Exact Solution of First Order Constraints Quadratic Optimal Control Problem

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Abstract

In this paper, we construct an algorithm to get the exact solution of the following linear quadratic optimal control problem of first order ordinary equation:

$$\textbf{(QF)} \qquad \begin{cases} \min J(x,u) = \displaystyle\int_0^T \left[x^2 + u^2\right] dt \\ \text{with constraints} \\ x' = ax + bu, \quad x(0) = c, \quad 0 \le t \le T, \end{cases}$$

where $x(t)$, $u(t)$, are functions from $[0,T] \to \Re$, a, b, c are real parameters and T is the final process time. We input the values of a, b, c, T and output the exact optimal solutions of $x(t)$ and $u(t)$.

1 Introduction

Optimal control problems governed by ordinary or partial differential equations arise in a wide range of applications [1]- [5]. Of a special interest is the linear quadratic optimal control problem (LQ), which appear in many different fields in engineering [6]- [9] as well as in quantum mechanics [10]- [12] and in other fields [13]- [19] due to its interesting features and its wider applicabilities. Sargent [19] gave historical survey of optimal control and went on to review the different approaches to the numerical solution of optimal control problem. It is well known that generally optimal quadratic control problems are difficult to solve. In particular, their analytical solutions are given for some cases [20]. In this paper, we find the exact solution of the quadratic optimal control of first order ordinary differential equation.

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2 Necessary and sufficient optimality conditions

Theorem 1. *Given the optimal control u and corresponding solution $x(t)$ of the the problem **(QF)**, then there exists adjoint variable $p(t) \in \Re$ such that the following equations are satisfied.*

$$\left.\begin{aligned} x' &= ax + \frac{b^2}{2}p \\ p' &= 2x - ap \end{aligned}\right\} \tag{1}$$

with initial and final conditions

$$x(0) = c, \quad p(T) = 0. \tag{2}$$

Proof. By introducing adjoint variable $p(t) \in \Re$, the required augmented functional from problem **(QF)** can be formed. The Hamiltonian function is given as

$$H(x, u, p, x') = \left[x^2 + u^2\right] + p\left[x' - ax - bu\right]$$

From the knowledge of calculus of variation [13], we form the necessary conditions for optimal control problem using Euler-Lagrangian (E-L) equations for H regarded as function of four variables (x, u, p, x'). Thus, the E-L system can be written

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial H}{\partial x'}\right] &= \frac{\partial H}{\partial x}, \\ \frac{d}{dt}\left[\frac{\partial H}{\partial u'}\right] &= \frac{\partial H}{\partial u}, \\ \frac{d}{dt}\left[\frac{\partial H}{\partial p'}\right] &= \frac{\partial H}{\partial p}, \end{aligned}$$

which gives the result. □

Now equation (1) can be rewritten as first order homogeneous linear differential system;

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} a & \frac{b^2}{2} \\ 2 & -a \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \tag{3}$$

From linear algabra,(3) have a unique general solution

$$\begin{pmatrix} x \\ p \end{pmatrix} = e^{Mt} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad M = \begin{pmatrix} a & \frac{b^2}{2} \\ 2 & -a \end{pmatrix}$$

and the matrix M can be diagonalize as follows:

$$M = \begin{pmatrix} \frac{a+\lambda}{2} & \frac{a-\lambda}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} \frac{a+\lambda}{2} & \frac{a-\lambda}{2} \\ 1 & 1 \end{pmatrix}^{-1}, \quad \lambda = \sqrt{a^2 + b^2}$$

then

$$\begin{aligned} e^{tM} &= \begin{pmatrix} \frac{a+\lambda}{2} & \frac{a-\lambda}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda}t & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} \frac{a+\lambda}{2} & \frac{a-\lambda}{2} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{\lambda} \begin{pmatrix} a\sinh\lambda t + \lambda\cosh\lambda t & \frac{b^2}{2}\sinh\lambda t \\ 2\sinh\lambda t & -a\sinh\lambda t + \lambda\cosh\lambda t \end{pmatrix} \end{aligned}$$

so

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} a\sinh\lambda t + \lambda\cosh\lambda t & \frac{b^2}{2}\sinh\lambda t \\ 2\sinh\lambda t & -a\sinh\lambda t + \lambda\cosh\lambda t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{4}$$

Then the general solutions of $x(t)$, $p(t)$ are given by

$$\begin{aligned} x(t) &= \frac{C_1}{\lambda}\left[a\sinh\lambda t + \lambda\cosh\lambda t\right] + \frac{C_2 b^2}{2\lambda}\sinh\lambda t \\ p(t) &= \frac{2C_1}{\lambda}\sinh\lambda t + \frac{C_2}{\lambda}\left[-a\sinh\lambda t + \lambda\cosh\lambda t\right] \end{aligned} \tag{5}$$

where C_1 and C_2 are two constants, which can be calculated directly from initial and final conditions (2):

$$C_1 = c, \quad C_2 = \frac{-2c\sinh\lambda_1 T}{-a\sinh\lambda_1 T + \lambda\cosh\lambda_1 T}$$

By substituting in (5), we obtain

$$\left.\begin{aligned} x(t) &= \frac{c\left[-a\sinh\lambda(T-t) + \lambda\cosh\lambda(T-t)\right]}{-a\sinh\lambda T + \lambda\cosh\lambda T}, \\ p(t) &= \frac{2c\sinh\lambda(T-t)}{-a\sinh\lambda T + \lambda\cosh\lambda T} \end{aligned}\right\} \tag{6}$$

3 Riccati equation

In this section, we will construct (6) by using another method (the method of feedback control).

From (4), we have

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} a\sinh\lambda t + \lambda\cosh\lambda t & \frac{b^2}{2}\sinh\lambda t \\ 2\sinh\lambda t & -a\sinh\lambda t + \lambda\cosh\lambda t \end{pmatrix} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}, \tag{7}$$

Since we know $x(0) = c$, the task is to choose $p(0)$ so that $p(T) = 0$.

An elegant way to do so is to try to find optimal control in linear feedback form; that is, to look for a function $q(.) : [0,T] \to \Re$ for which

$$u(t) = q(t)x(t).$$

We henceforth suppose that an optimal feedback control of this form exists, and attempt to calculate $q(t)$. Now

$$\frac{b^2}{2}p(t) = u(t) = q(t)x(t)$$

whence

$$\frac{b^2 p(t)}{2x(t)} = q(t)$$

Define now

$$r(t) = \frac{p(t)}{x(t)}$$

so that

$$q(t) = \frac{b^2}{2}r(t), \quad r(T) = 0$$

and from (1), we have a nonlinear first-order ODE for $r(.)$ with a terminal boundary condition:

$$(\mathbf{R}) \qquad \begin{cases} r' = 2 - 2a\,r - \dfrac{b^2}{2}r^2, & 0 \le t \le T \\ r(T) = 0. \end{cases}$$

This is called the Riccati equation

In summary so far, to solve our linear-quadratic regulator problem, we need to first solve the Riccati equation.

By taking $r(t) = \frac{2}{b^2}\frac{d'}{d}$, we can convert **(R)** into the following second-order, linear ODE:

$$\begin{cases} d'' = b^2 d - 2ad', & 0 \le t \le T \\ d'(T) = 0, & d(T) = 1. \end{cases} \tag{8}$$

The solution of (8) is given by

$$d(t) = \frac{e^{-a(t-T)}}{\lambda}\left[a \sinh \lambda(t-T) + \lambda \cosh \lambda(t-T)\right]$$

Then the solution $r(t)$ and $q(t)$ are given respectively by

$$r(t) = \frac{-2 \sinh \lambda(T-t)}{-a \sinh \lambda(T-t) + \lambda \cosh \lambda(T-t)}$$

$$q(t) = \frac{-b^2 \sinh \lambda(T-t)}{-a \sinh \lambda(T-t) + \lambda \cosh \lambda(T-t)}$$

and

$$p(0) = \frac{2c}{b^2} q(0) = \frac{-2c \sinh \lambda(T)}{-a \sinh \lambda(T) + \lambda \cosh \lambda(T)}$$

by substituting in (7), we obtain (6)

4 Algorithm

From the above sections, we have the following algorithm:

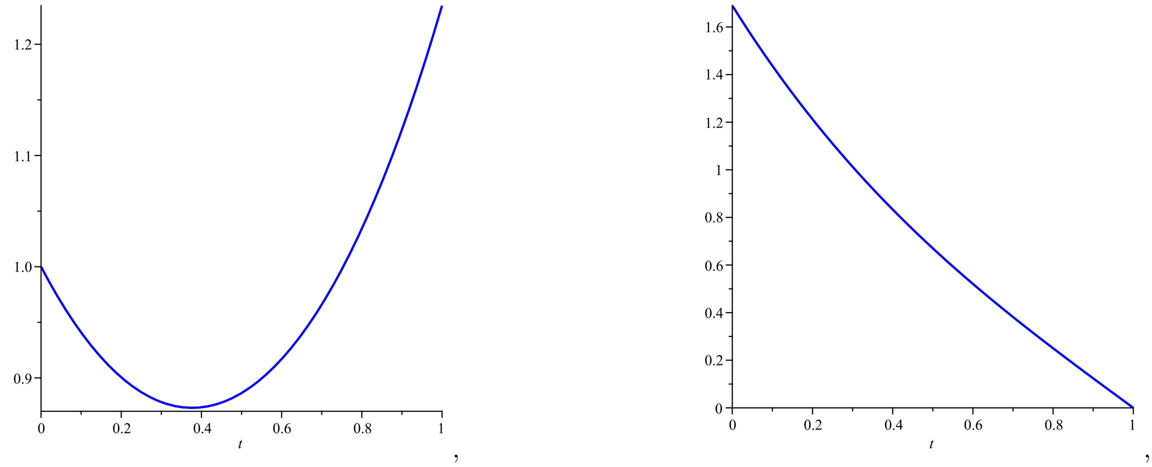
Algorithm 1 Compute exact solution of first order constraints quadratic optimal control problem **QF**

1. Input data(a, b, c and $T.$).
2. Compute $\lambda = \sqrt{a^2 + b^2}$.
3. Compute $x(t)$ and $u(t)$ from the following equations:

$$\left.\begin{aligned} x(t) &= \frac{c\left[-a \sinh \lambda(T-t) + \lambda \cosh \lambda(T-t)\right]}{-a \sinh \lambda T + \lambda \cosh \lambda T}, \\ u(t) &= \frac{b^2 p(t)}{2} = \frac{b^2 c \sinh \lambda(T-t)}{-a \sinh \lambda T + \lambda \cosh \lambda T} \end{aligned}\right\}$$

4. Graph $x(t)$ and $u(t)$.

To illustrate the method, we give some examples.



(a) x(t) (b) u(t)

Figure 1: Example 1

Example 1

$$\min J(x,u) = \int_0^1 \left[x^2 + u^2\right] dt$$

with constraints

$$x'(t) = x(t) + u(t), \quad x(0) = 1, \quad 0 \le t \le 1.$$

Example 2

$$\min J(x,u) = \int_0^{\ln 2} \left[x^2 + u^2\right] dt$$

with constraints

$$x'(t) = 3x(t) + 4u(t), \quad x(0) = 1, \quad 0 \le t \le \ln 2.$$

5 Conclusion

An algorithm for the exact optimal control solutions for first order ordinary equations is obtained. The result obtained by two methods, the first is the direct method and the second is feedback controls method. Also we get the solution of Riccati equation.

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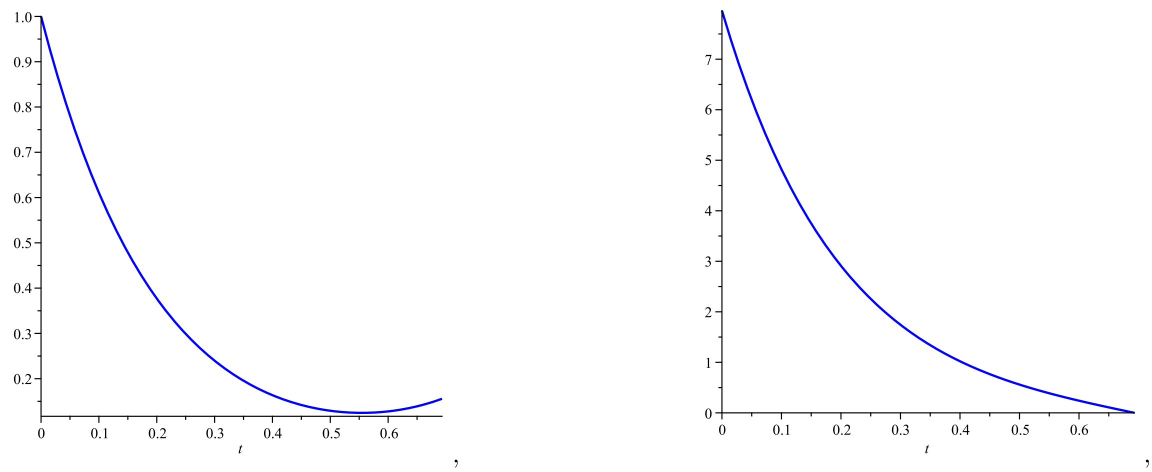

(a) x(t) (b) u(t)

Figure 2: Example 2

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