AMO - Advanced Modeling and Optimization, Volume 17, Number 1, 2015

A GEOMETRIC REDUCTION OF THE ERDŐS-STRAUS CONJECTURE

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Abstract

In this paper we will explore the solutions of the diophantine equation related the well known Erdős-Straus conjecture about unit fractions. For each prime p, we are discussing the relationship between the values x, y, and $z \in \mathbb{N}$ satisfying

$$\frac{4}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ x \le y \le z.$$

We will separate these solutions into two classes. We show that the most common relationship found is

$$x = \left\lfloor \frac{py}{4y-p} \right\rfloor + 1.$$

Finally, we will make a few conjectures to motivate further research in this area.

1. Introduction

The Erdős-Straus conjecture asserts that for every $n \ge 2$, there there exist natural numbers x, y, and z so that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \tag{1.1}$$

Naturally, this claim reduces to be shown correct for prime numbers n. We may assume that $x \le y \le z$, without loss of generality, since one of these values will be the largest and one will be the smallest. The solutions to (1.1) need not be unique.

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

For example, one can check that

$$\begin{aligned}\frac{4}{17} &= \frac{1}{5} + \frac{1}{34} + \frac{1}{170} \\ &= \frac{1}{5} + \frac{1}{30} + \frac{1}{510} \\ &= \frac{1}{6} + \frac{1}{15} + \frac{1}{510} \\ &= \frac{1}{6} + \frac{1}{17} + \frac{1}{102}.\end{aligned} \tag{1.2}$$

The Erdős-Straus conjecture dates back to the 1940s and early 1950s [7, 16, 18]. There have been attempts to solve this problem in many different ways. For example, some people used algebraic geometry techniques to bring a certain structure to this problem (see [4]), or analytic number theory techniques to find asymptotic results (see [5, 6, 11, 19, 20, 25, 26, 30]). Others looked into the study of related fractions, such as k/n for $k \geq 2$ (see [1, 5, 13, 17, 27, 28]) or using computational methods (see [23]). Less elementary methods were used in [2, 6, 19, 20]). The current authors have made attempts to make equivalent conjectures in different number fields [3]. The best-known approach was developed by Rosati [18]. Mordell [14] has a great description of this method and many attempts use the techniques in his paper (see [9, 21, 24, 29]).

The purpose of this paper is to classify each solution based on its geometric location. Figure 1 shows the geometric location of the solutions listed in (1.2) as pink cells where the cells represent the standard xy integer lattice when both $x > 0$ and $y > 0$. This image was made with a Microsoft excel worksheet by using conditional formatting of the cell colors. The pink cells that border the yellow cells in figure 1 will be of particular interest. In this case we see that all the pink cells border the yellow cells. To define the border between the yellow and blue cells in figure 1 we need to relate x and y. We let the cells be white if $x > y$ or if $y > z$. When $p = 17$ we will see that $y > z$ if $y > 34x/(4x - 17)$. We will let the cells be yellow if $z < 0$. The cells will be yellow if $y < 17x/(4x - 17)$ and the cells will be blue or pink if $17x/(4x-17) \leq y \leq 34x/(4x-17)$. The cells are pink only if z is an integer. Our main argument will be that a overwhelming majority of the solutions fall along the boundary of all (x, y) values that give $z > 0$.

We will also see that for all primes $p \neq 2$ and $p \neq 2521$ there exists at least one solution to (1.1) so that $x = \lfloor py/(4y - p) \rfloor + 1$, $\gcd(p, y) = 1$ and $z = p \cdot \operatorname{lcm}(x, y)$. For $p = 17$ we see that there are two solutions with this pattern.

These solutions are

	1	2	3	4	5	6	7	8	9	10	11	12
1	-0.5666667	-0.7906977	-0.9107143	-0.9855072	-1.0365854	-1.0736842	-1.1018519	-1.1239669	-1.141791	-1.1564626	-1.16875	-1.1791908
2	-0.7906977	-1.3076923	-1.6721311	-1.9428571	-2.1518987	-2.3181818	-2.4536082	-2.5660377	-2.6608696	-2.7419355	-2.8120301	-2.8732394
3	-0.9107143	-1.6721311	-2.3181818	-2.8732394	-3.3552632	-3.7777778	-4.1511628	-4.4835165	-4.78125	-5.049505	-5.2924528	-5.5135135
4	-0.9855072	-1.9428571	-2.8732394	-3.7777778	-4.6575342	-5.5135135	-6.3466667	-7.1578947	-7.9480519	-8.7179487	-9.4683544	-10.2
5	-1.0365854	-2.1518987	-3.3552632	-4.6575342	-6.0714286	-7.6119403	-9.296875	-11.147541	-13.189655	-15.454545	-17.980769	-20.816327
6	-1.0736842	-2.3181818	-3.7777778	-5.5135135	-7.6119403	-10.2	-13.471698	-17.73913	-23.538462	-31.875	-44.88	-68
7	-1.1018519	-2.4536082	-4.1511628	-6.3466667	-9.296875	-13.471698	-19.833333	-30.709677	-53.55	-132.22222	654.5	109.846154
8	-1.1239669	-2.5660377	-4.4835165	-7.1578947	-11.147541	-17.73913	-30.709677	-68	-1224	97.1428571	51.5862069	37.0909091
9	-1.141791	-2.6608696	-4.78125	-7.9480519	-13.189655	-23.538462	-53.55	-1224	76.5	41.3513514	30.0535714	24.48
10	-1.1564626	-2.7419355	-5.049505	-8.7179487	-15.454545	-31.875	-132.22222	97.1428571	41.3513514	28.3333333	22.5301205	19.245283
11	-1.16875	-2.8120301	-5.2924528	-9.4683544	-17.980769	-44.88	654.5	51.5862069	30.0535714	22.5301205	18.7	16.379562
12	-1.1791908	-2.8732394	-5.5135135	-10.2	-20.816327	-68	109.846154	37.0909091	24.48	19.245283	16.379562	14.5714286
13	-1.188172	-2.9271523	-5.7155172	-10.91358	-24.021739	-120.54545	64.4583333	29.9661017	21.1595745	17.1317829	14.8231707	13.3266332
14	-1.1959799	-2.975	-5.9008264	-11.609756	-27.674419	-357	47.6	25.7297297	18.9557522	15.6578947	13.7068063	12.4173913
15	-1.2028302	-3.0177515	-6.0714286	-12.289157	-31.875	510	38.8043478	22.9213483	17.3863636	14.5714286	12.8669725	11.7241379
16	-1.2088889	-3.0561798	-6.2290076	-12.952381	-36.756757	163.2	33.4035088	20.9230769	16.2119205	13.7373737	12.2122449	11.1780822
17	-1.2142857	-3.0909091	-6.375	-13.6	-42.5	102	29.75	19.4285714	15.3	13.0769231	11.6875	10.7368421
18	-1.2191235	-3.122449	-6.5106383	-14.232558	-49.354839	76.5	27.1139241	18.2686567	14.5714286	12.5409836	11.2575251	10.3728814
19	-1.2234848	-3.1512195	-6.6369863	-14.850575	-57.678571	62.516129	25.1222222	17.3422819	13.9759615	12.0973783	10.898773	10.0675325
20	-1.2274368	-3.1775701	-6.7549669	-15.454545	-68	53.6842105	23.5643564	16.5853659	13.4801762	11.7241379	10.5949008	9.80769231
21	-1.2310345	-3.2017937	-6.8653846	-16.044944	-81.136364	47.6	22.3125	15.9553073	13.0609756	11.4057508	10.3342105	9.58389262
22	-1.2343234	-3.2241379	-6.9689441	-16.622222	-98.421053	43.1538462	21.2845528	15.4226804	12.7018868	11.1309524	10.1081081	9.38912134
23	-1.2373418	-3.2448133	-7.0662651	-17.186813	-122.1875	39.7627119	20.4253731	14.9665072	12.3908451	10.8913649	9.91013825	9.21807466
24	-1.2401216	-3.264	-7.1578947	-17.73913	-156.92308	37.0909091	19.6965517	14.5714286	12.1188119	10.6806283	9.73535792	9.06666667
25	-1.2426901	-3.2818533	-7.2443182	-18.27957	-212.5	34.9315068	19.0705128	14.2259414	11.878882	10.4938272	9.57991803	8.93169877
26	-1.2450704	-3.2985075	-7.3259669	-18.808511	-315.71429	33.15	18.5269461	13.9212598	11.6656891	10.3271028	9.4407767	8.81063123
27	-1.2472826	-3.3140794	-7.4032258	-19.326316	-573.75	31.6551724	18.0505618	13.6505576	11.475	10.1773836	9.31549815	8.7014218
28	-1.2493438	-3.3286713	-7.4764398	-19.833333	-2380	30.3829787	17.6296296	13.4084507	11.3034301	10.0421941	9.20210896	8.60240964
29	-1.251269	-3.3423729	-7.5459184	-20.329897	1232.5	29.2871287	17.255	13.1906355	11.1482412	9.9195171	9.09899329	8.51223022
30	-1.2530713	-3.3552632	-7.6119403	-20.816327	510	28.3333333	16.9194313	12.9936306	11.0071942	9.80769231	9.00481541	8.42975207
31	-1.2547619	-3.3674121	-7.6747573	-21.292929	329.375	27.4956522	16.6171171	12.8145897	10.8784404	9.7053407	8.91846154	8.35402906
32	-1.256351	-3.378882	-7.7345972	-21.76	247.272727	26.7540984	16.3433476	12.6511628	10.7604396	9.61130742	8.83899557	8.28426396
33	-1.2578475	-3.3897281	-7.7916667	-22.217822	200.357143	26.0930233	16.0942623	12.5013928	10.6518987	9.524618	8.765625	8.21978022
34	-1.2592593	-3.4	-7.8461538	-22.666667	170	25.5	15.8666667	12.3636364	10.5517241	9.44444444	8.69767442	8.16
35	-1.2605932	-3.4097421	-7.8982301	-23.106796	148.75	24.965035	15.6578947	12.2365039	10.4589844	9.37007874	8.63456464	8.10442679
36	-1.2618557	-3.4189944	-7.9480519	-23.538462	133.043478	24.48	15.465704	12.1188119	10.3728814	9.30091185	8.57579618	8.05263158
37	-1.2630522	-3.4277929	-7.9957627	-23.961905	120.961538	24.0382166	15.2881944	12.0095465	10.2927273	9.23641703	8.52093596	8.00424178
38	-1.2641879	-3.4361702	-8.0414938	-24.377358	111.37931	23.6341463	15.1237458	11.9078341	10.2179262	9.17613636	8.46960667	7.95893224
39	-1.2652672	-3.4441558	-8.0853659	-24.785047	103.59375	23.2631579	14.9709677	11.8129176	10.1479592	9.11966988	8.42147806	7.91641791
40	-1.2662942	-3.4517766	-8.12749	-25.185185	97.1428571	22.9213483	14.8286604	11.7241379	10.0823723	9.06666667	8.3762598	7.87644788
41	-1.2672727	-3.4590571	-8.1679688	-25.577982	91.7105263	22.6054054	14.6957831	11.6409186	10.0207668	9.01681759	8.33369565	7.83880037
42	-1.268206	-3.4660194	-8.2068966	-25.963636	87.0731707	22.3125	14.5714286	11.562753	9.9627907	8.96984925	8.29355861	7.80327869
43	-1.2690972	-3.4726841	-8.2443609	-26.342342	83.0681818	22.040201	14.4548023	11.4891945	9.90813253	8.92551893	8.25564682	7.76970771
44	-1.2699491	-3.4790698	-8.2804428	-26.714286	79.5744681	21.7864078	14.3452055	11.4198473	9.85651537	8.88361045	8.21978022	7.73793103
45	-1.2707641	-3.4851936	-8.3152174	-27.079646	76.5	21.5492958	14.2420213	11.3543599	9.80769231	8.84393064	8.18579767	7.70780856
46	-1.2715447	-3.4910714	-8.3487544	-27.438596	73.7735849	21.3272727	14.1447028	11.2924188	9.76144244	8.80630631	8.1535545	7.6792144
47	-1.272293	-3.4967177	-8.3811189	-27.791304	71.3392857	21.1189427	14.0527638	11.2337434	9.71756757	8.77058178	8.12292052	7.65203512

Figure 1. *For $p = 17$, the cells contain the z value in terms of $x \in \{1, 2, .., 12\}$ and $y \in \{1, 2, ...47\}$. If $x > y$ or $y > z$ the cell color is white. If z is negative the cell color is yellow. If z is a solution of (1.1), i.e. z is a positive integer and $x \le y \le z$, the cell color is red (see (1.2)). Otherwise the cell color is blue.*

$$\begin{aligned}\frac{4}{17} &= \frac{1}{5} + \frac{1}{30} + \frac{1}{510} \\ &= \frac{1}{6} + \frac{1}{15} + \frac{1}{510}.\end{aligned}$$

Finally we will see that for all primes $p \notin \{2, 3, 7, 47, 193, 2521\}$ there exists at least one solution to (1.1) so that $y = \lfloor px/(4x - p) \rfloor + 1$, $\gcd(p, y) = 1$ and $z = p \cdot \mathrm{lcm}\,(x, y)$. For $p = 17$ we see that there is only one solution with this requirement. This solution is

$$\frac{4}{17} = \frac{1}{6} + \frac{1}{15} + \frac{1}{510}.$$

The rest of the paper is organized as follows: in Section 2 we will describe the main results without proof and in Section 3 we will fill in the necessary details.

2. Main Results

We would like to generalize the observations made in the introduction for 17, to any prime p. Our first goal in this endeavor is to define the boundary between the yellow cells and the blue or pink cells as in Figure 1 for a general prime p. We notice that if

$$y < \frac{px}{4x-p}$$

then

$$\frac{4}{p} < \frac{1}{x} + \frac{1}{y}. \tag{2.1}$$

To solve (1.1) when (2.1) holds, we necessarily need z be negative. Because this cannot happen, this implies that

$$y \geq \frac{px}{4x-p}.$$

To solve (1.1), the equation $4xy - p(x+y) = 0$ cannot hold because if it did hold, then

$$\frac{4}{p} = \frac{1}{x} + \frac{1}{y}$$

and necessarily z cannot be an integer. This equation will, however, define the boundary between the yellow cells and the blue or red cells mentioned from Figure 1 and it will apply to any prime p. To be on the correct side of this boundary we see that

$$4xy - p(x+y) > 0. \tag{2.2}$$

To be along the boundary, yet satisfy (2.2), we need to select the integer values of x and y so that the left hand side of the inequality (2.2) is the smallest possible positive value. The following definition will describe two ways that a solution to (1.1) can be along this boundary.

Definition 2.1. A solution to (1.1) is a type I(a) solution if

$$y = \left\lfloor \frac{px}{4x-p} \right\rfloor + 1. \tag{2.3}$$

A solution to (1.1) is a type I(b) solution if

$$x = \left\lfloor \frac{py}{4y-p} \right\rfloor + 1. \tag{2.4}$$

A solution is called a type I solution if it is a type I(a) solution, a type I(b) solution or both.

If we relate this to Figure 1, then type I solutions are given by the red cells that border a yellow cell from the bottom or from the right. In particular, a type I(a) solution is given by a red cell that borders a yellow cell from the bottom and a type I(b) solution is given by a pink cell that borders a yellow cell from the right.

We quickly find a relationship between type I(a) solutions and type I(b) solutions, which we outline in the following proposition.

Proposition 2.2. *If a solution is a type I(a) solution then it is a type I(b) solution.*

This means that if a solution to (1.1) is of type I, then it is of type I(b). We can use the two terms interchangeably. There is computational evidence to suggest that the only prime p where there is no solution of type I(a) is when $p = 193$. This computation evidence is through all primes less that 10^8. We summarize this conclusion in the following conjecture.

Conjecture 2.3. *The only prime p where there is no solution of type I(a) is $p =$ 193.*

Because all type I(a) solutions are type I(b) solutions, we can make a stronger statement about type I(b) solutions. Because

$$\frac{4}{193} = \frac{1}{50} + \frac{1}{1930} + \frac{1}{4825}$$

is a type I(b) solution, there is computational evidence to suggest that every prime p has a solution of type I(b). This computational evidence is through all primes less that 10^8. We summarize this conclusion in the following conjecture.

Conjecture 2.4. *Every prime p has a solution of type I(b).*

The fact that every prime has at least one solution of type I(b) gives the authors of this paper the impression that the proof of the Erdős-Straus conjecture reduces to finding a solution of type I(b) for every prime p. This may not be true, but it leads us to ask the natural question: "for which primes p, we can prove that there exists at least one decomposition of type I(b)?" In this direction, first we recall a theorem from [9].

Theorem 2.5. (Ionascu-Wilson) *Equation (1.1) has at least one solution for every prime number p, except possible for those primes of the form $p \equiv r \pmod{9240}$ where r is one of the 34 entries in the table:*

1	169	289	361	529	841
961	1369	1681	1849	2041	2209
2521	2641	2689	2809	3361	3481
3529	3721	4321	4489	5041	5161
5329	5569	6169	6241	6889	7561
7681	7921	8089	8761		

The decompositions created to prove Theorem 2.5 were given in [9] and can be tested to determine whether or not they were of type I(b). The following theorem tells us that every solution provided is of type I(b).

Theorem 2.6. *Every prime p that is guaranteed a solution by Theorem 2.5 has at least one solution of type I(b).*

Although we believe, every prime has at least one solution of type I(b), we were curious to know whether or not every solution was of type I(b). We can see for $p = 17$ that every solution was of type I(b), however, for other primes there exist solutions that are not of type I. For example, we have that

$$\frac{4}{71} = \frac{1}{20} + \frac{1}{284} + \frac{1}{355}$$

where we see that

$$x = \left\lfloor \frac{71 \cdot 284}{4 \cdot 284 - 71} \right\rfloor + 2 = 20.$$

To account for the remaining solutions, we make the following definition.

Definition 2.7. A solution to (1.1) that is not a type I solution is a type II solution.

It is natural to ask if there is a pattern within the class of type II solutions. Although there is most likely no upper bound to the number of type II solutions that exist for a given prime, it appears that as the number of type II solutions grow, the number of type I solutions grow as well. They do not, however, appear to grow at a uniform rate. Figure 2 shows the proportion of type II solutions for each prime less than 4000. There is no prime less than 4000 that has less than 80% of its solutions of type I, but this proportion seems sporadic.

We can see from Figure 2 that most primes have no type II solutions at all, so our next goal was to make an empirical distribution for the solutions to (1.1) based on the proximity of the solution to the boundary. For example, there are 38434 solutions to (1.1) for primes $p \leq 4000$. We will separate the number of solutions to (1.1) for prime numbers p into categories based on whether the solutions satisfy

$$x = \left\lfloor \frac{py}{4y - p} \right\rfloor + i$$

for $1 \leq i \leq 5$. Table 1 and Figure 3 summarize what we have found for primes $p \leq 4000$.

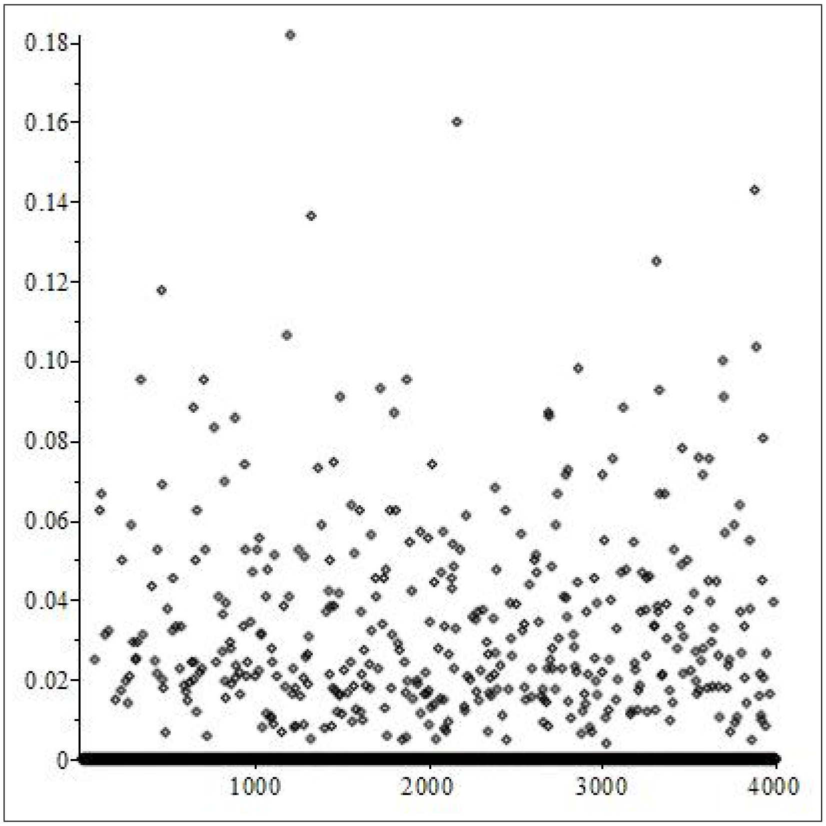

FIGURE 2. This graph shows the proportion of type II solutions for each prime p.

This distribution shows our point very well. If we are to describe a pattern for solutions to (1.1) for a general prime p, it appears that it is a safe assumption to let

$$x = \left\lfloor \frac{py}{4y-p} \right\rfloor + 1.$$

Next we turn our attention to another pattern one can easily identify for solutions of (1.1). As mentioned in the introduction, we can see that for all primes p such that $p \neq 2$ and $p \neq 2521$ there exists a solution so that $x = \lfloor py/(4y-p) \rfloor + 1$, $\gcd(p, y) = 1$ and $z = p \cdot \operatorname{lcm}(x, y)$. This has been checked computationally for all primes less than 10^8. Instead of trying to explain why the two primes $p = 2$ and $p = 2521$ do not follow this pattern, we argue that it suffices to find a prime p^* large enough so that every prime larger than p^* has the pattern we describe above. This brings up two conjectures. We believe that these conjectures govern at least one way to find a general pattern for the solutions of (1.1).

i	# solutions	proportion
1	37612	0.9786
2	517	0.0135
3	170	0.0044
4	64	0.0017
5	71	0.0018

TABLE 1. This table shows the empirical probability distribution function of the solutions to (1.1) based on their proximity to the boundary values that make z positive. The solutions are accumulated for primes less than 4000 and separated into categories based on whether the solutions satisfy $x = \lfloor py/(4y-p) \rfloor + i$.

First we mention that for any prime $p \neq 2$ and $y \in \mathbb{N}$ that satisfy (1.1) we have that $\lfloor py/(4y-p) \rfloor + 1 = \lceil py/(4y-p) \rceil$. Similarly for any prime $p \neq 2$ and $x \in \mathbb{N}$ that satisfy (1.1) we have that$\lfloor px/(4x-p) \rfloor + 1 = \lceil px/(4x-p) \rceil$. This will help simplify how we express our work. We now state our conjecture and provide a corollary to show the nature of our solution.

Conjecture 2.8. *Consider a prime $p^* \geq 2521$. Given any prime $p > p^*$ there exists $y \in \mathbb{N}$ so that $\lceil p/2 \rceil \leq y \leq \lfloor p(p+3)/6 \rfloor$, $\gcd(p, y) = 1$ and*

$$\frac{y}{(4y-p)-m} \in \mathbb{N}$$

where $m \equiv py \mod (4y-p)$.

Corollary 2.9. *Consider a prime $p^* \geq 2521$. Given any prime $p > p^*$ there exists $y \in \mathbb{N}$ so that $\lceil p/2 \rceil \leq y \leq \lfloor p(p+3)/6 \rfloor$, $\gcd(p, y) = 1$ and*

$$\frac{4}{p} = \frac{1}{\left\lceil \frac{py}{4y-p} \right\rceil} + \frac{1}{y} + \frac{1}{p \cdot \operatorname{lcm}\left(\left\lceil \frac{py}{4y-p} \right\rceil, y \right)}.$$

There are some scenarios for the prime p that are guaranteed a solution of this type. We outline the cases that have are guaranteed a solution in the following tables. These results are incomplete and rather difficult to show in general.

p	y
$3 \mod 4$	$(p(p+1)/4)+1$
$5 \mod 8$	$(3p+1)/4$
$17 \mod 24$	$(7p+1)/4$
$97 \mod 120$	$(7p+1)/8$
$73 \mod 840$	$(23p+1)/8$

p	y
$241 \mod 840$	$(23p+1)/8$
$409 \mod 840$	$(23p+1)/8$
$433 \mod 840$	$(15p+1)/4$
$601 \mod 840$	$(15p+1)/4$
$769 \mod 840$	$(15p+1)/4$

We next make an analogue to conjecture 2.8 when the solutions are of type I(a). This is much more enlightening for programming reasons. We only need to check that the following conjecture holds for values of $x \in \mathbb{N}$ so that $\lceil p/4 \rceil \leq x \leq \lfloor p/2 \rfloor$.

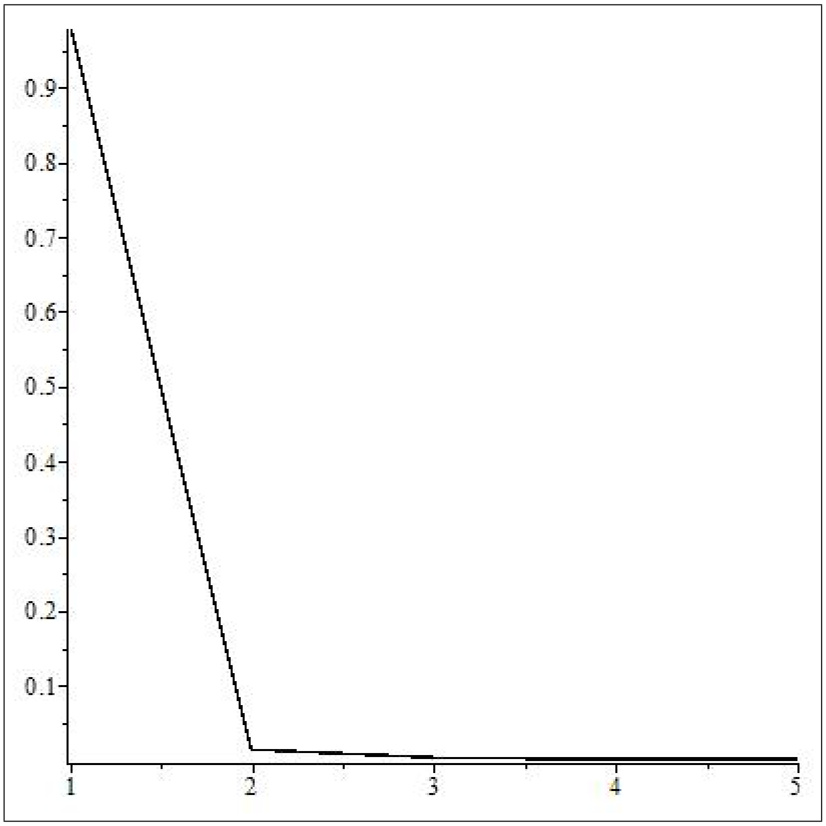

FIGURE 3. This graph draws the probability distribution function defined from table 1. The points in the pdf are connected with lines.

The first conjecture will require us to search for a solution to (1.1) for values of y on the boundary locations. As p gets large, the number of boundary locations grow at an asymptotic rate of $\mathcal{O}(p^2)$. For this next conjecture, when considering type I(a) solutions, as p gets large, the number of boundary locations grow at an asymptotic rate of $\mathcal{O}(p)$. This suggests that the result in [23] showing that every prime less than 10^{14} has a solution can be improved by searching for type I(a) solutions with $z = p \cdot \mathrm{lcm}(x, y)$.

Here we see that for all primes $p \notin \{2, 3, 7, 47, 193, 2521\}$ there exists a solution so that $y = \lceil px/(4x - p)\rceil$, $\gcd(p, y) = 1$ and $z = p \cdot \mathrm{lcm}(x, y)$. We provide the foundation of this in the following conjecture.

Conjecture 2.10. *Consider a prime $p^* \geq 2521$. Given any prime $p > p^*$ there exists $x \in \mathbb{N}$ so that $\lceil p/4\rceil \leq x \leq \lfloor p/2\rfloor$, $\gcd(p, \lceil px/(4x - p)\rceil) = 1$ and*

$$\frac{x}{(4x - p) - m} \in \mathbb{N}$$

where $m \equiv px \mod (4x-p)$.

Much like conjecture 2.8, this conjecture will lead to a solution of (1.1). Now we will have the denominators of our unit fractions x, $\lceil px/(4x-p)\rceil$ and $p{\cdot}\mathrm{lcm}(x, \lceil px/(4x-p)\rceil)$. We conclude our paper with proofs for some of our main points.

3. Detailed analysis

3.1. **Proposition 2.2.**

Proof. Suppose that for a prime p there exist values $x, y, z \in \mathbb{N}$ that make a solution to (1.1). Further suppose that this solution is of type I(a).
This will imply that

$$y = \left\lfloor \frac{px}{4x-p} \right\rfloor + 1.$$

We can clearly see that being a solution will imply that

$$\frac{4}{p} \geq \frac{1}{x} + \frac{1}{y}$$

but to begin we will prove is that

$$\frac{4}{p} \leq \frac{1}{x-1} + \frac{1}{y}.$$

Proving this claim will lead us to show that it is a type I(b) solution.

First notice that for any prime p and any $x \in \mathbb{N}$ such that $(p/4)+1 < x \leq (p/2)$ we have that

$$\begin{aligned}\frac{p(x-1)}{4(x-1)-p} - \frac{px}{4x-p} &= \frac{p^2}{(4(x-1)-p)(4x-p))}\\ &\geq \frac{p^2}{(4x-p)^2}\\ &\geq 1.\end{aligned}$$

This tells us that

$$\begin{aligned}\frac{p(x-1)}{4(x-1)-p} &\geq \frac{px}{4x-p} + 1\\ &\geq \left\lfloor \frac{px}{4x-p} \right\rfloor + 1.\end{aligned}$$

This will imply that

$$\begin{aligned}\frac{4}{p} &\leq \frac{1}{x-1} + \frac{1}{\left\lfloor \frac{px}{4x-p} \right\rfloor + 1}\\ &= \frac{1}{x-1} + \frac{1}{y}.\end{aligned}$$

To finish the proof we prove the following claim: if $x, y, z \in \mathbb{N}$ is a solution to (1.1) for a prime p and

$$\frac{4}{p} \leq \frac{1}{x-1} + \frac{1}{y}$$

then the solution is of type I(b).

Because

$$\frac{4}{p} \geq \frac{1}{x} + \frac{1}{y}$$

and

$$\frac{4}{p} \leq \frac{1}{x-1} + \frac{1}{y}$$

we see that

$$\frac{py}{4y-p} \leq x \leq \frac{py}{4y-p} + 1.$$

Because $4xy - p(x+y) \neq 0$ for any $x, y \in \mathbb{N}$ that will make a solution to (1.1), we see that $py/(4y-p)$ is not an integer for the possible values of y and p. Because x is a positive integer, we see then it must be true that

$$x = \left\lfloor \frac{py}{4y-p} \right\rfloor + 1.$$

This shows that the solution is of type I(b). □

3.2. **Theorem 2.6.**

Proof. This theorem is proved by the following selections of the value of y:

p	y
2	$p(p+2)/4$
3 mod 4	$(p(p+1)/4)+1$
5 mod 8	$p(p+3)/8$
17 mod 24	$p(p+7)/24$
73 mod 120	$p(p+7)/20$
97 mod 120	$p(p+3)/10$
4561 mod 9240	$3p$
4729 mod 9240	$3p$
5881 mod 9240	$3p$
6049 mod 9240	$3p$
6409 mod 9240	$3p$
6841 mod 9240	$3p$
7081 mod 9240	$3p$
7729 mod 9240	$3p$
8401 mod 9240	$3p$
3049 mod 9240	$p(p+31)/44$
4369 mod 9240	$p(p+31)/44$
7009 mod 9240	$p(p+31)/44$
1201 mod 9240	$5p(p+31)/616$

p	y
241 mod 840	$p(p+11)/42$
409 mod 840	$p(p+11)/42$
481 mod 840	$p(p+11)/84$
649 mod 840	$p(p+11)/84$
601 mod 840	$p(p+15)/56$
769 mod 840	$p(p+15)/56$
1009 mod 9240	$3p$
1129 mod 9240	$3p$
1801 mod 9240	$3p$
2881 mod 9240	$3p$
3649 mod 9240	$3p$
4201 mod 9240	$3p$
8521 mod 9240	$3p$
8689 mod 9240	$3p$
8929 mod 9240	$3p$
3889 mod 9240	$p(p+71)/44$
5209 mod 9240	$p(p+71)/44$
7849 mod 9240	$p(p+71)/44$
6001 mod 9240	$p(p+159)/616$

From this information one can derive the value of z that solves equation (1.1).

For example, if $p \equiv 5 \mod 8$, then there exists a value k so that $p = 8k+5$. We would see then that $y = (k+1)(8k+5)$.

Because

$$\begin{aligned}\frac{py}{4y-p} &= \frac{(k+1)(8k+5)}{4k+3} \\ &= 2(k+1) - \frac{k+1}{4k+3}\end{aligned}$$

and $0 < (k+1)/(4k+3) < 1$ for all $k \geq 0$, we see that $x = 2(k+1) = (p+3)/4$. Letting $x = (p+3)/4$ and $y = p(p+3)/8$ we see that necessarily $z = p(p+3)/4$. For every prime p listed above, the given selection of y will provide the values of x and z through the same process. $\square$

3.3. **Corollary 2.9.**

Proof. If conjecture 2.8 holds then we necessarily have that $py/((4y-p)-m) \in \mathbb{N}$ and one fact about every natural number $a \in \mathbb{N}$ is that $\gcd(a, a+1) = 1$, this will imply that

$$\gcd\left(\frac{py}{(4y-p)-m}, \frac{py}{(4y-p)-m} + 1\right) = 1.$$

In particular, this would imply that

$$\gcd\left(py, py + (4y-p) - m\right) = (4y-p) - m.$$

Because $m \equiv py \mod (4y-p)$, we see that

$$(4y-p)\left\lceil \frac{py}{4y-p} \right\rceil = py + (4y-p) - m.$$

This would imply that

$$\gcd\left(py, (4y-p)\left\lceil \frac{py}{4y-p} \right\rceil\right) = (4y-p)\left\lceil \frac{py}{4y-p} \right\rceil - py.$$

Because $\gcd(p, y) = 1$ we see that $\gcd((4y-p), py) = 1$. This will necessarily imply that

$$\gcd\left(py, \left\lceil \frac{py}{4y-p} \right\rceil\right) = (4y-p)\left\lceil \frac{py}{4y-p} \right\rceil - py.$$

Because $\lceil p/4 \rceil \leq \lceil py/(4y-p) \rceil \leq \lfloor p/2 \rfloor$ we see that $\gcd\left(\lceil py/(4y-p) \rceil, p\right) = 1$. This will imply that

$$\gcd\left(y, \left\lceil \frac{py}{4y-p} \right\rceil\right) = (4y-p)\left\lceil \frac{py}{4y-p} \right\rceil - py.$$

We can express this as

$$4y\left\lceil \frac{py}{4y-p} \right\rceil = py + p\left\lceil \frac{py}{4y-p} \right\rceil + \gcd\left(\left\lceil \frac{py}{4y-p} \right\rceil, y\right).$$

Dividing both sides of the equation by $py\left\lceil py/(4y-p)\right\rceil$, we have that

$$\frac{4}{p}=\frac{1}{\left\lceil\frac{py}{4y-p}\right\rceil}+\frac{1}{y}+\frac{1}{p\cdot\mathrm{lcm}\left(\left\lceil\frac{py}{4y-p}\right\rceil,y\right)}.$$

□

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