AMO[*]- Advanced Modeling and Optimization, Volume 16, Number 3, 2014*

Modeling asset prices based on two-factor stochastic volatility

Farshid Mehrdoust [(1)] and Naghmeh Saber [(2)]

[(1), (2)] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Abstract

In this paper, we present an extension of the double Heston's stochastic volatility model by adding jumps to financial modeling for stock prices in the double Heston model. We assume that the underlying asset price follows the double Heston's stochastic volatility model with jumps. We demonstrate the effective use of the Fast Fourier transform approach as an effective tool in the valuation of European options under the proposed model.

Key words: stochastic volatility model; double Heston model; jumps; European option pricing.

1 Introduction

The Black–Scholes model assumes that the probability distribution of the stock price at any given future time is lognormal. If this assumption is not true we shall get biases in the prices produced by the model [1]. On the other hand, in Black-Scholes-Merton models assumed a deterministic volatility. The stochastic volatility models have been proposed to resolve shortcoming. Typical stochastic volatility models commonly include Hull and White (1987), Scott (1987), Stein and Stein (1991) and Heston (1993). There are various extensions, the empirical performances of which have been presented and compared quite extensively by Bakshi et al. [5]. The standard stochastic volatility models fail to capture the smile slope and level movements [2]. In the mathematical finance, there are many papers on option pricing under stochastic volatility model [2-5]. Recently, many researchers have been suggested stochastic volatility models with jumps [6, 7].
In this paper, the double Heston's stochastic volatility model with jumps is considered rather than the general double Heston's stochastic volatility model [8]. We evaluate European options prices by applying the Fast Fourier transform (FFT) method.

2 The double Heston stochastic volatility model

Let (Ω, F, Q) be a probability space on which are defined several Brownian motion processes $W_1 = W_1(t), W_2 = W_2(t), B_1 = B_1(t)$ and $B_2 = B_2(t)$ for $0 \le t \le T$. Let $\{F_t\}_{t\ge 0}$ be the filtration generated by these Brownian motions and Q is a risk neutral probability under the asset price process [1]

[*]**AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

$S_t = \{S(t)\}_{t\geq 0}$ and two volatility processes $V_1 = V_1(t)$ and $V_2 = V_2(t)$. The double Heston's stochastic volatility (DH) model under the risk neutral probability space is specified as follows [6]

$$dS(t) = rS_t dt + \sqrt{V_1} S_t dW_1 + \sqrt{V_2} S_t dW_2, \quad (1)$$
$$dV_1 = \alpha_1(\theta_1 - V_1)dt + \sigma_1 \sqrt{V_1} dB_1, \quad (2)$$
$$dV_2 = \alpha_2(\theta_2 - V_2)dt + \sigma_2 \sqrt{V_2} dB_2, \quad (3)$$

where W_1 and B_1 are correlated Brownian motion processes with correlation parameter ρ_1. Also, ρ_2 is the correlation coefficient between W_2 and B_2. The parameters α_1 and α_2 are the mean reversion speed of the volatility. Also, θ_1 and θ_2 are long-run mean and the constant values σ_1 and σ_2 are the volatility coefficients of the volatility processes. The constant r is the interest rate of the model. Moreover, W_1 and W_2 are independent and also B_1 and B_2 are independent processes.
Now, let $X_t = \ln S_t$, by applying Ito-Doeblin formula [8], we have

$$dX_t = S_t^{-1} dS_t - \frac{1}{2} S_t^{-2} dS_t\, dS_t \quad (4)$$

such that

$$dS_t = rS_t dt + \sqrt{V_1} S_t dW_1 + \sqrt{V_2} S_t dW_2 \quad (5)$$
$$dS_t\, dS_t = S_t^2 (V_1 + V_2) dt. \quad (6)$$

If we substitute (5) and (6) into (4) we will get the asset price with both stochastic volatilities

$$\begin{aligned} dX_t &= [r - \frac{1}{2}(V_1 + V_2)]dt + \sqrt{V_1} dW_1 + \sqrt{V_2} dW_2, \\ dV_1 &= \alpha_1(\theta_1 - V_1)dt + \sigma_1 \sqrt{V_1} dB_1, \\ dV_2 &= \alpha_2(\theta_2 - V_2)dt + \sigma_2 \sqrt{V_2} dB_2, \\ dW_1 dB_1 &= \rho_1 dt, \\ dW_2\, dB_2 &= \rho_2 dt, \\ dW_1 dW_2 &= 0, \\ dB_1\, dB_2 &= 0. \end{aligned} \quad (7)$$

3 The double Heston model with jumps

Suppose that on probability space (Ω, F, Q) we define a Poisson process $\{N_t\}_{t\geq 0}$ for all $0 \leq t \leq T$ with constant intensity $\lambda \geq 0$. Moreover, it is assumed that the Poisson process $\{N_t\}_{t\geq 0}$ is independent of all Brownian motions. Let us consider a sequence of identically and independently distributed random variables e^{J_i} for each $1 \leq i \leq N_t$, denote the jump sizes, where $J_i \sim N(\mu, \gamma^2)$. Thus, the random variables e^{J_i} are log-normally. If we assume that in underlying asset price under the double Heston model occurs a lognormal jump with jump size e^{J_1} at time u, then we have $S_u = e^{J_1} S_{u-}$ where that S_{u-} is the value of the process S_u immediately before a jump. Therefore, we write

$$S_u - S_{u-} = (e^{J_1} - 1) S_{u-}. \quad (8)$$

If there is no jump at time u then $S_u - S_{u-} = 0$. In other cases, we will have

$$\sum_{u=0}^{t}(S_u - S_{u-}) = \sum_{u=0}^{t}(e^{J_i} - 1)S_{u-}\Delta N_u = \int_0^t (e^{J_i} - 1)S_{u-}dN_u , \tag{9}$$

for $i = 1,2,...,N_t$,where $(\sum_{i=1}^{N_t} e^{J_i} - 1)$ is a compound Poisson process. Therefore, the double Heston model for asset price process with compound Poisson process is as follows

$$dS_t = r\,S_t dt + \sqrt{V_1} S_t dW_t + \sqrt{V_2} S_t dW_2 + (e^{J_i} - 1)S_{t-}dN_t - E(\sum_{i=1}^{N_t} e^{J_i} - 1), \tag{10}$$

for $i = 1,2,...,N_t$ that $E(\sum_{i=1}^{N_t} e^{J_i} - 1)$ is expected relative jumps size Also, we can write

$$E(\sum_{i=1}^{N_t} e^{J_i} - 1) = \sum_{i=1}^{N_t} E(e^{J_i}) - 1 = \lambda m \tag{11}$$

$$m = e^{\mu + \frac{1}{2}\gamma^2} - 1 \tag{12}$$

Now, by substituting (11) into (10) we will have

$$dS_t = (r - \lambda m)S_t dt + \sqrt{V_1} S_t dW_1 + \sqrt{V_2} S_t dW_2 + (e^J - 1)S_t dN_t \tag{13}$$

Let us consider $X_t = \ln S_t$. According to Ito-Doeblin formula on X_t, we get

$$X_t = X_0 + \int_0^t S_t^{-1} dS_u^c - \frac{1}{2}\int_0^t S_t^{-2} dS_u^c dS_u^c + \sum_{u=0}^{t}(X_u - X_{u-}) \tag{14}$$

where

$$dS_t^c = (r - \lambda m)S_t dt + \sqrt{V_1}\, S_t\, dW_1 + \sqrt{V_2}\, S_t\, dW_2 \tag{15}$$

$$dS_t^c\, dS_t^c = S_t^2 (V_1 + V_2) dt \tag{16}$$

and S_t^c represents the continuous component of SDE (13). Also, X_{u-} is the value of X_t immediately before a jump at time u . Since the jump size is e^J and $X_t = \ln S_t$ if a jump occurs at time u we get $X_u = J\,X_{u-}$. Therefore, we have

$$X_u - X_{u-} = (J - 1)X_{u-} \tag{17}$$

If there is no jump at time u then $X_u - X_{u-} = 0$. In other cases, we will have

$$\sum_{u=0}^{t}(X_u - X_{u-}) = \sum_{u=0}^{t}(J - 1)X_{u-}\Delta N_u = \int_0^t (J - 1)X_{u-}dN_u \tag{18}$$

Note that the size of the jump of the X_t is independent. Plugging (18) into (14) and differentiating of X_t gives us

$$dX_t = S_t^{-1} dS_t^c - \frac{1}{2} S_t^{-2} dS_t^c\, dS_t^c + (J - 1)X_{t-}dN_t \tag{19}$$

If we substitute (15) and (16) into (19) we obtain the following asset price under double Heston's stochastic volatilities with jumps (DHJ) model

$$
\begin{aligned}
&dX_t = [r - \lambda m - \frac{1}{2}(V_1 + V_2)]dt + \sqrt{V_1}\, dW_1 + \sqrt{V_2}\, dW_2 + (J-1)X_{t-} dN_t \\
&dV_1 = \alpha_1(\theta_1 - V_1)dt + \sigma_1 \sqrt{V_1}\, dB_1 \\
&dV_2 = \alpha_2(\theta_2 - V_2)dt + \sigma_2 \sqrt{V_2}\, dB_2 \\
&dW_1 dB_1 = \rho_1 dt \\
&dW_2\, dB_2 = \rho_2 dt \\
&dW_1 dW_2 = 0 \\
&dB_1 dB_2 = 0
\end{aligned} \tag{20}
$$

4 European option pricing under DH and DHJ models

In this section, we employ the FFT method for pricing European call options under DH and DHJ models. Suppose $C(T,K)$ is the price of a European call option with maturity T and the strike price K [7]

$$C(T,K) = e^{-r(T-t)} E[(S_T - K)^+ \mid F_t] \tag{21}$$

Let $t = 0, X_T = \ln S_T, k = \ln K$. Carr and Madan [9] define a modified call price function as following to

$$c(T,k) = e^{\beta k} C(T,K)\ , \beta > 0 \tag{22}$$

β is The modified call price coefficient .The choice of β may depend on the model for S_t. The Fourier transform and inverse Fourier transform of $c(T,k)$ is as follows:

$$F_{c_T}(\varphi) = \int_{-\infty}^{+\infty} e^{i\varphi k} c(T,k)dk \tag{23}$$

$$c(T,k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\varphi k} F_{c_T}(\varphi)d\varphi \tag{24}$$

$$
\begin{aligned}
F_{c_T}(\varphi) &= \int_{-\infty}^{+\infty} e^{i\varphi k} e^{\beta k} e^{-rT} \int_{k}^{+\infty} (e^{x_T} - e^k)\, g_T(x_T)dx_T dk \\
&= \int_{-\infty}^{+\infty} e^{-rT} g_T(x_T) \int_{-\infty}^{x_T} (e^{x_T + \beta k} - e^{(\beta+1)k}) e^{i\varphi k} dk\, dx_T \\
&= \frac{e^{-rT}}{(1+\beta+i\varphi)(\beta+i\varphi)} \int_{-\infty}^{+\infty} e^{(\varphi-(\beta+1)i)i\, x_T}\, g_T(x_T)dx_T \\
&= \frac{e^{-rT}\Psi_T\left(\varphi-(\beta+1)i\right)}{(1+\beta+i\varphi)(\beta+i\varphi)}
\end{aligned} \tag{25}
$$

Where $g_T(x_T)$ is the density function of the process X_T and $\Psi_T(u)$ is the characteristic function of X_T under the risk neutral probability.It is shown that the European Call option using FFT method is as follows [7]

$$C(T,k) = \frac{e^{-\beta k_u}}{\pi}\sum_{j=1}^{N} \exp\{ib\upsilon_j - i\frac{2\pi}{N}(j-1)(u-1)\} F_{C_T}(\upsilon_j)\frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}) \tag{26}$$

where N is generally a power of two and η depends on N Also, the other parameters are as follow

$$\upsilon_j = \eta(j-1)\ , b = \frac{\pi}{\eta}\ , k_u = -b + \frac{2b}{N}(u-1),\ u = 1,2,...,N+1.$$

We can see that an obvious idea in FFT method is to use characteristic function. Using this idea we obtain the characteristic function of DHJ model. The characteristic function is given by[6]

$$F(T-t,x,\varphi,v_1,v_2)=\Psi(\varphi,T-t,x)=E(e^{i\varphi X_t}\,|X_t=x,V_1=v_1,V_2=v_2)$$

By applying to the multivariate Ito-Doeblin formula for $F(T-t,x,v_1,v_2)$. we have :

$$dF(T-t,x,\varphi,v_1,v_2)=f_t dt+f_x dx+f_{v_1}dv_1+f_{v_2}dv_2+\frac{1}{2}f_{xx}dx\,dx+\frac{1}{2}f_{v_1v_1}dv_1\,dv_1$$
$$+\frac{1}{2}f_{v_2v_2}dv_2\,dv_2+f_{xv_1}dx\,dv_1+f_{xv_2}dx\,dv_2 \tag{27}$$

where $f_t=\frac{\partial F}{\partial t}, f_\omega=\frac{\partial F}{\partial \omega}, f_{\omega\omega}=\frac{\partial F}{\partial \omega^2}$ for $\omega=x,v_1,v_2$ and $f_{xv_1}=\frac{\partial F}{\partial x\,\partial v_1}, f_{xv_2}=\frac{\partial F}{\partial x\,\partial v_2}$.

$$dF=f_t\,dt+\{[r-\frac{1}{2}(V_1+V_2)]\,dt+\sqrt{V_1}\,dW_1+\sqrt{V_2}\,dW_2\}\,f_x$$
$$+\{\alpha_1(\theta_1-V_1)dt+\sigma_1\sqrt{V_1}dB_1\}f_{v_1}+\{\alpha_2(\theta_2-V_2)dt+\sigma_2\sqrt{V_2}dB_2\}f_{v_2}+\frac{1}{2}(V_1+V_2)f_{xx}dt$$
$$+\frac{1}{2}\sigma_1^2V_1f_{v_1v_1}dt+\frac{1}{2}\sigma_2^2V_2f_{v_2v_2}dt+\sigma_1\rho_1V_1f_{xv_1}dt+\sigma_2\rho_2V_2f_{xv_2}dt \tag{28}$$

Duffie et al. [4] proposed a generalized Feynman-kac theorem for affine jump diffusion processes. Based on the theory of generalized Feynman-Kac, we obtain $F(T-t,x,\varphi,v_1,v_2)$ by solving the following partial integral-differential equation

$$f_t+[r-\lambda m-\frac{1}{2}(V_1+V_2)\,]f_x+\alpha_1(\theta_1-V_1)\,f_{v_1}+\alpha_2(\theta_2-V_2)\,f_{v_2}+\frac{1}{2}(V_1+V_2)f_{xx}$$
$$+\frac{1}{2}\sigma_1^2V_1\,f_{v_1v_1}+\frac{1}{2}\sigma_2^2V_2\,f_{v_2v_2}+\sigma_1\,\rho_1V_1f_{xv_1}+\sigma_2\,\rho_2V_2\,f_{xv_2}+$$
$$\lambda\int_{-\infty}^{+\infty}[F(T-t,x+J,\varphi,v_1,v_2)-F(T-t,x,\varphi,v_1,v_2)]q(J)dJ=0 \tag{29}$$

where $q(J)$ is the distribution function of random variable J .We can write the integral in the above equation as follows (for easiness ,we suppress the conditional portion of the expectations)

$$\lambda\int_{-\infty}^{+\infty}[F(T-t,x+J,k,v_1,v_2)-F(t,x,\varphi,v_1,v_2)]\,q(J)\,dJ=$$
$$\lambda\int_{-\infty}^{+\infty}[E(e^{i\varphi(X_t+J)})-E(e^{i\varphi X_t})]\,q(J)dJ=\lambda\int_{-\infty}^{+\infty}E[e^{i\varphi X_t}(e^{i\varphi J}-1)]q(J)dJ=$$
$$\lambda\int_{-\infty}^{+\infty}E(e^{i\varphi X_t})E(e^{i\varphi J}-1)q(J)dJ=\lambda E(e^{i\varphi X_t})E[E(e^{i\varphi J}-1)]=$$
$$\Lambda(\varphi)F(t,x,\varphi,v_1,v_2) \tag{30}$$

where $\Lambda(\varphi)=\lambda(e^{i\mu\varphi-\frac{1}{2}\varphi^2\gamma^2}-1)$.

Now, we define

$$F(T-t,x,\varphi,v_1,v_2)=\exp\{G(T-t,\varphi)+H_1(T-t,\varphi)v_1+H_2(T-t,\varphi)v_2+i\varphi x\} \tag{31}$$

With the boundary condition of (29)

$$G(0,\varphi)=0\ ,\ H_1(0,\varphi)=0\ ,\ H_2(0,\varphi)=0$$

By differential of (31) and substituting in(29) , we have

$$\begin{aligned}&[\frac{\partial G(T-t,\varphi)}{\partial t}+i(r-\lambda m)\varphi+\alpha_1\theta_1 H_1(T-t,\varphi)+\alpha_2\theta_2 H_2(T-t,\varphi)+\Lambda(\varphi)]+\\&[\frac{\partial H_1(T-t,\varphi)}{\partial t}+\frac{1}{2}\sigma_1^2 H_1^2(T-t,\varphi)+(i\varphi\sigma_1\rho_1-\alpha_1)H_1(T-t,\varphi)-\frac{\varphi}{2}(i+\varphi)]v_1+\\&[\frac{\partial H_2(T-t,\varphi)}{\partial t}+\frac{1}{2}\sigma_2^2 H_2^2(T-t,\varphi)+(i\varphi\sigma_2\rho_2-\alpha_2)H_2(T-t,\varphi)-\frac{\varphi}{2}(i+\varphi)]v_2=0\end{aligned} \tag{32}$$

Thus, we obtain the following equations system

$$\begin{aligned}&\frac{\partial H_1(T-t,\varphi)}{\partial t}+\frac{1}{2}\sigma_1^2 H_1^2(T-t,\varphi)+(i\sigma_1\rho_1\varphi-\alpha_1)H_1(T-t,\varphi)-\frac{\varphi}{2}(i+\varphi)=0\\&\frac{\partial H_2(T-t,\varphi)}{\partial t}+\frac{1}{2}\sigma_2^2 H_2^2(T-t,\varphi)+(i\sigma_2\rho_2\varphi-\alpha_2)H_2(T-t,\varphi)-\frac{\varphi}{2}(i+\varphi)=0\\&\frac{\partial G(T-t,\varphi)}{\partial t}+i(r-\lambda m)\varphi+\alpha_1\theta_1 H_1(T-t,\varphi)+\alpha_2\theta_2 H_2(T-t,\varphi)+\Lambda(\varphi)=0\end{aligned} \tag{33}$$

By solving the first two equations of one-dimensional Ricatti, and the third one only requires a simple integration. we have

$$\begin{aligned}G(\tau,\varphi)=i(r-\lambda)\varphi\tau+\frac{\alpha_1\theta_1}{\sigma_1^2}[(\alpha_1-\rho_1\sigma_1\varphi i+d_1)\tau-2\ln(\frac{1-g_1\exp(d_1\tau)}{1-g_1})]+\frac{\alpha_2\theta_2}{\sigma_2^2}[(\alpha_2-\rho_2\sigma_2\varphi i+d_2)\tau\\-2\ln(\frac{1-g_2\exp(d_2\tau)}{1-g_2})]+\Lambda(\varphi)\tau\end{aligned}$$

with

$$H_j(\tau,\varphi)=\frac{\alpha_j-\rho_j\sigma_j\varphi i+d_j}{\sigma_j^2}\left[\frac{1-\exp(d_j\tau)}{1-g_j\exp(d_j\tau)}\right],$$

$$g_j=\frac{\alpha_j-\rho_j\sigma_j\varphi i+d_j}{\alpha_j-\rho_j\sigma_j\varphi i-d_j}\ ,\ d_j=\sqrt{(\rho_j\sigma_j\varphi i-\alpha_j)^2+\sigma_j^2(\varphi i+\varphi^2)},\qquad j=1,2.$$

where $\tau=T-t$.

5 Computational Results

In this section, we present some computational results to solve the evaluation of European call options based on FFT method and for DH and DHJ models. We use the following parameters [6]

$S_0 = 61.9, T = 1, r = 0.03, v_{01} = 0.6, v_{02} = 0.7, \sigma_1 = 0.1, \sigma_2 = 0.2, \alpha_1 = 0.9, \alpha_2 = 1.2, \theta_1 = 0.1, \theta_2 = 0.15, \rho_1 = -0.5,$
$\rho_2 = -0.5.$

Also, we consider the following jumps parameters $\lambda = 0.22, \mu = 0.22, \gamma = 0.25.$

In Table 1, the call option prices for three strikes K by employing FFT method with N=256 are presented. Figures 1 and 2 show implied volatility surface regarding to DHJ and DH models, respectively. These results show that DHJ model is able to nicely reproduce a wide range of the volatility surface implied from option prices in the markets.

TABLE 1: EUROPEAN CALL OPTION PRICES UNDER DH AND DHJ MODELS BY FFT METHOD

STRIKE PRICE	FFT BASED OPTION PRICES UNDER DH MODEL	CPU TIME DH(SEC.)	FFT BASED OPTION PRICES UNDER DHJ MODEL	CPU TIME DHJ(SEC.)
43.33	27.18	0.013	27.36	0.035
61.9	19.14	0.013	19.45	0.035
80.47	13.44	0.013	14.13	0.035

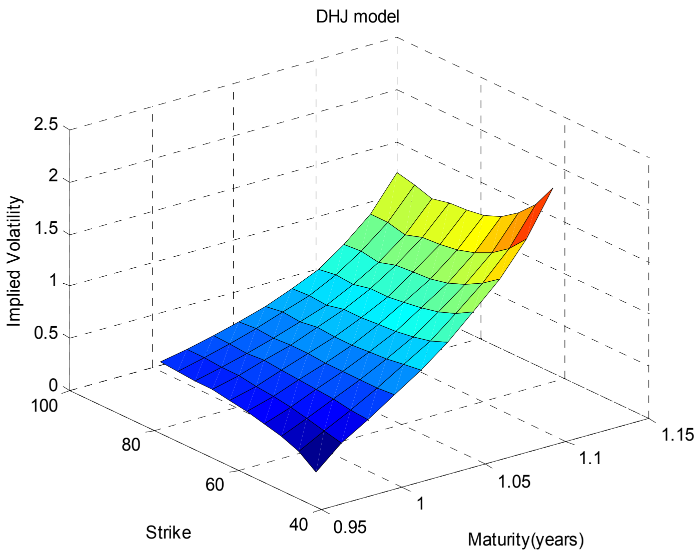

Figure 1: Implied volatility surface of DHJ model:
by using FFT method for strikes: 43.33-80.47 and Maturity:0.9-1.1.

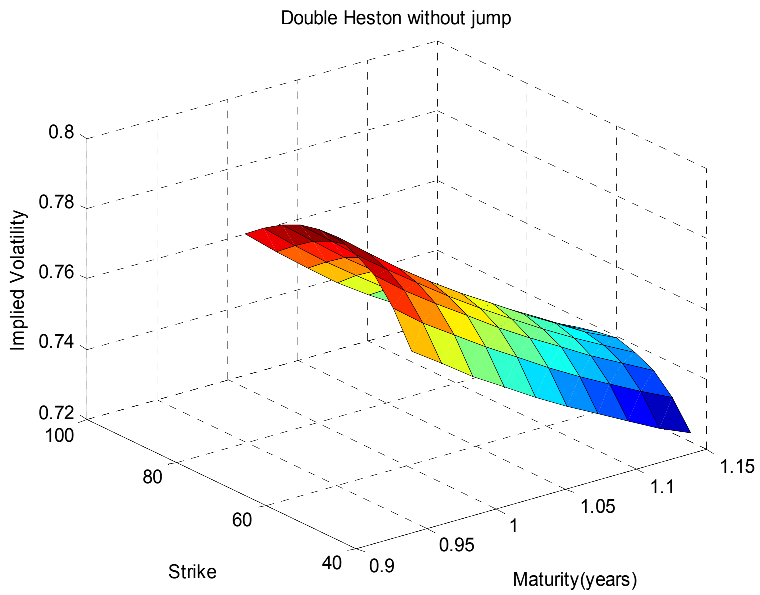

Figure 2: Implied volatility surface of DH model:
by using FFT method for strikes: 43.33-80.47 and Maturity: 0.9-1.1.

Discussion and conclusion

In this paper, by adding jumps in the double Heston's stochastic volatility model, we propose a flexible stochastic volatility model, namely DHJ model. We also employed FFT method with high accuracy and efficiency for pricing European call options under DHJ and DH models.

References

[1] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities, Journal of Political Economy, 81(1973) 637-659.
[2] D. Duffie, J. Pan, K. Singleton, Transform analysis and asset pricing for affine jump-diffusion, Econometrica 68 (2000) 1343–1376..
[3] J. C. Hull, A. White, The Pricing of Options on Assets with Stochastic Volatilities, Journal of Finance, 42 (1987) 281–300.
[4] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bonds and currency options, The Review of Financial Studies 6(2) (1993) 327–343.
[5] B. Kim and I. Wee, Pricing of geometric Asian options under Heston's stochastic volatility model, Quantitative Finance, 2011.
[6] P. Gauthier and D. Possama, Effcient simulation of the double Heston model, E-copy vailable at: http://ssrn.com/abstract=1434853.
[7] E. Pillay and J.G. O'Hara, FFT based option pricing under a mean reverting process with stochastic volatility and jumps, Journal of Computational an Applied Mathematics, 235(2011) 3378-3384.
[8] Christoffersen, Heston, Jacobs. The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work So Well, Creates Research Paper, 2009.
[9] P. Carr, D.B. Madan, Option valuation using the fast Fourier transform, Journal of Computational Finance 2 (4) (1999) 61–73