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On pricing European options under HCIR model: A comparative study

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Abstract

This paper concerns the numerical solution of the three-dimensional Heston- Cox- Ingersoll- Ross (HCIR) partial differential equation for the fair values of European- style financial options. Adomian decomposition method and Homotopy perturbation method are two powerful methods which consider an approximate solution of the HCIR stochastic differential equation which is reformed to PDE form. Also, the theoretical analysis of these methods shows two methods are equivalent in solving nonlinear equations.

Keywords: Heston- Cox- Ingersoll- Ross (HCIR) partial differential equation, option pricing, Adomian decomposition method (ADM), Homotopy perturbation method (HPM).

1 Introduction

One of the most problems in numerical analysis is that of finding the solution of the equation $u(x) = 0$. for a given function u, which is sufficiently smooth in the neighborhood of a simple root α. In most cases it is difficult to obtain an analytical solution of $u(x) = 0$. Therefore the exploitation of numerical techniques for solving such equations becomes a main subject of considerable interests. In recent years, there have been some developments in the study of Newton- like iterative methods. To obtain these iterative methods, Adomian s decomposition method (ADM) [1- 3], the Homotopy perturbation method (HPM) [4-6] as well as the other more general methods such as the homotopy analysis method [7] play an important role in the process of numerical approximation. Over the past few years, the two methods- ADM and HPM- have been applied to solve a wide range of problems, deterministic and stochastic, linear and nonlinear, in different disciplines such as physics, chemistry, biology, engineering, and etc.
In the original Black- Scholes option valuation theory the volatility and the interest rate are both assumed constant. In practice however, these quantities vary through time. A popular extension of the Black- Scholes model has been proposed by Heston [8]. Here, besides the asset price, the volatility is modeled itself as a stochastic process.

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In this paper, we consider the following model, given by a system of three stochastic differential equations (SDEs)

$$\begin{cases} dS_\tau = R_\tau S_\tau d\tau + S_\tau \sqrt{V_\tau} dW_\tau^1, \\ dV_\tau = \kappa(\eta - V_\tau)d\tau + \delta_1 \sqrt{V_\tau} dW_\tau^2, \\ dR_\tau = a(b(\tau) - R_\tau)d\tau + \delta_2 \sqrt{R_\tau} dW_\tau^3. \end{cases} \tag{1}$$

for $0 < \tau \le T$ with $T > 0$ the given maturity time of the option. S_τ, V_τ and R denote random variables that represent the asset price, its variance, and the interest rate, respectively at time τ. The variance process V_τ was proposed by Heston and the interest rate process R_τ by Cox, Ingersoll and Ross [9-10]. In Eq. (1), parameters $\kappa, \eta, \delta_1, \delta_2$ and a are given positive real numbers and $dW_\tau^1, dW_\tau^2, dW_\tau^3$ are Brownian motions, with given correlation factors $\rho_1, \rho_2, \rho_3 \in [-1,1]$. Further, b is a given, deterministic, positive function of time, which can be chosen such that to match the current term structure of interest rates.
The partial differential equation (PDE) associated with (1) for the fair values of European- style options forms a time- dependent connection- diffusion- reaction equation with mixed spatial-derivative terms.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \delta_1^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2} \delta_2{}^2 r \frac{\partial^2 u}{\partial r^2} + \rho_{12} \delta_1 s v \frac{\partial^2 u}{\partial s \partial v} + \rho_{13} \delta_2 s \sqrt{vr} \frac{\partial^2 u}{\partial s \partial r} \\ &+ \rho_{23} \delta_1 \delta_2 \sqrt{vr} \frac{\partial^2 u}{\partial v \partial r} + rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b(T-t) - r) \frac{\partial u}{\partial r} - ru. \end{aligned} \tag{2}$$

On the unbounded three- dimensional spatial domain $s > 0, v > 0, r > 0$ with $0 < t \le T$. $u(s,, v, r, t)$ denotes the fair value of a European- style option if at the time $\tau = T - t$, the asset price, its variance, and the interest rate be equal to s, v, r, respectively. Eq. (2) is known as the Heston- Cox- Ingersoll-Ross (HCIR) PDE. This equation usually appears with initial and boundary value conditions which are determined by the specific option under consideration. Initial conditions in European call options equation is denoted by,

$$u(s, v, r, t) = \max(0, s - K). \tag{3}$$

with a given strike price $K > 0$.

2 ADM for HCIR partial differential equation

To solve equation (2), by ADM, well addressed in [11-12], let's take the following canonical form of the equation, Lets the Eq. (2) have applied and if we used $u(s, v, r, 0) = \max(0, s - K)$ as the initial condition by using ADM structure then we can construct

$$\begin{aligned} u(s, v, r, t) &= u_0(s, v, r, 0) - \int_0^t \Big(\frac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \delta_1^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2} \delta_2{}^2 r \frac{\partial^2 u}{\partial r^2} \\ &+ \rho_{12} \delta_1 s v \frac{\partial^2 u}{\partial s \partial v} + \rho_{13} \delta_2 s \sqrt{vr} \frac{\partial^2 u}{\partial s \partial r} + \rho_{23} \delta_1 \delta_2 \sqrt{vr} \frac{\partial^2 u}{\partial v \partial r} + rs \frac{\partial u}{\partial s} \\ &+ \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b(T-t) - r) \frac{\partial u}{\partial r} - ru\Big) dt. \end{aligned} \tag{4}$$

Let's take the solution as a series, say $u = u_0 + u_1 \lambda + u_2 \lambda^2 + \ldots$ and following an alternate algorithm for Adomian polynomials [13] we get

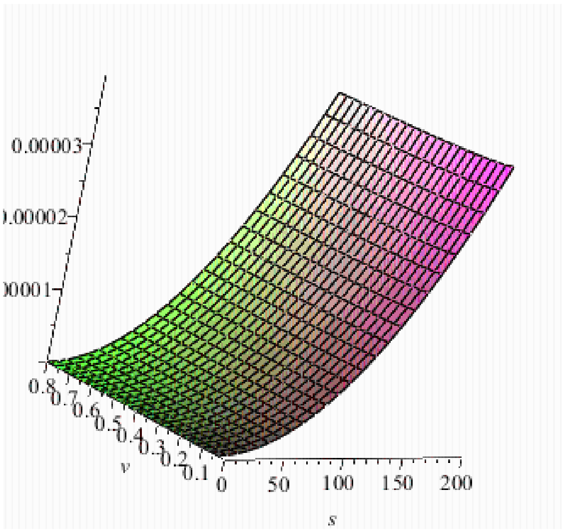
$$
\begin{aligned}
&A_0 = rs - 3v + 0.36,\\
&A_1 = -3(0.12 - v)(r^2 st - 3vrt + 0.36rt),\\
&\vdots
\end{aligned}
\tag{5}
$$

By using parameters as:

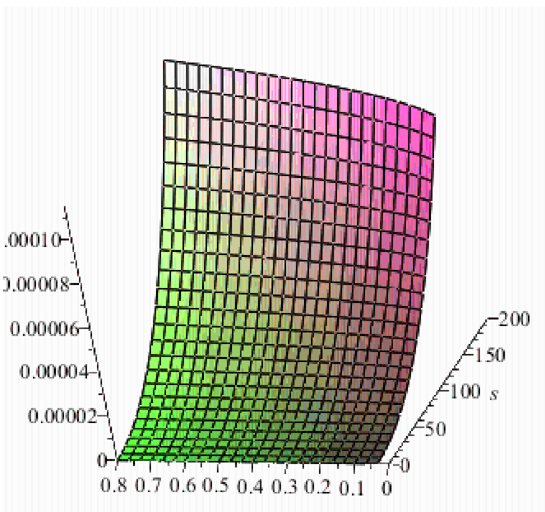
$$
\text{Case 1:}\ \begin{array}{l} T = 1, K = 100,\ \kappa = 3,\ \eta = 0.12,\ a = 0.2,\ b = 0.05,\ \delta_1 = 0.04, \\ \delta_2 = 0.03,\ \rho_{12} = 0.6,\ \rho_{13} = 0.2,\ \rho_{23} = 0.4 \end{array}
\tag{6}
$$

$$
\text{Case2:}\ \begin{array}{l} T = 15,\ K = 100,\ \kappa = 0.3,\ \eta = 0.04,\ a = 0.16,\ b = 0.055,\ \delta_1 = 0.9, \\ \delta_2 = 0.03,\ \rho_{12} = -0.5,\ \rho_{13} = 0.2,\ \rho_{23} = 0.1 \end{array}
$$

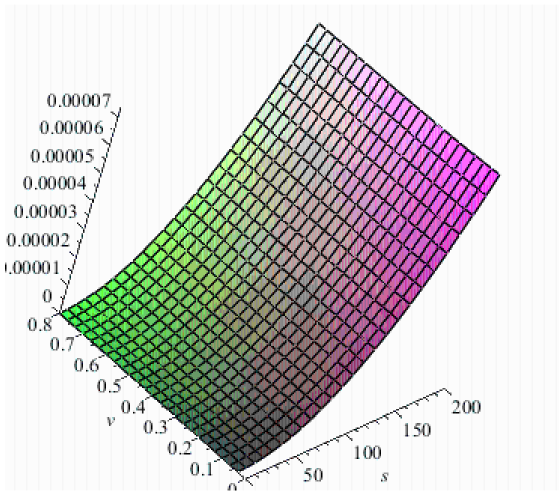
$$
\text{Case3:}\ \begin{array}{l} T = 5,\ K = 100,\ \kappa = 1,\ \eta = 0.09,\ a = 0.22,\ b = 0.034,\ \delta_1 = 1, \\ \delta_2 = 0.11,\ \rho_{12} = -0.3,\ \rho_{13} = -0.5,\ \rho_{23} = -0.2 \end{array}
$$



Case1: ADM for HCIR PDE



Case2: ADM for HCIR PDE



Case3: ADM for HCIR PDE

3 HCIR partial differential equation by HPM

For solving equation (2), by HPM, well addressed in [14-16], the basic idea is to construct a homotopy $H(v,p):\mathbb{R}\times[0,1]\to\mathbb{R}$ which satisfies

$$H(v,p)=pf(v)+(1-p)(f(v)-f(x_0))=0,\quad v\in\mathbb{R},$$

or

$$H(v,p)=f(v)-f(x_0)+pf(x_0)=0,\quad v\in\mathbb{R},\tag{7}$$

where $p\in[0,1]$ is an embedding parameter, and x_0 is an initial approximation.

Consider Eq. (2) as follow

$$\frac{\partial u}{\partial t}=\frac{1}{2}s^2v\frac{\partial^2 u}{\partial s^2}+\frac{1}{2}\delta_1^2v\frac{\partial^2 u}{\partial v^2}+\frac{1}{2}\delta_2^2r\frac{\partial^2 u}{\partial r^2}+\rho_{12}\delta_1 sv\frac{\partial^2 u}{\partial s\partial v}+\rho_{13}\delta_2 s\sqrt{vr}\frac{\partial^2 u}{\partial s\partial r}$$
$$+\rho_{23}\delta_1\delta_2\sqrt{vr}\frac{\partial^2 u}{\partial v\partial r}+rs\frac{\partial u}{\partial s}+\kappa(\eta-v)\frac{\partial u}{\partial v}+a(b(T-t)-r)\frac{\partial u}{\partial r}-ru.$$

Now, we construct a homotopy in the following form

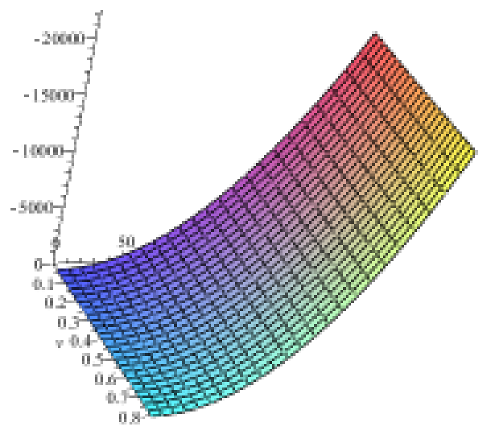
$$\begin{aligned}H(s,v,r,t):=&(1-p)(L(u)-L(v_0))+p(\frac{\partial u}{\partial t}-\frac{1}{2}s^2v\frac{\partial^2 u}{\partial s^2}-\frac{1}{2}\delta_1^2v\frac{\partial^2 u}{\partial v^2}-\frac{1}{2}\delta_2^2r\frac{\partial^2 u}{\partial r^2}\\&-\rho_{12}\delta_1 sv\frac{\partial^2 u}{\partial s\partial v}-\rho_{13}\delta_2 s\sqrt{vr}\frac{\partial^2 u}{\partial s\partial r}-\rho_{23}\delta_1\delta_2\sqrt{vr}\frac{\partial^2 u}{\partial v\partial r}-rs\frac{\partial u}{\partial s}-\kappa(\eta-v)\frac{\partial u}{\partial v}\\&-a(b(T-t)-r)\frac{\partial u}{\partial r}+ru)=0.\end{aligned}\tag{8}$$

Consider $v_0(s,v,r,0)=2(r-2)s^2t^2$ as an initial approximation that satisfies in initial condition. Substituting solution series, into Eq. (8) and equating the terms with identical powers of p, leads to

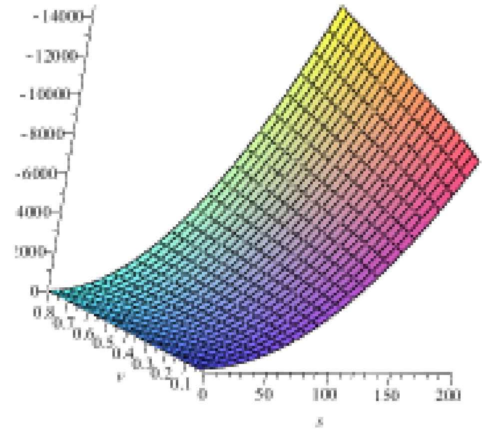
$$
\begin{aligned}
&p^0: \frac{\partial}{\partial t}u_0(s,v,r,t) - (\frac{\partial}{\partial t}v_0(s,v,r,t)) = 0, \\
&p^1: \frac{\partial}{\partial t}u_1(s,v,r,t) - 0.08s\sqrt{vr}(\frac{\partial^2}{\partial s \partial r}u_0(s,v,r,t)) - 0.024sv(\frac{\partial^2}{\partial v \partial s}u_0(s,v,r,t)) \\
&-0.5s^2v(\frac{\partial^2}{\partial s^2}u_0(s,v,r,t)) - rs(\frac{\partial}{\partial s}u_0(s,v,r,t)) + \frac{\partial}{\partial t}v_0(s,v,r,t) - 0.36(\frac{\partial}{\partial v}u_0(s,v,r,t)) \\
&-0.010(\frac{\partial}{\partial r}u_0(s,v,r,t)) - 0.0008v(\frac{\partial^2}{\partial v^2}u_0(s,v,r,t)) - 0.00045r(\frac{\partial^2}{\partial r^2}u_0(s,v,r,t)) \\
&+0.2r(\frac{\partial}{\partial r}u_0(s,v,r,t)) + 0.010t(\frac{\partial}{\partial r}u_0(s,v,r,t)) + ru_0(s,v,r,t) \\
&-0.00048\sqrt{vr}(\frac{\partial^2}{\partial r \partial v}u_0(s,v,r,t)) + 3v(\frac{\partial}{\partial v}u_0(s,v,r,t)) = 0, \\
&\vdots
\end{aligned} \tag{9}
$$

By using the following parameters, we are obtained the numerical results which are outlined in Table 1.

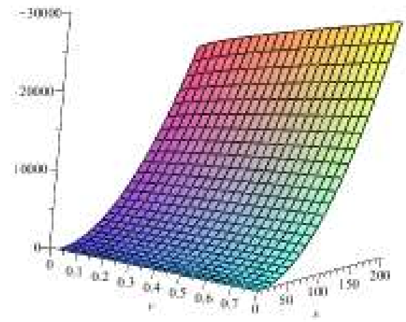
$$
\begin{aligned}
&\text{Case 1:} \begin{array}{l} T=1,\ K=100,\ \kappa=3,\ \eta=0.12,\ a=0.2,\ b=0.05,\ \delta_1=0.04, \\ \delta_2=0.03,\ \rho_{12}=0.6,\ \rho_{13}=0.2,\ \rho_{23}=0.4 \end{array} \\
&\text{Case2:} \begin{array}{l} T=15,\ K=100,\ \kappa=0.3,\ \eta=0.04,\ a=0.16,\ b=0.055,\ \delta_1=0.9, \\ \delta_2=0.03,\ \rho_{12}=-0.5,\ \rho_{13}=0.2,\ \rho_{23}=0.1 \end{array} \\
&\text{Case3:} \begin{array}{l} T=5,\ K=100,\ \kappa=1,\ \eta=0.09,\ a=0.22,\ b=0.034,\ \delta_1=1, \\ \delta_2=0.11,\ \rho_{12}=-0.3,\ \rho_{13}=-0.5,\ \rho_{23}=-0.2 \end{array}
\end{aligned} \tag{10}
$$



Case1: HPM for HCIR PDE



Case2: HPM for HCIR PDE



Case3: HPM for HCIR PDE

Table 1: ADM and HPM values

$U(t,r,s,v)$	U_{ADM}	U_{HPM}	$\lVert U_A - U_H \rVert$
t=0.001	0.1218828527E-11	0.1218828519E-11	0.000000008E-11
t=0.002	0.9642989987E-9	0.9642989946E-9	0.000000041E-9
t=0.003	0.8069392633E-7	0.8069392632E-7	0.0000000001E-7
t=0.004	0.7599628844E-6	0.7599628723E-6	0.0000000121E-6
t=0.005	0.2125185999E-5	0.2125186000E-5	0.0000000001E-5
t=0.006	0.5709457955E-5	0.5709457935E-5	0.0000000020E-5
t=0.007	0.1159414510E-4	0.1159414501E-4	0.0000000009E-5
t=0.008	0.2386021778E-4	0.2386021764E-4	0.0000000014E-4

4 Equivalence between ADM and HPM

Theorem 4.1 The Homotopy perturbation method is the Adomian decomposition method with the Adomian polynomials given by

$$\begin{cases} A_1 = \tilde{x}_1 - A_0 + x_0 - c, \\ A_n = \tilde{x}_n, \quad (n \geq 2). \end{cases} \tag{11}$$

Proof. Let $f(x) = N(x) - x + c$, then $f(x) = 0 \Leftrightarrow x = N(x) + c$, if

$v = x_0 + \tilde{x}_1 p + \tilde{x}_2 p^2 + ... = \sum_{n=0}^{\infty} \tilde{x}_n p^n$ $(\tilde{x}_0 = x_0)$ in (5) is a solution of (7), then

$$f(x) = N(x) - x + c, \; f(\sum_{n=0}^{\infty} \tilde{x}_n p^n) - f(x_0) + pf(x_0) = 0 \tag{12}$$

Which yields

$$\frac{d}{dp}[\; f(\sum_{n=0}^{\infty} \tilde{x}_n p^n) \;]_{p=0} + f(x_0) = 0. \tag{13}$$

Also, we have

$$\frac{d^n}{dp^n}[\; f(\sum_{n=0}^{\infty} \tilde{x}_n p^n) \;]_{p=0} = 0, \; (n \geq 2). \tag{14}$$

Now, from Adomian polynomials and $f(x) = N(x) - x + c$, we have

$$\frac{1}{n!}\frac{d^n}{dp^n}[\; f(\sum_{n=0}^{\infty} \tilde{x}_n p^n) \;]_{p=0} = \frac{1}{n!}\frac{d^n}{dp^n}[\; N(\sum_{n=0}^{\infty} \tilde{x}_n p^n) \;]_{p=0} - \frac{1}{n!}\frac{d^n}{dp^n}[\; (\sum_{n=0}^{\infty} \tilde{x}_n p^n) \;]_{p=0}$$
$$= A_n(\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_n) - \tilde{x}_n = A_n - \tilde{x}_n, \; (n \geq 1). \tag{15}$$

From Eq. (13) and Eq. (14) we conclude that

$$\begin{cases} A_1 - \tilde{x}_1 + f(x_0) = 0, \\ A_n - \tilde{x}_n = 0, \quad (n \geq 2). \end{cases} \tag{16}$$

From $f(x_0) = N(x_0) - x_0 + c = A_0 - x_0 + c$ and Eq. (16), we obtain the desired Adomian polynomials as in theorem 4.1. Also, it follows from the method ADM that

$$c + A_0 + A_1 + A_2 + ... = x_0 + \tilde{x}_1 + \tilde{x}_2 + ... \tag{17}$$

Is the solution of $f(x) = 0$, where the Adomian polynomials are only of the following form:

$$A = A(x, x) = x - A - c, \quad A = A(x, x, ..., x) = x, \; (n \geq 2). \tag{18}$$

This completes the proof of Theorem 4.1.

Theorem 4.2. The Adomian s decomposition method is the Homotopy perturbation method with the homotopy $H(u, p)$ given by

$$H(u, p) = pN(u) - u + c. \tag{19}$$

Proof. Let $u(p) = \sum_{i=0}^{\infty} x_i p^i$. Then the convergence of series $x = \sum_{i=0}^{\infty} x_i$ implies

$$\lim_{p \to 1} u(p) = u(1) = \sum_{i=0}^{\infty} x_i \quad \text{(Abel Theorem)}, \tag{20}$$

Thus the series solution $x = \sum_{i=0}^{\infty} x_i$ can be written as

$$x = \lim_{p \to 1} u(p). \tag{21}$$

Similarly, the convergence of series $N(x)=\sum_{i=0}^{\infty} A_i$ implies

$$N(x)=\sum_{n=0}^{\infty} A_n = \lim_{p\to 1}\sum_{n=0}^{\infty} A_n p^n. \tag{22}$$

From the Taylor s expansion of a function near the origin, we know form Adomian polynomials that

$$\sum_{n=0}^{\infty} A_n p^n = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{dp^n}[N(\sum_{i=0}^{\infty} x_i p^i)]_{p=0}\, p^n = N(\sum_{i=0}^{\infty} x_i p^i) = N(u(p)). \tag{23}$$

Hence, from (22) and (23), we have

$$N(x)=\lim_{p\to 1} N(u(p)). \tag{24}$$

From Eq. (21) and Eq. (24) we conclude that

$$N(u(p)) - u(p) + c = 0 \tag{25}$$

Holds for p sufficiently close to 1. We shall construct a homotopy $H(u,p)$ such that $H(u,0)=-u+c$ and $H(u,1)=N(u)-u+c$.

In view the above discussion we see that

$$u(p)=\sum_{n=0}^{\infty} x_n p^n = x + p\sum_{n=0}^{\infty} x_{n+1} p^n = x + \sum_{n=0}^{\infty} A_n p^n = c + pN(u(p)). \tag{26}$$

That is, $pN(u(p)) - u(p) + c = 0.$ Therefore, by letting

$$H(u,p) = pN(u) - u + c, \tag{29}$$

We observe from (26) that the power series $\sum_{n=0}^{\infty} x_n p^n$ corresponds to the solution of the equation $H(u,p)=pN(u)-u+c=0,$ and becomes the approximate solution if $p\to 1.$ This showes that the Adomian decomposition method is the Homotopy perturbation method, and the proof is completed.

5 Conclusion

The main concern of this article is to construct numerical solution for Heston-Cox-Ingersoll-Ross partial differential equation. We have achieved this goal by applying Adomian s decomposition method and the Homotopy perturbation method. Theorems 4.1 and 4.2 illustrate that two methods of ADM and HPM are equivalent. This is very important for the further applications of the two methods. Firstly, the convergence of HPM can be obtained from the convergence of ADM; secondly, the approximation solution by applying ADM can also be obtained by applying HPM and vice versa.

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