Modeling asset prices based on two-factor stochastic volatility

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Abstract

In this paper, we present an extension of the double Heston’s stochastic volatility model by adding jumps to financial modeling for stock prices in the double Heston model. We assume that the underlying asset price follows the double Heston’s stochastic volatility model with jumps. We demonstrate the effective use of the Fast Fourier transform approach as an effective tool in the valuation of European options under the proposed model.

Key words: stochastic volatility model; double Heston model; jumps; European option pricing.

1 Introduction

The Black–Scholes model assumes that the probability distribution of the stock price at any given future time is lognormal. If this assumption is not true we shall get biases in the prices produced by the model [1]. On the other hand, in Black-Scholes-Merton models assumed a deterministic volatility. The stochastic volatility models have been proposed to resolve shortcoming. Typical stochastic volatility models commonly include Hull and White (1987), Scott (1987), Stein and Stein (1991) and Heston (1993). There are various extensions, the empirical performances of which have been presented and compared quite extensively by Bakshi et al. [5]. The standard stochastic volatility models fail to capture the smile slope and level movements [2]. In the mathematical finance, there are many papers on option pricing under stochastic volatility model [2-5]. Recently, many researchers have been suggested stochastic volatility models with jumps [6, 7]. In this paper, the double Heston’s stochastic volatility model with jumps is considered rather than the general double Heston’s stochastic volatility model [8]. We evaluate European options prices by applying the Fast Fourier transform (FFT) method.

2 The double Heston stochastic volatility model

Let \( \{ \Omega, F, Q \} \) be a probability space on which are defined several Brownian motion processes \( W_1 = W_1(t), W_2 = W_2(t), B_1 = B_1(t) \) and \( B_2 = B_2(t) \) for \( 0 \leq t \leq T \). Let \( \{ F_t \}_{t \geq 0} \) be the filtration generated by these Brownian motions and \( Q \) is a risk neutral probability under the asset price process.
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\[ S_t = \{S(t)\}_{t \geq 0} \] and two volatility processes \( V_1 = V_1(t) \) and \( V_2 = V_2(t) \). The double Heston’s stochastic volatility (DH) model under the risk neutral probability space is specified as follows [6]

\[
\begin{align*}
    dS_t &= rS_t dt + \sqrt{V_1(t)} S_t dW_1 + \sqrt{V_2(t)} S_t dW_2, \\
    dV_1 &= \alpha_1 (\theta_1 - V_1) dt + \sigma_1 \sqrt{V_1} dB_1, \\
    dV_2 &= \alpha_2 (\theta_2 - V_2) dt + \sigma_2 \sqrt{V_2} dB_2,
\end{align*}
\]

where \( W_1 \) and \( B_1 \) are correlated Brownian motion processes with correlation parameter \( \rho_1 \). Also, \( \rho_2 \) is the correlation coefficient between \( W_2 \) and \( B_2 \). The parameters \( \alpha_1 \) and \( \alpha_2 \) are the mean reversion speed of the volatility. Also, \( \theta_1 \) and \( \theta_2 \) are long-run mean and the constant values \( \sigma_1 \) and \( \sigma_2 \) are the volatility coefficients of the volatility processes. The constant \( r \) is the interest rate of the model. Moreover, \( W_1 \) and \( W_2 \) are independent and also \( B_1 \) and \( B_2 \) are independent processes.

Now, let \( X_t = \ln S_t \), by applying Ito-Doeblin formula [8], we have

\[
\begin{align*}
    dX_t &= S_t^{-1} dS_t - \frac{1}{2} S_t^{-2} dS_t dS_t \\
    &= dS_t - \frac{1}{2} S_t^{-2} dS_t dS_t,
\end{align*}
\]

such that

\[
\begin{align*}
    dS_t &= rS_t dt + \sqrt{V_1} S_t dW_1 + \sqrt{V_2} S_t dW_2 \\
    &= S_t^2 (V_1 + V_2) dt.
\end{align*}
\]

If we substitute (5) and (6) into (4) we will get the asset price with both stochastic volatilities

\[
\begin{align*}
    dX_t &= [r - \frac{1}{2} (V_1 + V_2)] dt + \sqrt{V_1} dW_1 + \sqrt{V_2} dW_2, \\
    dV_1 &= \alpha_1 (\theta_1 - V_1) dt + \sigma_1 \sqrt{V_1} dB_1, \\
    dV_2 &= \alpha_2 (\theta_2 - V_2) dt + \sigma_2 \sqrt{V_2} dB_2, \\
    dW_1 dB_1 &= \rho_1 dt, \\
    dW_2 dB_2 &= \rho_2 dt, \\
    dW_1 dW_2 &= 0, \\
    dB_1 dB_2 &= 0.
\end{align*}
\]

3 The double Heston model with jumps

Suppose that on probability space \((\Omega, F, Q)\) we define a Poisson process \( \{N_t\}_{t \geq 0} \) for all \( 0 \leq t \leq T \) with constant intensity \( \lambda \geq 0 \). Moreover, it is assumed that the Poisson process \( \{N_t\}_{t \geq 0} \) is independent of all Brownian motions. Let us consider a sequence of identically and independently distributed random variables \( e^{J_i} \) for each \( 1 \leq i \leq N_t \), denote the jump sizes, where \( J_i \sim N(\mu, \gamma^2) \). Thus, the random variables \( e^{J_i} \) are log-normally. If we assume that in underlying asset price under the double Heston model occurs a lognormal jump with jump size \( e^{J_i} \) at time \( u \), then we have \( S_u = e^{J_i} S_{u-} \) where that \( S_{u-} \) is the value of the process \( S_u \) immediately before a jump. Therefore, we write

\[
S_u - S_{u-} = (e^{J_i} - 1)S_{u-}.
\]

If there is no jump at time \( u \) then \( S_u - S_{u-} = 0 \). In other cases, we will have
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\[
\sum_{u=0}^{i} (S_u - S_{u-}) = \sum_{u=0}^{i} (e^{i \lambda} - 1) S_u \Delta N_u = \int_{0}^{i} (e^{i \lambda} - 1) S_u \, dN_u,
\]

for \( i = 1, 2, ..., N_i \), where \( \sum_{i=1}^{N_i} (e^{i \lambda} - 1) \) is a compound Poisson process. Therefore, the double Heston model for asset price process with compound Poisson process is as follows

\[
dS_t = r S_t \, dt + \sqrt{V_1 t} S_t \, dW_1 + \sqrt{V_2 S_t} \, dW_2 + (e^{i \lambda} - 1) S_t \, dN_i - E \left( \sum_{i=1}^{N_i} (e^{i \lambda} - 1) \right),
\]

for \( i = 1, 2, ..., N_i \) that \( E \left( \sum_{i=1}^{N_i} (e^{i \lambda} - 1) \right) \) is expected relative jumps size. Also, we can write

\[
E \left( \sum_{i=1}^{N_i} (e^{i \lambda} - 1) \right) = \sum_{i=1}^{N_i} E (e^{i \lambda} - 1) = \lambda m
\]

\[
m = e^{\mu \gamma \lambda - 1}
\]

Now, by substituting (11) into (10) we will have

\[
dS_t = (r - \lambda m) S_t \, dt + \sqrt{V_1 S_t} \, dW_1 + \sqrt{V_2 S_t} \, dW_2 + (e^{i \lambda} - 1) S_t \, dN_i \]

Let us consider \( X_t = \ln S_t \). According to Ito-Doeblin formula on \( X_t \), we get

\[
X_t = X_0 + \int_{0}^{t} S_{i-1}^{c} \, dS_{i}^{c} - \frac{1}{2} \int_{0}^{t} S_{i-2}^{c} \, dS_{i}^{c} + \sum_{u=0}^{i} (X_u - X_{u-})
\]

where

\[
dS_{i}^{c} = (r - \lambda m) S_{i} \, dt + \sqrt{V_{1} S_{i}} \, dW_{1} + \sqrt{V_{2} S_{i}} \, dW_{2}
\]

\[
dS_{i}^{c} \, dS_{i}^{c} = S_{i}^{2} (V_{1} + V_{2}) \, dt
\]

and \( S_{i}^{c} \) represents the continuous component of SDE (13). Also, \( X_{u-} \) is the value of \( X_t \) immediately before a jump at time \( u \). Since the jump size is \( e^{i \lambda} \) and \( X_t = \ln S_t \) if a jump occurs at time \( u \) we get \( X_u = J X_{u-} \). Therefore, we have

\[
X_u - X_{u-} = (J - 1) X_{u-}
\]

If there is no jump at time \( u \) then \( X_u - X_{u-} = 0 \). In other cases, we will have

\[
\sum_{u=0}^{i} (X_u - X_{u-}) = \sum_{u=0}^{i} (J - 1) X_{u-} \Delta N_u = \int_{0}^{i} (J - 1) X_{u-} \, dN_u
\]

Note that the size of the jump of the \( X_t \) is independent. Plugging (18) into (14) and differentiating of \( X_t \) gives us

\[
dx_t = S_{i-1}^{c} \, dS_{i}^{c} - \frac{1}{2} S_{i-2}^{c} \, dS_{i}^{c} \, dS_{i}^{c} + (J - 1) X_{u-} \, dN_u
\]

If we substitute (15) and (16) into (19) we obtain the following asset price under double Heston's stochastic volatilities with jumps (DHJ) model.
\[ dX_t = [r - \lambda m - \frac{1}{2} (V_1 + V_2)] dt + \sqrt{V_1} dW_1 + \sqrt{V_2} dW_2 + (J - 1) X_t \, dN_t \]

\[ dV_1 = \alpha_1 (\theta_1 - V_1) dt + \sigma_1 \sqrt{V_1} dB_1 \]

\[ dV_2 = \alpha_2 (\theta_2 - V_2) dt + \sigma_2 \sqrt{V_2} dB_2 \]

\[ dW_1 dB_1 = \rho_1 dt \]

\[ dW_2 dB_2 = \rho_2 dt \]

\[ dW_1 dB_2 = 0 \]

\[ dV_1 dB_1 = \rho dB dt \]

\[ dV_2 dB_2 = \rho dB dt \]

\[ \lambda \alpha \theta \sigma \rho \]

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4 European option pricing under DH and DHJ models

In this section, we employ the FFT method for pricing European call options under DH and DHJ models. Suppose \( C(T, K) \) is the price of a European call option with maturity \( T \) and the strike price \( K \) [7]

\[ C(T, K) = e^{-r(T-t)} [E[(S_T - K)^+ | F_t]] \]  

(21)

Let \( t = 0, X_T = \ln S_T, k = \ln K \). Carr and Madan [9] define a modified call price function as following to

\[ c(T, k) = e^{-\beta k} C(T, K), \beta > 0 \]  

(22)

\( \beta \) is the modified call price coefficient. The choice of \( \beta \) may depend on the model for \( S_T \). The Fourier transform and inverse Fourier transform of \( c(T, k) \) is as follows:

\[ F_{c_{T}} (\varphi) = \int_{-\infty}^{\infty} e^{i \varphi k} c(T, k) dk \]  

(23)

\[ c(T, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \varphi k} F_{c_{T}} (\varphi) d\varphi \]  

(24)

\[ F_{c_{T}} (\varphi) = \int_{-\infty}^{\infty} e^{i \varphi k} e^{-\beta k} e^{\beta t} \int_{-\infty}^{\infty} (e^{x_T} - e^{-k}) g_T(x_T) dx_T dk \]

\[ = \int_{-\infty}^{\infty} e^{-\beta t} g_T(x_T) \int_{-\infty}^{\infty} (e^{x_T + \beta k} - e^{(\beta + 1)k}) e^{i \varphi k} dk dx_T \]

\[ = e^{-\beta t} \Psi_T (\varphi - (\beta + 1)i) \int_{-\infty}^{\infty} e^{(\varphi - (\beta + 1)i)x_T} g_T(x_T) dx_T \]

(25)

Where \( g_T(x_T) \) is the density function of the process \( X_T \) and \( \Psi_T (\varphi) \) is the characteristic function of \( X_T \) under the risk neutral probability. It is shown that the European Call option using FFT method is as follows [7]

\[ C(T, k) = \frac{e^{-\beta u}}{\pi} \sum_{j=-\frac{N}{2}}^{\frac{N}{2}} \exp \{ibv_j - i \frac{2\pi}{N} (j - 1)(u - 1)\} F_{c_{T}} (v_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j,-}) \]  

(26)

where \( N \) is generally a power of two and \( \eta \) depends on \( N \). Also, the other parameters are as follow

\[ v_j = \eta(j - 1), b = \frac{\pi}{\eta}, k_u = -b + \frac{2b}{N} (u - 1), u = 1, 2, \ldots, N + 1. \]
We can see that an obvious idea in FFT method is to use characteristic function. Using this idea we obtain the characteristic function of DHJ model. The characteristic function is given by[6]

\[ F(T - t, x, \varphi, \nu_1, \nu_2) = \Psi(\varphi, T - t, x) = E(e^{ix\varphi \nu_1} | X_t = x, \nu_1 = \nu_1, \nu_2 = \nu_2) \]

By applying to the multivariate Ito-Doeblin formula for \( F(T - t, x, \varphi, \nu_1, \nu_2) \), we have:

\[
dF(T - t, x, \varphi, \nu_1, \nu_2) = f_x dt + f_v dx + f_{\nu_1} d\nu_1 + f_{\nu_2} d\nu_2 + \frac{1}{2} f_{xx} dx \times dx + \frac{1}{2} f_{\nu_1\nu_1} d\nu_1 d\nu_1 + \frac{1}{2} f_{\nu_1\nu_2} d\nu_1 d\nu_2 \]

\[+ \frac{1}{2} f_{\nu_2\nu_2} d\nu_2 d\nu_2 + f_{\nu_1\nu_2} d\nu_1 d\nu_2 + f_{\nu_2\nu_1} d\nu_2 d\nu_2 \]

(27)

where \( f_x = \frac{\partial F}{\partial t}, f_\omega = \frac{\partial F}{\partial \omega}, f_{\omega\omega} = \frac{\partial F}{\partial \omega^2} \) for \( \omega = x, \nu_1, \nu_2 \) and \( f_{\nu_1\nu_1} = \frac{\partial F}{\partial \nu_1}, f_{\nu_1\nu_2} = \frac{\partial F}{\partial \nu_1 \partial \nu_2} \).

\[
dF = f_x dt + \left\{ r - \left( \frac{1}{2}(V_1 + V_2) \right) \right\} dt + \sqrt{V_1} dW_1 + \sqrt{V_2} dW_2 \]

\[+ \{ \alpha_1(\theta_1 - V_1) dt + \sigma_1 \sqrt{V_1} dB_1(V_1) \} f_{\nu_1} + \{ \alpha_2(\theta_2 - V_2) dt + \sigma_2 \sqrt{V_2} dB_2(V_2) \} f_{\nu_2} + \frac{1}{2}(V_1 + V_2) f_{xx} dt \]

\[+ \frac{1}{2} \sigma_1^2 V_1 f_{\nu_1\nu_1} dt + \frac{1}{2} \sigma_2^2 V_2 f_{\nu_2\nu_2} dt + \sigma_1 \rho_1 V_1 f_{\nu_1\nu_1} dt + \sigma_2 \rho_2 V_2 f_{\nu_2\nu_2} dt \]

(28)

Duffie et al. [4] proposed a generalized Feynman-kac theorem for affine jump diffusion processes. Based on the theory of generalized Feynman-Kac, we obtain \( F(T - t, x, \varphi, \nu_1, \nu_2) \) by solving the following partial integral-differential equation

\[
f_t + \left[ r - \lambda m - \frac{1}{2}(V_1 + V_2) \right] f_x + \alpha_1(\theta_1 - V_1) f_{\nu_1} + \alpha_2(\theta_2 - V_2) f_{\nu_2} + \frac{1}{2}(V_1 + V_2) f_{xx}
\]

\[+ \frac{1}{2} \sigma_1^2 V_1 f_{\nu_1\nu_1} + \frac{1}{2} \sigma_2^2 V_2 f_{\nu_2\nu_2} + \sigma_1 \rho_1 V_1 f_{\nu_1\nu_1} + \sigma_2 \rho_2 V_2 f_{\nu_2\nu_2} + \]

\[\lambda \int_{-\infty}^{t} \left[ F(T - t, x + J, \varphi, \nu_1, \nu_2) - F(T - t, x, \varphi, \nu_1, \nu_2) \right] q(J) dJ = 0 \]

(29)

where \( q(J) \) is the distribution function of random variable \( J \). We can write the integral in the above equation as follows (for easiness, we suppress the conditional portion of the expectations)

\[
\lambda \int_{-\infty}^{t} \left[ F(T - t, x + J, \varphi, \nu_1, \nu_2) - F(t, x, \varphi, \nu_1, \nu_2) \right] q(J) dJ = \lambda \int_{-\infty}^{t} \left[ E(e^{i\varphi X_t + J}) - E(e^{i\varphi X_t}) \right] q(J) dJ = \lambda \int_{-\infty}^{t} E[e^{i\varphi X_t} \left( e^{i\varphi J} - 1 \right)] q(J) dJ = \]

\[
\lambda \{ E(e^{i\varphi X_t}) E(e^{i\varphi J} - 1) q(J) dJ \} = \lambda E(e^{i\varphi X_t}) \left( E(e^{i\varphi J}) - 1 \right) = \Lambda(\varphi) F(t, x, \varphi, \nu_1, \nu_2) \]

(30)
where \( \Lambda(\varphi) = \lambda(\varphi) \). 

Now, we define

\[
F(T-t,x,\varphi,v_1,v_2) = \exp \{G(T-t,\varphi)+H_1(T-t,\varphi)v_1+H_2(T-t,\varphi)v_2+i\varphi x \} 
\] (31)

With the boundary condition of (29)

\[
G(0,\varphi) = 0, \quad H_1(0,\varphi) = 0, \quad H_2(0,\varphi) = 0
\]

By differential of (31) and substituting in(29) , we have

\[
\begin{align*}
\frac{\partial G(T-t,\varphi)}{\partial t} &+ i(r-\lambda m)\varphi + \alpha_1\varphi_1H_1(T-t,\varphi) + \alpha_2\varphi_2H_2(T-t,\varphi) + \Lambda(\varphi) + \\
\frac{\partial H_1(T-t,\varphi)}{\partial t} &+ \frac{1}{2}\sigma_1^2H_1^2(T-t,\varphi) + (i\varphi_1\varphi_1 - \alpha_1)H_1(T-t,\varphi) - \frac{\varphi}{2}(i + \varphi)v_1 + \\
\frac{\partial H_2(T-t,\varphi)}{\partial t} &+ \frac{1}{2}\sigma_2^2H_2^2(T-t,\varphi) + (i\varphi_2\varphi_2 - \alpha_2)H_2(T-t,\varphi) - \frac{\varphi}{2}(i + \varphi)v_2 = 0
\end{align*}
\] (32)

Thus, we obtain the following equations system

\[
\begin{align*}
\frac{\partial H_1(T-t,\varphi)}{\partial t} &+ \frac{1}{2}\sigma_1^2H_1^2(T-t,\varphi) + (i\varphi_1\varphi_1 - \alpha_1)H_1(T-t,\varphi) - \frac{\varphi}{2}(i + \varphi) = 0 \\
\frac{\partial H_2(T-t,\varphi)}{\partial t} &+ \frac{1}{2}\sigma_2^2H_2^2(T-t,\varphi) + (i\varphi_2\varphi_2 - \alpha_2)H_2(T-t,\varphi) - \frac{\varphi}{2}(i + \varphi) = 0 \\
\frac{\partial G(T-t,\varphi)}{\partial t} &+ i(r-\lambda m)\varphi + \alpha_1\varphi_1H_1(T-t,\varphi) + \alpha_2\varphi_2H_2(T-t,\varphi) + \Lambda(\varphi) = 0
\end{align*}
\] (33)

By solving the first two equations of one-dimensional Ricatti, and the third one only requires a simple integration. we have

\[
G(\tau,\varphi) = i(r-\lambda)\varphi \tau + \frac{\alpha_1\varphi_1}{\sigma_1^2}[(\alpha_1 - \rho_1\varphi_1 + d_1)\tau - 2\ln\left(\frac{1-g_1\exp(d_1\tau)}{1-g_1}\right)] + \frac{\alpha_2\varphi_2}{\sigma_2^2}[(\alpha_2 - \rho_2\varphi_2 + d_2)\tau - 2\ln\left(\frac{1-g_2\exp(d_2\tau)}{1-g_2}\right)] + \Lambda(\varphi)\tau
\]

with

\[
H_j(\tau,\varphi) = \frac{\alpha_j - \rho_j\varphi_j + d_j}{\sigma_j^2} \left[ \frac{1-\exp(d_j\tau)}{1-g_j\exp(d_j\tau)} \right],
\]

\[
g_j = \frac{\alpha_j - \rho_j\varphi_j + d_j}{\alpha_j - \rho_j\varphi_j - d_j}, \quad d_j = \sqrt{(\rho_j\varphi_j - \alpha_j)^2 + \sigma_j^2(\varphi_j^2 + \varphi_j^2)}, \quad j = 1,2.
\]

where \( \tau = T-t \).

5 Computational Results

In this section, we present some computational results to solve the evaluation of European call options based on FFT method and for DH and DHJ models. We use the following parameters [6]
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\[
S_0 = 61.9, T = 1, r = 0.03, \nu_0 = 0.6, \nu_1 = 0.7, \sigma_0 = 0.1, \sigma_1 = 0.2, \alpha_1 = 0.9, \alpha_2 = 1.2, \theta_1 = 0.1, \theta_2 = 0.15, \rho_1 = -0.5, \\
\rho_2 = -0.5.
\]

Also, we consider the following jumps parameters \( \lambda = 0.22, \mu = 0.22, \gamma = 0.25. \)

In Table 1, the call option prices for three strikes \( K \) by employing FFT method with \( N=256 \) are presented. Figures 1 and 2 show implied volatility surface regarding to DHJ and DH models, respectively. These results show that DHJ model is able to nicely reproduce a wide range of the volatility surface implied from option prices in the markets.

**TABLE 1: EUROPEAN CALL OPTION PRICES UNDER DH AND DHJ MODELS BY FFT METHOD**

<table>
<thead>
<tr>
<th>STRIKE PRICE</th>
<th>FFT BASED OPTION PRICES UNDER DH MODEL</th>
<th>CPU TIME DH(SEC.)</th>
<th>FFT BASED OPTION PRICES UNDER DHJ MODEL</th>
<th>CPU TIME DHJ(SEC.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>43.33</td>
<td>27.18</td>
<td>0.013</td>
<td>27.36</td>
<td>0.035</td>
</tr>
<tr>
<td>61.9</td>
<td>19.14</td>
<td>0.013</td>
<td>19.45</td>
<td>0.035</td>
</tr>
<tr>
<td>80.47</td>
<td>13.44</td>
<td>0.013</td>
<td>14.13</td>
<td>0.035</td>
</tr>
</tbody>
</table>

**Figure 1:** Implied volatility surface of DHJ model: by using FFT method for strikes: 43.33-80.47 and Maturity:0.9-1.1.
Figure 2: Implied volatility surface of DH model: by using FFT method for strikes: 43.33-80.47 and Maturity: 0.9-1.1.

Discussion and conclusion

In this paper, by adding jumps in the double Heston's stochastic volatility model, we propose a flexible stochastic volatility model, namely DHJ model. We also employed FFT method with high accuracy and efficiency for pricing European call options under DHJ and DH models.

References