(t)$ axis, moving in $x_1(t)$ direction or moving in both $x_1(t)$ and $x_2(t)$ direction. It is also desired to minimize time for each of the above scenario via optimal input $\alpha(t)$ and $\beta(t)$. Figure 1 can be represented with the following time varying nonlinear state-space model [1] as

$$\begin{cases} \dot{x}_1 = x_3(t), \\ \dot{x}_2 = x_4(t), \\ \dot{x}_3 = \alpha(t)\cos\beta(t), \\ \dot{x}_4 = \alpha(t)\sin\beta(t) - g, \end{cases} \tag{32}$$

where $x_1(t), x_2(t), x_3(t), x_4(t), \alpha(t)$ and $\beta(t)$ are the horizontal position, vertical position, horizontal velocity, vertical velocity, acceleration, and angular position of the missile, respectively, and $g$ is the gravitational force.

The time optimal problem here is to guide the system (32) from a given state $x_0 \in \mathbb{R}^4$ to a specified target $x_f \in \mathbb{R}^4$. Guidance to the target must be done in such a way that the capture time is minimized.

**6.2 An illustrative example**

In order to test the proposed methodology, we consider the time optimal control (1)-(4)

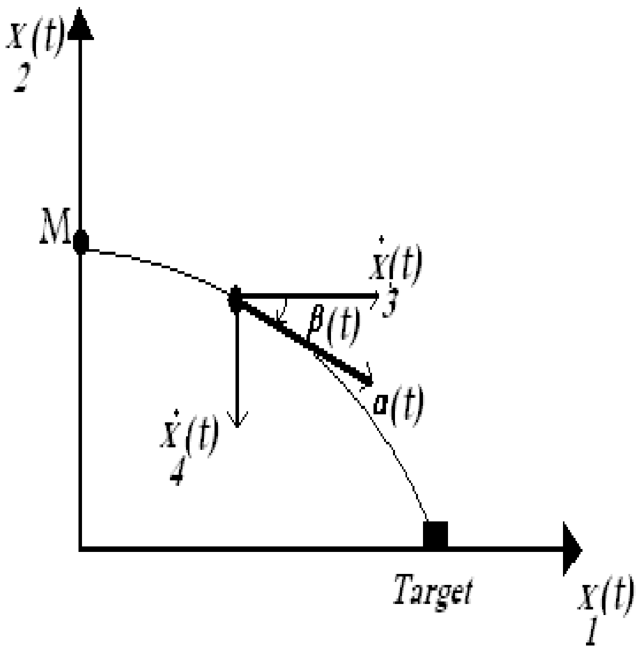


Figure 1: A simplified block diagram for missile target scenario.

with the following data [1] as

$$\begin{cases} x_1(0) = x_3(0) = x_4(0) = 0, \\ x_2(0) = 10, \ x_1(T) = 110, \ x_2(T) = 0, \\ x_3(T) = 141.4, x_4(T) = -14.14, \\ \alpha(t) = 100. \end{cases}$$

In this example, we choose

$$U = [-0.14, 0.22], \ A = [0, 110] \times [0, 10] \times [0, 150] \times [-15, 0].$$

$\Omega = J \times A \times U$ is divided into $N = 184800$ partitions by dividing $U$ to 10 partitions, $A = A_1 \times A_2 \times A_3 \times A_4$ to 1540 partitions and $J$ to 12 partitions. Thus set $\Omega$ will be covered with a type grid, where this grid will be defined by taking all points in $\Omega$ as $\iota_l = (t_l, x_l, u_l)$. The points in these grids will be numbered sequentially from 1 to $N$. We also choose $M_1 = 4$,

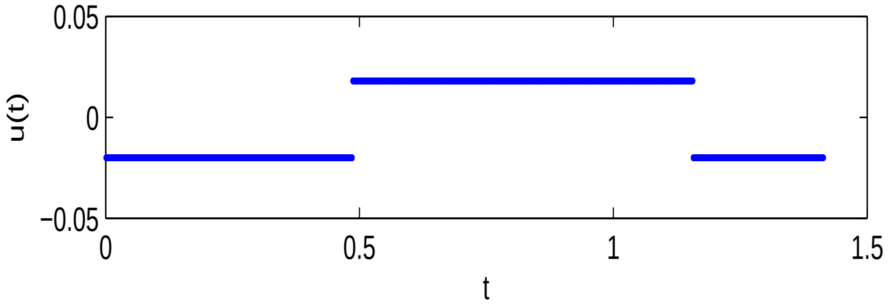


Figure 2: Nearly piecewise constant optimal control.

$M_2 = 4$ and $L_1 = 12$. For example, functions appearing in (27) are chosen as

$$\phi_1 = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \phi_2 = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix}, \phi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix}.$$

Implementing the corresponding LP model, by Algorithm 5.1 with $T_2 = 10$ as initial upper bound, the best lower bound is found $T_1 = 1.403$ and the nearly optimal capture time is $T = 1.414$. Figure 2 shows the nearly optimal control function that is obtained from the solution of the final LP problem. This piecewise constant function is substituted in the systems equation (2)-(4) and the corresponding response is found by solving an initial value problem numerically. The approximate suboptimal trajectories are also shown in Figure 3.

To end this section, we answer a natural question: are there advantages of the proposed method compared to the existing ones? To answer this, we summarize what we have observed from numerical experiments and theoretical results as below.

• The main advantages of the proposed method are that the method is not iterative, it is self-starting, and it does not need to solve corresponding boundary value problems.

• In this approach, the nonlinearity of the constraints and objective function has not serious effects on the solution.

• Because of its flexibility, this method has been extended to solve a variety of control problems.

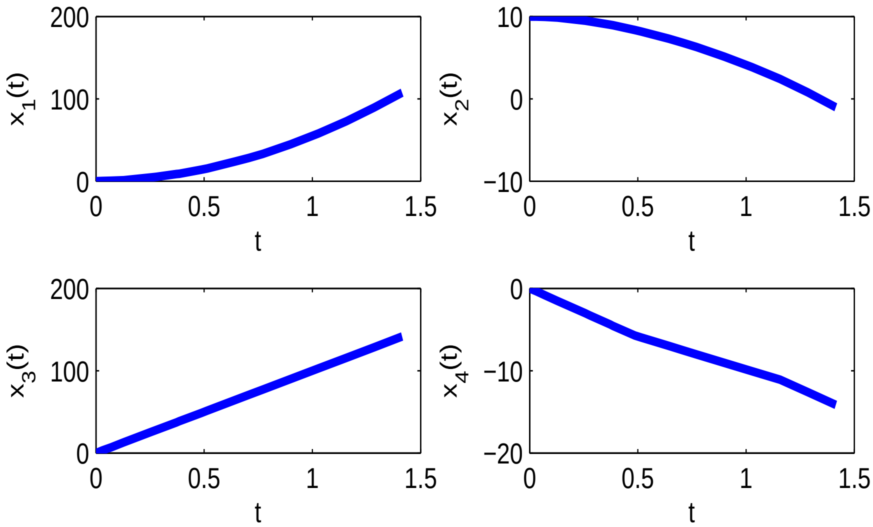


Figure 3: The approximate suboptimal trajectories.

## 7. Conclusion

A numerical method for solving optimal control problem of a missile-target intercept scenario under minimum time has been presented. When time is minimized, since acceleration $\alpha(t)$ is assumed to be constant and the gravitational force $g$ existed, the angle $\beta(t)$ is selected in a manner so that we would aim to hit to target. This is physically reasonable since $g$ affects the motion of the mass. The used numerical approach in this problem is based on some principles of measure theory, functional analysis and LP. In fact, the mathematical procedure used in this paper, is based on three steps:

(a) Any admissible pair is first replaced by exactly one point in a geometry.

(b) Then any point in this geometry is injected to a functional in a functional space.

(c) Any functional in functional space is embedded by a measure in some measure spaces.

Now if one can find optimal measure, obviously the time optimal path is obtained to missile guidance to hit the fixed target.

# References

[1] O. Ansary, A. Rahrooh, Optimial control of a moving missile tracking a moving target, Southcon/95. Conference Record, Fort Lauderdale, FL , USA, 1995; 123 - 126.

[2] D. N. Burghes, A. Graham, Introduction to Control Theory, Including Optimal Control, E.Horwood, Halsted Press: New Yourk , 1980.

[3] Y. Jun, Optimum control MLRS with forecasting model by criterion of generalized activity, Ballistic Journal, 12 (2000) 11 - 18.

[4] Z. Yuchun, Z. Yongqiang, The research on optimal control method of missiles' flight based on variable calculus, Third International Conference on Intelligent Networks and Intelligent Systems, 2010, 136 - 138.

[5] K. Balachandran, N. Rajagopal, Time - optimal synthesis for a special class of second order nonlinear control system, Elektronnoe Modelirovanie 1 (1999) 106 - 117.

[6] E. R. Pinch, Optimal Control and the Calculus of Variations, Oxford University Press: Oxford, 1993.

[7] C. Y. Kaya, J. L. Noakes, A global control law with implications in time-optimal control, Proceedings of the 3rd IEEE, Conference on Decision and Control, Orlando, FL, U.S.A., 1994, 3823 - 3824.

[8] R. R. Mohler, Bilinear Control Processes, Academic Press: New York, 1973.

[9] R. R. Mohler, Applications to bilinear control, Nonlinear Systems, Prentice-Hall: Englewood Cliffs, New Jersey, 1991.

[10] E. B. Meier, A. E. Bryson, Jr Efficient algorithm for time optimal control of a two-link manipulator, Journal of Guidance, Control, and Dynamics, 13 (1990) 859 - 866.

[11] V. M. Aleksandrov, Numerical solution for a linear optimal control problem. Fundamental and Applied Mathematics 6 (2000) 33-42.

[12] A. H. Borzabadi, H. H. Mehne, MILP modeling for time optimal guidance to a moving target, Journal of Applied Mathematics and Computing, 20 (2006) 293 - 303.

[13] Y. Chen , X. Dong, E. Zu, K. Liao, Adjacent grids algorithm and dynamics programming realization for minimum time-to-climb trajectories International Conference on Information and Automation, 2009 Zhuhai, Macau, 703 - 707.

[14] L. Doyen, M. Quincampoix, Multi-target control problems, Journal of Optimization Theory and Applications 93 (1997) 121 - 139.

[15] B. Farhadinia, K. L. Teo, R. C. Loxton, A computational method for a class of non-standard time optimal control problems involving multiple time horizons, Mathematical and Computer Modelling, 49 (2009) 1682 - 1691.

[16] M. E. Fisher, J. L. Noakes, K. L. Teo, A minimum trapping time problem in optimal control theory, Journal of Australian Mathematical Society: Series B 32 (1990) 100 - 114.

[17] C. Y. Kaya, J. L. Noakes, Computational method for time-optimal switching control, Journal of Optimization Theory and Applications, 117 (2003) 69 - 92.

[18] H. W. J. Lee, L. S. Jennings, K. L. Teo, V. Rehbock, Control parametrization enhancing technique for time optimal control problems, Dynamic Systems and Applications 6 (1997) 243 - 262.

[19] S. Lyashevskiy, Y. Chen, Time-optimal control for advanced aircraft in the presence of uncertainties, Decision and Control, Proceedings of the 35th IEEE, Kobe , 1996 Japan, 3204 - 3205.

[20] R. Naldi, L. Marconi, Minimum time trajectories for a class of V/STOL aircrafts Control and Decision Conference (CCDC), 2010 Chinese 2097 - 2102.

[21] H. H. Mehne, M. H. Farahi, A. V. Kamyad, MILP modelling for the time optimal control problem in the case of multiple targets, Optimal Control Applications and Methods 27 (2006) 77 - 91.

[22] Y. O. Shaw, B. L. Pierson, L. Ching-Fang, Aircraft model comparison for minimum time-to-climb problem, The First IEEE Regional Conference on Aerospace Control Systems, 1993, 463 - 468.

[23] K. L. Teo, C. C. Lim, Time optimal control computation with application to ship steering, Journal of Optimization Theory and Applications, 56 (1988) 145 - 156.

[24] J. E. Rubio, Control and Optimization; The Linear Treatment of Non-linear Problems Manchester, U. K., Manchester University Press, (1986).

[25] S. Effati, M. Hoseini, A. R. Nazemi, A new numerical method by revised measure theory for solving the nonlinear initial value problems, Applied Mathematics and Computation, 186 (2007), 780–788.

[26] M. Zamirian, M. H. Farahi, and A. R. Nazemi, An applicable method for solving the shortest path problems, Applied Mathematics and Computation, 190 (2007) 1479–1486.

[27] A. R. Nazemi, M. H. Farahi and M. Zamirian, Filtration problem in inhomogeneous dam by using embedding method, Applied Mathematics & Computing, 28 (2008) 313–332.

[28] A. R. Nazemi, M. H. Farahi and H. H. Mehne, Optimal shape design of iron pole section of electromagnet, Physics Letters A, 372 (2008) 3440–3451.

[29] A. R. Nazemi and M. H. Farahi Control the fibre orientation distribution at the outlet of contraction, Acta Applicandae Mathematicae, 106 (2009) 279–292.

[30] A. R. Nazemi, LP modelling for the time optimal control problem of the heat equation, Journal of Mathematical Modelling and Algorithm, 10 (2011) 227–244.

[31] M. H. Farahi, H. H. Mehne and A. H. Borzabadi, Wing drag minimization by using measure theory, Optimization Methods and Software, 21 (2006) 169-177.

[32] M. H. Farahi, A. H. Borzabadi, H. H. Mehne, A. V. Kamyad, Measure theoretical approach for optimal shape design of a nozzle, Applied Mathematics & Computing, 17 (2005), 315-328.

[33] H. H. Mehne, M. H. Farahi, J. A. Esfahani, On an optimal shape design problem to control a thermoelastic deformation under a prescribed thermal treatment, Applied Mathematical Modelling, 168 (2005) 1258–1272.

[34] A. V. Kamyad, J. E. Rubio and D. A. Wilson, The optimal control of the multidimensional diffusion equation, Journal of Optimization Theory and Applications, 70 (1991), 191–209.

[35] W. Rudin, Functional Analysis, second ed., McGraw-Hill, New York, (1991).

[36] P. C. Rosenbloom, Quelques classes de problems extremaux, Bull. Soc. Math. France 80 (1952) 183–216. England, (1973).