Mond-Weir type multiobjective second-order fractional symmetric duality with (ϕ, ρ)-invexity

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Abstract. In this paper, we start our discussion with a pair of multiobjective Mond-Weir type second-order symmetric dual fractional programming problem and derive weak, strong and strict duality theorems under second-order (ϕ, ρ)-invexity assumptions.

Keywords: Second-order symmetric duality, multiobjective fractional programming, efficient solution, second-order (ϕ, ρ)-invexity.

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1. Introduction

Optimization has been expanding in all directions at an astonishing rate during the last few decades. The theory of duality is elegant and important concept within the field of operations research. The concept of second-order duality was first introduced by Mangasarian [10], where he pointed out possible advantages of second-order dual over the first order dual. Due to the fact that there are more parameters involved, second-order dual provides tighter bounds for the value of objective function of the primal problem when approximations are used.

The problems in which objective functions are ratio of two functions are termed as fractional programming problems. It can be used in engineering and economics to minimize a ratio of functions between a given period of time and a utilized resource in order to measure the efficiency or productivity of a system (see Stancu-Minasian [14]). For more information on the fractional programs the readers are advised to see [13].

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The first symmetric dual formulation for quadratic program was proposed by Dorn [6], who defined the program and its dual to be symmetric if the dual program is recasted in the form of the primal, its dual is primal. Subsequently, the notion of symmetric duality was developed significantly by Dantzig *et al.* [5]. Chandra *et al.* [4] considered the symmetric dual fractional program and derived the appropriate duality theorems. Mond *et al.* [12], Weir [16] extended the results of Chandra *et al.* [4] to nondifferentiable fractional programs and to multiobjective fractional programs.

Suneja *et al.* [15] studied a pair of Mond-Weir type multiobjective second-order symmetric dual programs without non-negative constraints and established duality theorems under η-bonvexity and η-pseudobonvexity assumptions. They also discussed second-order self-duality theorems by taking the functions to be skew-symmetric. Ahmad and Husain [1] formulated a pair of multiobjective fractional symmetric dual programs over arbitrary cones and established appropriate duality results. Recently, Ahmad and Husain [2] focused on multiobjective second-order symmetric duality with cone constraints and usual duality results are established under second-order invexity assumptions.

Yang *et al.* [18] considered a pair of second-order symmetric dual programs and obtained duality results under F-convexity assumptions. Gupta and Kailey [8, 9] formulated second-order symmetric dual programs for a class of nondifferentiable multiobjective programming problem and established duality theorems for the aforementioned pair using the notion of second-order F-convexity/pseudoconvexity. Very recently, the work is further extended by Gulati *et al.* [7] by introducing a pair of symmetric dual second-order fractional programs to derive appropriate duality results. They also discussed minimax mixed integer symmetric dual fractional programs.

In this paper, a pair of multiobjective second-order fractional symmetric dual programs is formulated. Weak, strong and converse duality theorems are established under second-order (ϕ, ρ)-invexity assumptions. Moreover, a self dual programs is formulated and also self duality theorem is discussed. Some known models are the special case of the model considered in the present paper.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and let R^n_+ be its non-negative orthant. The following conventions for vectors in R^n will be used in the sequel of the paper:

$x < y$ if and only if $y - x \in \text{int } R^n$;

$x \leq y$ if and only if $y - x \in \text{int } R^n_+ \setminus \{0\}$;

$x \leqq y$ if and only if $y - x \in \text{int } R^n_+$;

$x \nleq y$ is the negation of $x \leq y$.

A general multiobjective programming problem can be expressed in the following form:

(P) Minimize $h(x) = (h_1(x), h_2(x), ..., h_k(x))$

subject to $r(x) \leqq 0$,

where $h : R^n \to R^k$ and $r : R^n \to R^m$. We shall denote the feasible set of (P) by $A = \{x | r(x) \leqq 0, x \in R^n\}$.

Definition 2.1 A feasible point x^* is said to be an weak efficient (or weak Pareto optimal) solution of (P), if there exists no other $x \in A$ such that $h(x) < h(x^*)$.

Definition 2.2 A feasible point x^* is said to be an efficient (or Pareto optimal) solution of (P), if there exists no other $x \in A$ such that $h(x) \leq h(x^*)$.

Definition 2.3 A feasible point x^* is said to be a properly efficient solution of (P), if it is an efficient solution of (P) and if there exists a scalar $M > 0$ such that for each i and $x \in A$ satisfying $h_i(x) < h_j(x^*)$, we have

$$\frac{h_i(x^*) - h_i(x)}{h_j(x) - h_j(x^*)} \leq M,$$

for some j satisfying $h_j(x) > h_j(x^*)$.

Let $S_1 \subset R^n$ and $S_2 \subset R^m$ and let $f(x, y)$ be a real valued twice differentiable function defined on $S_1 \times S_2$. Then $\nabla_x f$ and $\nabla_y f$ denote gradient vectors of f with respect to x and y, respectively and $\nabla_{xy} f$ denotes the $n \times m$ matrix of second-order partial derivatives.

Definition 2.4 A real valued function $f(., y) : S_1 \times S_2 \to R$ is said to be second-order (ϕ, ρ)-invex at $u \in S_1$ with respect to $q \in R^n$, if for all $\phi : S_1 \times S_1 \times R^{n+1} \to R$ with ρ as a real number, we have

$$f(x, y) - f(u, y) + \frac{1}{2} q^T \nabla_{xx} f(u, y) q \geqq \phi(x, u; \nabla_x f(u, y) + \nabla_{xx} f(u, y) q, \rho).$$

Definition 2.5 A real valued function $f(x, .) : S_1 \times S_2 \to R$ is said to be second-order (ϕ, ρ)-invex at $y \in S_2$ with respect to $p \in R^m$, if for all $\phi : S_2 \times S_2 \times R^{m+1} \to R$ with ρ as a real number, we have

$$f(x, v) - f(x, y) + \frac{1}{2} p^T \nabla_{yy} f(x, y) p \geqq \phi(v, y; \nabla_y f(x, y) + \nabla_{yy} f(x, y) p, \rho).$$

3. Second-order multiobjective fractional symmetric duality

In this paper, we consider the following pair of multiobjective Mond-Weir type fractional symmetric dual programs:

Primal problem (MWP)

$$\text{Minimize } L(x,y,p) = (L_1(x,y,p_1), L_2(x,y,p_2), ..., L_k(x,y,p_k))^t$$

subject to

$$\sum_{i=1}^{k} \lambda_i[(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - L_i(x,y,p_i)(\nabla_y g_i(x,y) + \nabla_{yy} g_i(x,y)p_i)] \leqq 0,$$

$$y^t \sum_{i=1}^{k} \lambda_i[(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - L_i(x,y,p_i)(\nabla_y g_i(x,y) + \nabla_{yy} g_i(x,y)p_i)] \geqq 0,$$

$\lambda > 0.$

Dual problem (MWD)

$$\text{Maximize } M(u,v,q) = (M_1(u,v,q_1), M_2(u,v,q_2), ..., M_k(u,v,q_k))^t$$

subject to

$$\sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - M_i(u,v,q_i)(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)] \geqq 0,$$

$$u^t \sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - M_i(u,v,q_i)(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)] \leqq 0,$$

$\lambda > 0,$

where

$$L_i(x,y,p_i) = \frac{f_i(x,y) - \frac{1}{2}p_i^t \nabla_{yy} f_i(x,y)p_i}{g_i(x,y) - \frac{1}{2}p_i^t \nabla_{yy} g_i(x,y)p_i},$$

$$M_i(u,v,q_i) = \frac{f_i(u,v) - \frac{1}{2}q_i^t \nabla_{xx} f_i(u,v)q_i}{g_i(u,v) - \frac{1}{2}q_i^t \nabla_{xx} g_i(u,v)q_i}.$$

Here $f_i,\ g_i : S_1 \times S_2 \to R$ are twice continuously differentiable functions for all $i = 1,2,...,k,\ p_i = (p_1,p_2,...,p_k) \in R^m,\ q_i = (q_1,q_2,...,q_k) \in R^n$ and $\lambda = (\lambda_1,\lambda_2,...,\lambda_k) \in R^k$. It is assumed that in the feasible regions the numerators are nonnegative and denominators are positive. Let $l = (l_1,l_2,...,l_k)^t,\ m = (m_1,m_2,...,m_k)^t \in R^k$. Then we can express the programs (MWP) and (MWD) equivalently to:

(EMWP) Minimize l

subject to

$$(f_i(x,y) - \frac{1}{2}p_i^t \nabla_{yy} f_i(x,y)p_i) - l_i(g_i(x,y) - \frac{1}{2}p_i^t \nabla_{yy} g_i(x,y)p_i) = 0,\ i = 1,2,...,k, \quad (1)$$

$$\sum_{i=1}^{k} \lambda_i[(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - l_i(\nabla_y g_i(x,y) + \nabla_{yy} g_i(x,y)p_i)] \leqq 0, \quad (2)$$

$$y^t \sum_{i=1}^{k} \lambda_i[(\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y)p_i) - l_i(\nabla_y g_i(x,y) + \nabla_{yy} g_i(x,y)p_i)] \geqq 0, \quad (3)$$

$\lambda > 0.$

(EMWD) Maximize m

subject to

$$f_i(u,v) - \frac{1}{2} q_i^t \nabla_{xx} f_i(u,v) q_i - m_i(g_i(u,v) - \frac{1}{2} q_i^t \nabla_{xx} g_i(u,v) q_i) = 0, \; i = 1,2,...,k, \quad (4)$$

$$\sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)] \geqq 0, \quad (5)$$

$$u^t \sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)] \leqq 0, \quad (6)$$

$\lambda > 0.$

Now we prove weak, strong and converse duality theorems for (EMWP) and (EMWD) but equally apply to (MFP) and (MFD).

Theorem 3.1 (Weak Duality). *Let* (x, y, l, λ, p) *be feasible to* (EMWP)*and* (u, v, m, λ, q) *be feasible to* (EMWD) *and* $g(x, v) > 0$. *Further, we assume that*

(*a*) $\sum_{i=1}^{k} \lambda_i(f_i(.,v) - m_i g_i(.,v))$ *be second-order* (ϕ_1, ρ)*-invex at* u,

(*b*) $\sum_{i=1}^{k} \lambda_i(-f_i(x,.) + l_i g_i(x,.))$ *be second-order* (ϕ_2, ρ)*-invex at* y,

(*c*) $\phi_1(x, u, (\xi_1, \rho)) + u^T \xi_1 \geqq 0, \; \forall \xi_1 \in R_+^n$ *and* $\phi_2(v, y, (\xi_2, \rho)) + y^T \xi_2 \leqq 0, \; \forall \xi_2 \in R_+^m$.

Then $l \geqq m$.

Proof. From the dual constraint (5) and the condition $\phi_1(x, u, (\xi_1, \rho)) + u^T \xi_1 \geqq 0, \; \forall \xi_1 \in R_+^n$ we have

$$\phi_1(x, u; (\sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)], \rho))$$

$$u^T \sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)] \geqq 0,$$

which by the dual constraint (6) becomes

$$\phi_1(x, u; (\sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v)q_i)], \rho)) \geqq 0.$$

From the second-order (ϕ_1,ρ)-invexity of $\sum_{i=1}^{k}\lambda_i(f_i(.,v)-m_ig_i(.,v))$ at u, the above inequality gives

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v)-m_ig_i(x,v)-\{f_i(u,v)-m_ig_i(u,v)\}+\frac{1}{2}q_i^T\{\nabla_{xx}(f_i(u,v)-m_ig_i(u,v))\}q_i]\geqq 0,$$

which by the dual constraint (4) becomes

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v)-m_ig_i(x,v)]\geqq 0. \tag{7}$$

On the other hand, from the dual constraint (2) and the condition $\phi_2(v,y,(\xi_2,\rho))+y^T\xi_2\leqq 0,\ \forall \xi_2\in R_+^m$, we have

$$\phi_2(v,y,(-\sum_{i=1}^{k}\lambda_i[(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)-l_i(\nabla_y g_i(x,y)+\nabla_{yy}g_i(x,y)p_i)],\rho))$$

$$-y^T\sum_{i=1}^{k}\lambda_i[(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)-l_i(\nabla_y g_i(x,y)+\nabla_{yy}g_i(x,y)p_i)]\geqq 0,$$

which by the dual constraint (3) becomes

$$\phi_2(v,y,(-\sum_{i=1}^{k}\lambda_i[(\nabla_y f_i(x,y)+\nabla_{yy}f_i(x,y)p_i)-l_i(\nabla_y g_i(x,y)+\nabla_{yy}g_i(x,y)p_i)],\rho))\geqq 0.$$

From second-order (ϕ_2,ρ)-invexity of $\sum_{i=1}^{k}\lambda_i(-f_i(x,.)+l_ig_i(x,.))$ at y, the above inequality gives

$$\sum_{i=1}^{k}\lambda_i[\{-f_i(x,v)+l_ig_i(x,v)\}-\{-f_i(x,y)+l_ig_i(x,y)\}+\frac{1}{2}p_i^T\nabla_{yy}\{-f_i(x,y)+l_ig_i(x,y)\}p_i]\geqq 0,$$

which by the dual constraint (1) becomes

$$\sum_{i=1}^{k}\lambda_i(-f_i(x,v)+l_ig_i(x,v))\geqq 0. \tag{8}$$

From (7) and (8), we get

$$\sum_{i=1}^{k}\lambda_i(l_i-m_i)g_i(x,v)\geqq 0.$$

Since $\lambda>0,\ g(x,v)>0$, it implies that

$$l\geqq m.$$

Hence the theorem.

Theorem 3.3 (Strong Duality). *Let f be thrice differentiable function on $R^n\times R^m$. Let $(\bar{x},\bar{y},\bar{l},\bar{\lambda},\bar{p})$ be a weak efficient solution of*(EMWP) *and suppose that*

(a) $(\nabla_{yy} f_i - \bar{l}_i \nabla_y g_i)$ *is positive definite and* $\bar{p}_i^T(\nabla_y f_i - \bar{l}_i \nabla_y g_i) \geqq 0,\ \forall i = 1, 2, ..., k,$ *or* $(\nabla_{yy} f_i - \bar{l} \nabla_{yy} g_i)$ *is negative definite and* $\bar{p}_i^T(\nabla_y f_i - \bar{l}_i \nabla_y g_i) \leqq 0,\ \forall i = 1, 2, ..., k,$

(b) $\nabla_y g_i\ \bar{p}_i + \bar{p}_i^T \nabla_{yy} g_i\ \bar{p}_i \geqq 0,\ i = 1, 2, ..., k,$

(c) *the set* $\{(\nabla_y f_1 + \nabla_{yy} f_1\ \bar{p}_1) - \bar{l}_1(\nabla_y g_1 + \nabla_{yy} g_1\ \bar{p}_1), (\nabla_y f_2 + \nabla_{yy} f_2\ \bar{p}_2) - \bar{l}_2(\nabla_y g_2 + \nabla_{yy} g_2\ \bar{p}_2), ..., (\nabla_y f_k + \nabla_{yy} f_k\ \bar{p}_k) - \bar{l}_k(\nabla_y g_k + \nabla_{yy} g_k\ \bar{p}_k)\}$ *is linearly independent,*

where $f_i = f_i(\bar{x}, \bar{y}),\ i = 1, 2, ..., k$. *Then* $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$ *is feasible for* (EMWD). *Furthermore, if the hypotheses of Theorem 3.1 are satisfied, then* $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$ *is a properly efficient solution of* (EMWD) *and the two objective values are equal.*

Proof. Since $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{p})$ is a weak efficient solution of (EMWP) by Fritz John type necessary optimality conditions, there exists $\alpha \in R^k, \beta \in R^k, \gamma \in R^m, \delta \in R, \mu \in R^k$ such that

$$\sum_{i=1}^{k} \beta_i[(\nabla_x f_i - \frac{1}{2}(\nabla_{yy} f_i \bar{p}_i)_x \bar{p}) - \bar{l}_i(\nabla_x g_i - \frac{1}{2}(\nabla_{yy} g_i \bar{p}_i)_x \bar{p}_i) + (\gamma - \delta\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i[(\nabla_{yx} f_i + (\nabla_{yy}\ f_i \bar{p}_i)_x \bar{p}_i) - \bar{l}_i(\nabla_{yx} g_i + (\nabla_{yy} g_i\ \bar{p}_i)_x)] - \eta = 0, \tag{9}$$

$$\sum_{i=1}^{k} (\beta_i - \delta\lambda_i)[(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) - \bar{l}_i(\nabla_y g_i + \nabla_{yy} g_i \bar{p}_i)] + \sum_{i=1}^{k} [(\nabla_{yy} f_i - \bar{l}_i \nabla_{yy} g_i) \{(\gamma - \delta\bar{y})\lambda_i - \beta_i \bar{p}_i\}] + \sum_{i=1}^{k} [(\nabla_{yy} f_i \bar{p}_i)_y - \bar{l}_i(\nabla_{yy} g_i \bar{p}_i)_y (\gamma - \delta\bar{y})\lambda_i - \frac{\beta_i \bar{p}_i}{2}] = 0, \tag{10}$$

$$(\gamma - \delta\bar{y})^t[(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) - \bar{l}_i(\nabla_y g_i + \nabla_{yy} g_i \bar{p}_i)] - \mu_i = 0,\ i = 1, 2, ..., k, \tag{11}$$

$$[(\gamma - \delta\bar{y})^t \bar{\lambda}_i - \beta_i \bar{p}_i]^t (\nabla_{yy} f_i - \bar{l}_i \nabla_{yy} g_i) = 0,\ i = 1, 2, ..., k, \tag{12}$$

$$\alpha_i - \beta_i\Big(g_i - \frac{1}{2}\bar{p}_i^t \nabla_{yy} g_i \bar{p}_i\Big) - (\gamma - \delta\bar{y})^t(\bar{\lambda}_i(\nabla_y g_i + \nabla_{yy} g_i \bar{p}_i)) = 0,\ i = 1, 2, ..., k, \tag{13}$$

$$\gamma^t \sum_{i=1}^{k} \bar{\lambda}_i[(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) - \bar{l}_i(\nabla_y g_i + \nabla_{yy} g_i \bar{p}_i)] = 0, \tag{14}$$

$$\delta\bar{y}^t \sum_{i=1}^{k} \bar{\lambda}_i[(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) - \bar{l}_i(\nabla_y g_i + \nabla_{yy} g_i \bar{p}_i)] = 0, \tag{15}$$

$$\mu^t \bar{\lambda} = 0, \tag{16}$$

$$\eta^T \bar{x} = 0, \tag{17}$$

$$(\alpha, \beta, \gamma, \delta, \mu) \neq 0,\ (\alpha, \beta, \gamma, \delta, \mu) \geqq 0. \tag{18}$$

Since $\bar{\lambda} > 0$, it follows from (16) that $\mu = 0$. Therefore from (11) we get

$$(\gamma - \delta\bar{y})^t[(\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) - \bar{l}_i(\nabla_y g_i + \nabla_{yy} g_i \bar{p}_i)] = 0,\ i = 1, 2, ..., k. \tag{19}$$

From (12) and the condition (a), we have

$$(\gamma-\delta\bar{y})^t\bar{\lambda}_i=\beta_i\bar{p}_i,\ i=1,2,...,k. \tag{20}$$

Using (20) in (10), we get

$$\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)[(\nabla_y f_i+\nabla_{yy}f_i\bar{p}_i)-\bar{l}_i(\nabla_y g_i+\nabla_{yy}g_i\bar{p}_i)]$$

$$+\frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i[((\nabla_{yy}f_i\bar{p}_i)_y-\bar{l}_i(\nabla_{yy}g_i\bar{p}_i)_y)(\gamma-\delta\bar{y})]=0. \tag{21}$$

We claim that $\beta_i\neq 0,\ \forall i\in 1,2,...,k$. If possible, let us suppose that there exists $i\in 1,2,...,k$ such that $\beta_i=0$. Then from (20) implies that

$$\gamma-\delta\bar{y}=0, \tag{22}$$

therefore, (21) reduces to

$$\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)[(\nabla_y f_i+\nabla_{yy}f_i\bar{p}_i)-\bar{l}_i(\nabla_y g_i+\nabla_{yy}g_i\bar{p}_i)]=0. \tag{23}$$

By assumption (c), the above relation yields

$$\beta_i-\delta\bar{\lambda}_i=0,\ i=1,2,...,k, \tag{24}$$

Since $\bar{\lambda}>0$, the above relation implies that $\delta=0$ and $\beta=0$. From (9), (13) and (22), we get $\eta=0,\ \alpha=0$ and $\gamma=0$. It contradicts the fact that $(\alpha,\beta,\gamma,\delta,\mu)\geqq 0$. Hence $\beta_i\neq 0,\ \forall i\in\{1,2,...,k\}$. We now prove that $\beta>0$. To prove this, it suffices to show that $\beta\geqq 0$. From (13) and (20), we have

$$\alpha_i=\beta_i[(g_i-\frac{1}{2}\bar{p}_i^t\nabla_{yy}g_i\bar{p}_i)-\bar{p}_i(\bar{\lambda}_i(\nabla_y g_i+\nabla_{yy}g_i\bar{p}_i))]=0,\ i=1,2,...,k.$$

Since $\alpha\geqq 0,\ g_i-\frac{1}{2}\bar{p}_i^T\nabla_{yy}g_i\bar{p}_i>0,\ i=1,2,...k$, from assumption (b), we can obtain $\beta\geqq 0$, which by $\beta_i\neq 0,\ \forall i\in 1,2,...,k$ yields $\beta>0$.

From (14), (15) and (20), we have

$$\sum_{i=1}^{k}\beta_i(\bar{p}_i^T(\nabla_y f_i-\bar{l}_i\nabla_y g_i)+\bar{p}_i^T(\nabla_{yy}f_i-\bar{l}_i\nabla_{yy}g_i)\bar{p}_i)=0.$$

By $\beta>0$ and assumption (a), we obtain $\bar{p}_i=0,\ i=1,2,...,k$. Thus from (20) we get (22). Similarly, we obtain (24). Using (22) and the fact that $\bar{p}_i=0,\ i=1,2,...,k$ in (9), we get

$$\sum_{i=1}^{k}\beta_i[\nabla_x f_i-\bar{l}_i(\nabla_x g_i)]=\eta,$$

which by $\beta > 0,\ \bar{\lambda} > 0,\ \eta \geqq 0$ and (24) yields

$$\sum_{i=1}^{k} \beta_i[\nabla_x f_i - \bar{l}_i(\nabla_x g_i)] = \frac{\eta}{\delta} \geqq 0,$$

Combining the above result with (17), we obtain

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i[\nabla_x f_i - \bar{l}_i(\nabla_x g_i)] = 0,$$

Since $\delta > 0$, so from (22) we get

$$\bar{y} = \frac{\gamma}{\delta} \geqq 0.$$

Thus it follows from the above three inequalities that $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q})$ is feasible solution to (EMWD). Under the assumptions of Theorem 3.1, if $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$ is not an efficient solution of (EMWD), then there exists other feasible solution $(u, v, m, \bar{\lambda}, q)$ of (EMWD) such that $\bar{l} \leq m$. Since $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{p})$ is a feasible solution of (EMWP), by Theorem 3.1, we have $\bar{l} \nleq m$, hence the contradiction implies $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$ is an efficient solution of (EMWD).

If $(\bar{x}, \bar{y}, \bar{l}, \bar{\lambda}, \bar{q} = 0)$ is not a properly efficient solution of (EMWD), then there exists other feasible solution $(u, v, m, \bar{\lambda}, q)$ of (EMWD) such that for an index $i \in \{1, 2, ..., k\}$ and any real number $M > 0$, $m_i - \bar{l}_i > M(\bar{l}_j - m_j)$ for j satisfying $\bar{l}_j > m_j$ whenever $m_i > \bar{l}_i$. This implies $m_i > \bar{l}_i$ can be made arbitrarily large and this contradicts with Theorem 3.1. Also the two objective values are equal.

Theorem 3.4 (Converse Duality). *Let f be thrice differentiable function on $R^n \times R^m$. Let $(\bar{u}, \bar{v}, \bar{m}, \bar{\lambda}, \bar{q})$ be a weak efficient solution of* (EMWD) *and suppose that*

(a) $(\nabla_{xx} f_i - \bar{m}_i \nabla_x g_i)$ *is positive definite and* $\bar{q}_i^T(\nabla_x f_i - \bar{m}_i \nabla_x g_i) \geqq 0,\ \forall i = 1, 2, ..., k$, *or* $(\nabla_{xx} f_i - \bar{m} \nabla_{xx} g_i)$ *is negative definite and* $\bar{q}_i^T(\nabla_x f_i - \bar{m}_i \nabla_x g_i) \leqq 0,\ \forall i = 1, 2, ..., k,$

(b) $\nabla_x g_i\ \bar{q}_i + \bar{q}_i^T \nabla_{xx} g_i\ \bar{q}_i \geqq 0,\ i = 1, 2, ..., k,$

(c) *the set* $\{(\nabla_x f_1 + \nabla_{xx} f_1\ \bar{q}_1) - \bar{m}_1(\nabla_x g_1 + \nabla_{xx} g_1\ \bar{q}_1), (\nabla_x f_2 + \nabla_{xx} f_2\ \bar{q}_2) - \bar{m}_2(\nabla_x g_2 + \nabla_{xx} g_2\ \bar{q}_2), ..., (\nabla_x f_k + \nabla_{xx} f_k\ \bar{q}_k) - \bar{m}_k(\nabla_x g_k + \nabla_{xx} g_k\ \bar{q}_k)\}$ *is linearly independent,*

where $f_i = f_i(\bar{u}, \bar{v}),\ i = 1, 2, ..., k$. Then $(\bar{u}, \bar{v}, \bar{m}, \bar{\lambda}, \bar{p} = 0)$ is feasible for (EMWD). *Furthermore, if the hypotheses of Theorem 3.1 are satisfied, then $(\bar{u}, \bar{v}, \bar{m}, \bar{\lambda}, \bar{p} = 0)$ is a properly efficient solution of* (EMWD) *and the two objective values are equal.*

Proof. It follows on the lines of Theorem 3.3.

4. Self duality

A mathematical programming problem is said to be self-dual if it is formally identical with its dual, that is, the dual can be recast in the form of the primal. If we take the functions f_i as skew-symmetric and g_i as symmetric, that is,

$$f_i(x,y) = -f_i(y,x),\ g_i(x,y) = g_i(y,x)$$

for each $i = 1,2,...,k$, then we shall show that the programs (EMWP) and (EMWD) are self-dual. By recasting the dual problem (EMWD) as minimization problem, we have

$$\text{Minimize } -m = (-m_1, -m_2, ..., -m_k)$$

subject to

$$\Big[f_i(u,v) - \frac{1}{2}q_i^t \nabla_{xx} f_i(u,v) q_i - m_i(g_i(u,v) - \frac{1}{2}q_i^t \nabla_{xx} g_i(u,v) q_i)\Big] = 0,\ i = 1,2,...,k,$$

$$\sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v) q_i)] \geqq 0,$$

$$u^t \sum_{i=1}^{k} \lambda_i[(\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) q_i) - m_i(\nabla_x g_i(u,v) + \nabla_{xx} g_i(u,v) q_i)] \leqq 0,$$

$$\lambda > 0,$$

where $m_i = \left[\frac{f_i(u,v) - \frac{1}{2}q_i^t \nabla_{xx} f_i(u,v) q_i}{g_i(u,v) - \frac{1}{2}q_i^t \nabla_{xx} g_i(u,v) q_i}\right]$.

Since f_i and $g_i,\ i = 1,2,...,k$ are skew-symmetric and symmetric, respectively, we have

$$\nabla_x f_i(u,v) = -\nabla_x f_i(v,u), \nabla_{xx} f_i(u,v) = -\nabla_{xx} f_i(v,u),$$

$$\nabla_y f_i(u,v) = -\nabla_y f_i(v,u), \nabla_{yy} f_i(u,v) = -\nabla_{yy} f_i(v,u),$$

$$\nabla_x g_i(u,v) = \nabla_x g_i(v,u), \nabla_{xx} g_i(u,v) = \nabla_{xx} g_i(v,u),$$

$$\nabla_y g_i(u,v) = \nabla_y g_i(v,u), \nabla_{yy} g_i(u,v) = \nabla_{yy} g_i(v,u).$$

Hence the dual problem (EMWD) can be written as

$$\text{Minimize } z = (z_1, z_2, ..., z_k)$$

subject to

$$f_i(v,u) - \frac{1}{2}q_i^t \nabla_{yy} f_i(v,u) q_i - z_i(g_i(v,u) - \frac{1}{2}q_i^t \nabla_{yy} g_i(v,u) q_i) = 0,\ i = 1,2,...,k,$$

$$\sum_{i=1}^{k} \lambda_i[(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u) q_i) - z_i(\nabla_y g_i(v,u) + \nabla_{yy} g_i(v,u) q_i) \leqq 0,$$

$$u^t \sum_{i=1}^{k} \lambda_i[(\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u)q_i) - z_i(\nabla_y g_i(v,u) + \nabla_{yy} g_i(v,u)q_i)] \geqq 0,$$

$$\lambda > 0,$$

where $z_i = \frac{f_i(v,u) - \frac{1}{2}q_i^t \nabla_{yy} f_i(v,u)q_i}{g_i(v,u) - \frac{1}{2}q_i^t \nabla_{yy} g_i(v,u)q_i}$, for all $i = 1, 2, ..., k$.
This shows that the dual problem (EMWD) is identical to (EMWP). Hence if (u, v, λ, m, q) is feasible for (EMWD), then (v, u, λ, m, q) is feasible for (EMWP) and conversely.

We now state the self-duality theorem.

Theorem 4.1 *Let f_i and g_i, $i = 1, 2, ..., k$ be skew-symmetric and symmetric, respectively. Then* (EMWP)*is self-dual. Furthermore, if* (EMWP)*and*(EMWD)*are dual problems and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{m}, \bar{p})$ is a joint optimal solution, then so is $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{m}, \bar{p})$ and and the common optimal value of the objective functions is 0.*

Proof. It follows on the lines of the corresponding results by Weir and Mond [17].

5. Special Cases

(i) If $k = 1$, then the problem (MWP) and (MWD) are reduced to that presented in Gulati *et al.* [7].

(ii) If we set $k = 1$, $g(x, y) = 1$, then (MWP) and (MWD) becomes the programs studied in Bector and Chandra [3]. Also if $p = 0$ and $q = 0$, then they reduce to the problems presented in Mond and Weir [11].

(iii) If $g = 1$ for all x, y in (MWP) and (MWD), we get the programs studied in Suneja *et al.* [15].

6. Conclusion

In this article, a pair of Mond-Weir type multiobjective second-order fractional symmetric dual programs is presented and weak, strong and converse duality relations between primal and dual problems are discussed. It is the future tasks of the the authors to extend these results to higher-order fractional symmetric dual programs over cones. It will be interesting to check the validity of duality results for multiobjective second-order mixed integer programs, wherein some primal and dual variables are constrained to belong to some arbitrary sets.

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