Functional Variable Method for Solving the Generalized Reaction Duffing Model and the Perturbed Boussinesq Equation

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Abstract

In this paper, we employ the functional variable method to find the exact solutions of generalized reaction Duffing model and the shallow water waves along with its perturbation terms that are modeled by Boussinesq equation. The traveling wave solutions obtained via this method are expressed by hyperbolic functions and the trigonometric functions. We believe that this approach can also be used to solve other nonlinear partial differential equations.

Keywords: Functional variable method; Generalized reaction Duffing model; Perturbed Boussinesq equation.

1. Introduction

Searching for exact solutions of nonlinear partial differential equations (NPDEs) has become a more attractive topic in physical science and nonlinear science. The investigation of the travelling wave solutions for NPDEs plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as chemistry, physics and the engineering disciplines. Recently, many kinds of powerful methods have been proposed to find exact solutions of NPDEs, for example, Variational iteration method[1], Algebraic method [2], Jacobi elliptic function expansion method [3], F-expansion method [4], Auxiliary equation method [5], Tanh method [6] and Generalized hyperbolic function [7].

In the pioneering work, *Zerarka et al* introduced the so-called functional variable method to find the exact solutions for a wide class of linear and nonlinear wave equations [8-9]. This method was further developed by many authors [10-12]. The advantage of this method is that one treats nonlinear problems by essentially linear methods, based on which it is easy to construct in full the exact solutions such as soliton-like waves, compacton and noncompacton solutions, trigonometric function solutions, pattern soliton solutions, black solitons or kink solutions, and so on.

In this paper, we applied the functional variable method to obtain the exact solutions of the generalized reaction Duffing model in the form

$$u_{tt} + pu_{xx} + qu + ru^2 + su^3 = 0, \tag{1}$$

where p, q, r and s are all constants [13].

Eq. (1) reductions many well-known nonlinear wave equations such as

(i) Klein-Gordon equation

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0. \tag{2}$$

(ii) Landau-Ginzburg-Higgs equation

$$u_{tt} - pu_{xx} - m^2 u + g^2 u^3 = 0. \tag{3}$$

(iii) φ^4 equation

$$u_{tt} - u_{xx} + u - u^3 = 0. \tag{4}$$

(iv) Duffing equation

$$u_{tt} + bu + cu^3 = 0. \tag{5}$$

(v) Sine-Gordon equation

$$u_{tt} - u_{xx} + u - \frac{1}{6}u^3 = 0. \tag{6}$$

We also consider perturbed Boussinesq equation (BE) in the form

$$u_{tt} - k^2 u_{xx} + a(u^{2n})_{xx} + b u_{xxxx} = \beta u_{xx} + \rho u_{xxxx}, \tag{7}$$

where u represents the wave profile while the independent variables x and t represent the spatial and temporal coordinates respectively. a, b, k, β and ρ are all constants. If the right hand side of Eq. (7) is zero, it represents the Boussinesq equation. For the perturbation terms in Eq. (7), the coefficient of β is the dissipative term and ρ provides higher term [14]. On the left side of Eq. (7), a represents the coefficient of nonlinear term while the exponent n represents power law nonlinearity factor. Typically, n dictates the strength of nonlinearity.

The rest of this paper is organized as follows: in Section 2, we present the summary of the functional variable method. In Section 3, the applications of our method to the generalized reaction Duffing model and the perturbed BE are illustrated. Lastly, conclusions are given in Section 4.

2. The functional variable method

In this Section we describe the main steps of the functional variable method for finding exact solutions of NPDEs.

Consider a general nonlinear partial differential equation in the form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{8}$$

where $u = u(x,t)$ is the solution of nonlinear partial differential equation Eq. (8), the subscript denotes partial derivative.

Zerarka et al, in [8] has summarized the functional variable method in the following.

Using a wave transformation

$$u(x,t) = U(\xi), \qquad \xi = x - ct, \tag{9}$$

where c is constant to be determined later. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.), \qquad \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \qquad \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.), \quad \ldots.$$

Using Eq. (9), the nonlinear partial differential equation Eq. (8) can be converted to a nonlinear ordinary differential equation like

$$G(U, U_{\xi}, U_{\xi\xi}, U_{\xi\xi\xi}, \ldots) = 0. \tag{10}$$

Then we make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_{\xi} = F(U), \tag{11}$$

and some successive derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)''\sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \\ &\vdots \\ &. \end{aligned} \tag{12}$$

where "$'$" stands for $\frac{d}{dU}$.

The ODE (10) can be reduced in terms of U, F and its derivatives upon using the expressions of Eq. (12) into Eq. (10) gives

$$G(U, F, F', F'', F''', \ldots) = 0. \tag{13}$$

The key idea of this particular form Eq. (13) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq. (13) provides the expression of F, and this, together with Eq. (11), give appropriate solutions to the original problem.

3. Applications

In this Section, we demonstrate the application of functional variable method for finding the exact travelling wave solutions of the generalized reaction Duffing model and the perturbed BE.

***3.1.* Generalized reaction Duffing model.** Assume that Eq. (1) has an exact solution in the form of a travelling wave

$$u(x,t) = U(\xi), \qquad \xi = x - ct, \tag{14}$$

where c is wave velocity. Substituting (14) into (1), we get the following nonlinear ODE:

$$c^2 U_{\xi\xi} + pU_{\xi\xi} + qU + rU^2 + sU^3 = 0, \tag{15}$$

or

$$U_{\xi\xi} = -\frac{1}{c^2 + p}[qU + rU^2 + sU^3]. \tag{16}$$

Substituting Eq. (12) into Eq. (16) we obtain

$$(F(U)^2)' = -\frac{2}{c^2 + p}[qU + rU^2 + sU^3], \tag{17}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq. (17) with respect to U and after the mathematical manipulations, we have

$$F(U) = \sqrt{-\frac{1}{c^2 + p}}U\sqrt{q + \frac{2r}{3}U + \frac{s}{2}U^2}, \tag{18}$$

or

$$F(U) = \sqrt{-\frac{s}{2(c^2 + p)}}U\sqrt{(U + \frac{2}{3}\frac{r}{s})^2 + \frac{2q}{s} - \frac{4}{9}\frac{s^2}{r^2}}. \tag{19}$$

From Eq. (11) and Eq. (19) we deduce that

$$\int \frac{dU}{U\sqrt{(U - N)^2 - M^2}} = \sqrt{-\frac{s}{2(c^2 + p)}}\,\xi, \tag{20}$$

where $N = -\frac{2}{3}\frac{r}{s}$, $M = \sqrt{-\frac{2q}{s} + N^2}$. After integrating Eq. (20), with zero constant of integration, we have following exact solutions:

if $\frac{q}{c^2 + p} < 0$, we have the hyperbolic solutions:

$$u_1(x,t) = \frac{(M + N)\text{sech}^2\left[\frac{1}{2}\sqrt{-\frac{q}{c^2 + p}}(x - ct)\right]}{1 + (\frac{M + N}{M - N})\text{tanh}^2\left[\frac{1}{2}\sqrt{-\frac{q}{c^2 + p}}(x - ct)\right]}, \tag{21}$$

$$u_2(x,t)=\frac{(M-N)\mathrm{csch}^2\left[\frac{1}{2}\sqrt{-\frac{q}{c^2+p}}(x-ct)\right]}{1+(\frac{M-N}{M+N})\mathrm{coth}^2\left[\frac{1}{2}\sqrt{-\frac{q}{c^2+p}}(x-ct)\right]}. \tag{22}$$

If $\frac{q}{c^2+p}>0$, we have the periodic solutions:

$$u_3(x,t)=\frac{(M+N)\sec^2\left[\frac{1}{2}\sqrt{\frac{q}{c^2+p}}(x-ct)\right]}{1-(\frac{M+N}{M-N})\tan^2\left[\frac{1}{2}\sqrt{\frac{q}{c^2+p}}(x-ct)\right]}, \tag{23}$$

$$u_4(x,t)=\frac{-(M-N)\csc^2\left[\frac{1}{2}\sqrt{\frac{q}{c^2+p}}(x-ct)\right]}{1-(\frac{M-N}{M+N})\cot^2\left[\frac{1}{2}\sqrt{\frac{q}{c^2+p}}(x-ct)\right]}. \tag{24}$$

In Figure 1, $u_1(x,t)$ shows one of exact solutions of Eq. (1) for $p=1, q=-0.2, r=-1, s=2$ and $c=2$.

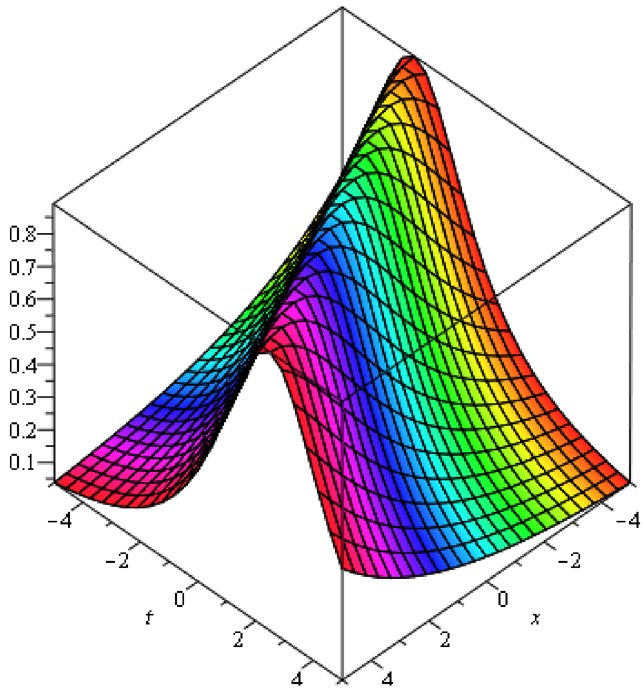

Fig. 1. Hyperbolic function solution (21) of the Generalized reaction Duffing model (1), for $p=1, q=-0.2, r=-1, s=2$ and $c=2$.

3.2. Perturbed Boussinesq equation. Assume that Eq. (7) has an exact solution in the form of a travelling wave

$$u(x,t) = U(\xi), \qquad \xi = x - ct. \tag{14}$$

Substituting Eq. (14) into (7), we get the following nonlinear ODE:

$$c^2U_{\xi\xi} - k^2U_{\xi\xi} + a(U^{2n})_{\xi\xi} + bU_{\xi\xi\xi\xi} = \beta U_{\xi\xi} + \rho U_{\xi\xi\xi\xi}. \tag{15}$$

Integrating Eq. (15) twice with respect to ξ and setting the constants of integration to be zero we get

$$c^2U - k^2U + aU^{2n} + bU_{\xi\xi} = \beta U + \rho U_{\xi\xi}, \tag{16}$$

or

$$U_{\xi\xi} = (\frac{c^2 - k^2 - \beta}{\rho - b})U + \frac{a}{\rho - b}U^{2n}. \tag{17}$$

Then we use the transformations

$$U_\xi = F(U), \tag{18}$$

and (12) to convert Eq. (17) to

$$\frac{1}{2}(F^2(U))' = (\frac{c^2 - k^2 - \beta}{\rho - b})U + \frac{a}{\rho - b}U^{2n}, \tag{19}$$

where the prime denotes differentiation with respect to ξ. According to Eq. (12), we get from Eq. (19) the expressions of the function $F(U)$ as

$$F(U) = \sqrt{\frac{(c^2 - k^2 - \beta)}{\rho - b}}U\sqrt{1 + \frac{2a}{(2n+1)(c^2 - k^2 - \beta)}U^{2n-1}}. \tag{20}$$

After changing the variables

$$Z = -\frac{2a}{(2n+1)(c^2 - k^2 - \beta)}U^{2n-1}, \tag{21}$$

and using the relation (11), the solution of Eq. (15) is in the following form:

$$U(\xi) = \left[\frac{(2n+1)(\beta + k^2 - c^2)}{2a}\operatorname{sech}^2\left(\frac{(2n-1)}{2}\sqrt{\frac{c^2 - k^2 - \beta}{\rho - b}}(\xi)\right)\right]^{\frac{1}{2n-1}}. \tag{22}$$

We can easily obtain the following hyperbolic solutions:

$$u_1(x,t)=\left[\frac{(2n+1)(\beta+k^2-c^2)}{2a}\operatorname{sech}^2\left(\frac{(2n-1)}{2}\sqrt{\frac{(c^2-k^2-\beta)}{\rho-b}}(x-ct)\right)\right]^{\frac{1}{2n-1}}, \quad (23)$$

and

$$u_2(x,t)=\left[-\frac{(2n+1)(\beta+k^2-c^2)}{2a}\operatorname{csch}^2\left(\frac{(2n-1)}{2}\sqrt{\frac{c^2-k^2-\beta}{\rho-b}}(x-ct)\right)\right]^{\frac{1}{2n-1}}. \quad (24)$$

For $\frac{c^2-k^2-\beta}{\rho-b}<0$, it is easy to see that solutions (23) and (24) can reduce to periodic solutions as follow

$$u_3(x,t)=\left[\frac{(2n+1)(\beta+k^2-c^2)}{2a}\sec^2\left(\frac{(2n-1)}{2}\sqrt{-\frac{c^2-k^2-\beta}{\rho-b}}(x-ct)\right)\right]^{\frac{1}{2n-1}}, \quad (25)$$

and

$$u_4(x,t)=\left[\frac{(2n+1)(\beta+k^2-c^2)}{2a}\csc^2\left(\frac{(2n-1)}{2}\sqrt{-\frac{c^2-k^2-\beta}{\rho-b}}(x-ct)\right)\right]^{\frac{1}{2n-1}}. \quad (26)$$

In Figure 2, $u_3(x,t)$ shows one of exact solutions of Eq. (7) for $a=1, k=1, b=1, \beta=1, \rho=1.5$ and $c=0.75$.

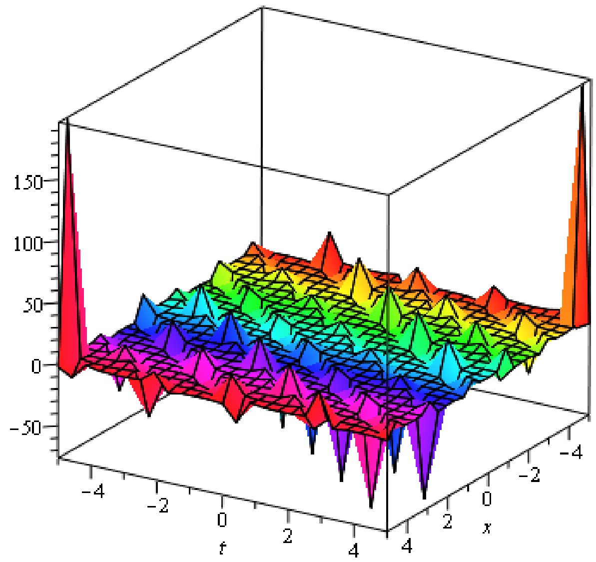

Fig. 2. Solitons solution (25) of the *Perturbed Boussinesq equation*, (7), for $a=1, k=1, b=1, \beta=1, \rho=1.5$ and $c=0.75$.

4 Conclusions

The functional variable method has been successfully used to obtain two traveling wave solutions of the Generalized reaction Duffing model and the perturbed Boussinesq equation. The main merits of the functional variable method over the other methods are as follows.

(i) There is no need to apply the initial and boundary conditions at the outset. The method yields a general solution with free parameters which can be identified by the above conditions.

(ii) The general solution obtained by functional variable method is without approximation.

(iii) The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions.

(iv) The solution procedure can be easily implemented in Mathematica or Maple.

Moreover, we conclude that the functional variable method is significant and important for finding the exact traveling wave solutions of nonlinear evolution equations which can be converted to a second-order ODE through the travelling wave transformation. The proposed method can be applied to many other nonlinear evolution equations in mathematical physics.

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