AMO - Advanced Modeling and Optimization, Volume 16, Number 3, 2014

A new and efficient method for solving the system of linear inequalities

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September 8, 2014

Abstract

In this paper, we use an approximate subgradient algorithm for solving unconstrained nonsmooth convex and nonconvex optimization problems. Descent directions in this algorithm are computed by solving a system of linear inequalities. In this paper, we present a new and efficient method for solving the system of linear inequalities based on an extension of Newton method. We also compare the proposed method with nonsmooth optimization approach using the results of numerical experiments. These results demonstrate the superiority of the proposed method for solving this system over the nonsmooth optimization method.

Keywords: Descent directions, Solving the system of linear inequalities, Nonsmooth optimization approach, Approximate subgradient method.
AMS Subject Classification: 90C30, 90C31, 15A39

1 Introduction

Nonsmooth, nonconvex unconstrained problems appears in many disciplines such as economic, general nonlinear programming and data mining. Among

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these, we mention the bundle-type methods [1, 2, 3, 4, 5, 6, 7, 8], a gradient sampling algorithm [9], algorithms based on smoothing techniques [10] and the discrete gradient method [11, 12, 13, 14, 15, 16, 17].

In this paper, we use an approximate subgradient algorithm for solving unconstrained nonsmooth convex and nonconvex optimization problems. Descent directions in this algorithm are computed by solving a system of linear inequalities. We present a new and efficient method for solving the system of linear inequalities. We also compare the proposed method with nonsmooth optimization approach in reference [11] using the results of numerical experiments.

The structure of the paper is as follows. Section 2 provides some necessary preliminaries and we describe the approximate subgradient method. In Section 3, we present a new and efficient method for solving the system of linear inequalities. In section 4, it is presented the results of numerical experiments. Concluding remarks are given in Section 5.

We now describe our notation. Let $x = [x_i] \in \mathbb{R}^n$. By x^T, we mean the transpose of vector x and the scalar product of two vectors x and y in the $n-$dimensional real space $\mathbb{R}^n$ will be denoted by $\langle x, y\rangle$. The value $\langle \nabla f(x), d\rangle$ is called directional derivative of f at x in direction d where $\nabla f(x)$ is the gradient of f at x. For $x \in \mathbb{R}^n$, $\|x\|$ and $|x|$ denote $2-$norm and the vector with absolute values of each component of x respectively.

1.1 The Clarke subdifferential

Let f be a locally Lipschitz function defined on $\mathbb{R}^n$. Clarke introduced the notion of subdifferential for such functions (see, e.g., [18]). Since these functions are differentiable almost everywhere, we can define for them a Clarke subdifferential as follows:

$$\partial f(x) = \text{co}\{\upsilon \in \mathbb{R}^n : \exists (x^k \in D(f), x^k \to x, k \to +\infty) : \\ \upsilon = \lim_{k\to+\infty} \nabla f(x^k)\}. \tag{1.1}$$

Here $D(f)$ denotes the set where f is differentiable, co denotes the convex hull of a set.

It is shown in [18] that the mapping $x \to \partial f(x)$ is upper semicontinuous and bounded on bounded sets.

The generalized directional derivative of f at x in the direction g is defined as

$$f^0(x,g) = \limsup_{y \to x, \alpha \to +0} \frac{f(y+\alpha g) - f(y)}{\alpha} \tag{1.2}$$

2 The approximate subgradient method

Consider the following unconstrained minimization problem:

$$\text{Minimize} \;\; f(x) \text{ subject } \;\; \text{to} \;\; x \in \mathbb{R}^n, \tag{2.1}$$

where the objective function f is locally Lipschitz. Recently, there were several methods to solve (2.1). In this section, we consider the approximate subgradient method and its various properties.
The approximate subgradient method is based on the notion of a discrete gradient. We can obtain a set of discrete gradients as follows (see [11]) :

$$x_{k+1} = x_k + \sigma_k g_k. \tag{2.2}$$

where g_k is any approximate subgradient and σ_k obtain by a line-search based on the Armijo rule.

Compute the direction $g \in R^n$ as a solution to the following system:

$$\langle v^i, g \rangle + \delta \leq 0, \;\; i = 1, \dots, k, \;\; g \in S_1. \tag{2.3}$$

Where $S_1 = \{g \in R^n : \;\; \| g \| = 1\}$, v is a discrete gradient of the function f at the point $x \in R^n$, δ is a small positive constant and the system (2.3) solve with nonsmooth optimization approach [11].

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Algorithm 1 *The approximate subgradient method.*

Let $z \in p, \lambda > 0, \alpha \in (0, 1]$ and real numbers $c_1 \in (0, 1)$, $c_2 \in (0, c_1]$, $\delta > 0$ be given and let p be the set of all univariate positive infinitesimal functions (see [11]).

Step 1 Choose any starting point $x^0 \in \mathbb{R}^n$ and set $k = 0$.

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Step 2 set $s = 0$ and $x^{k_s} = x^k$.

Step 3 Compute a descent direction at $x = x^{k_s}, \delta = \delta_k, z = z_k$, $\lambda = \lambda_k$ (see [11]). We get the system (2.3) and solve it.

Step 4 If this system is not solvable put $x^{k+1} = x^{k_s}, k = k+1$ and go to step 2. Otherwise we get the direction $g^{k_s} \in S_1$ which is a solution to this system

$$f(x^{k_s} + \lambda_k g^{k_s}) - f(x^{k_s}) \leq -c_1 \lambda_k \delta_k.$$

Step 5 Construct the following iteration $x^{k_{s+1}} = x^{k_s} + \sigma_s g^{k_s}$, where σ_s is defined as follows

$$\sigma_s = \text{argmax}\{\sigma \geq 0 : f(x^{k_s} + \sigma g^{k_s}) - f(x^{k_s}) \leq -c_2 \sigma \delta_k\}.$$

Step 6 Set $s = s + 1$ and go to Step 3.

More details on the convergence proof for Algorithm 1 can be found in [11]. The main step of the approximate subgradient method is to Compute the direction $g \in \mathbb{R}^n$ as a solution to the following system

$$\langle v^i, g\rangle + \delta \leq 0,\ i = 1, \ldots, k,\ g \in S_1.$$

Where δ is a small positive constant. In this section, we apply an efficient algorithm in order to compute the descent directions.

Proposition 2.1. *If the system* (2.3) *is not solvable, then*

$$\min_{v \in \tilde{D}_k} \| v \| < \delta. \tag{2.4}$$

Proof: refer to [11].

Proposition 2.2. *If* (2.4) *is satisfied then the system* (2.3) *is not solvable*

Proof: refer to [11].

3 Solving the inequality system

Step 3 is an important step in Algorithm 1, where we solve the system (2.3) to find search directions. Different methods have been developed to solve a system of linear inequalities [19, 20]. However, these methods cannot be applied directly to solve the system (2.3) because of the presence of the additional condition $g \in S_1$. The system (2.3) is solved with nonsmooth optimization approach in reference [11].
In this paper, we first reformulate the system (2.3) as the following optimization problem:

$$\text{Minimize} \quad Q_k(g) = (\langle v^i, g\rangle + \delta,)_+ \qquad i = 1, \dots, k \tag{3.1}$$

subject to

$$g \in B_1 = \{g \in \mathbb{R}^n : \|g\|^2 \leq 1\}. \tag{3.2}$$

Then, we use Tikhonov regularizing - which is a well-studied approach for solving the system 2.3. Instead of solving system 2.3, we consider the following problem :

$$\min_{g \in R^n} Q_k(g)^2 + \rho\|g\|^2, \tag{3.3}$$

where ρ is a positive constant value; it is called the regularizing parameter [21]. We will demonstrate the approximate subgradient algorithm when descent directions are computed by solving this system based on Tikhonov regularizing method, is more efficient than approximate subgradient when descent directions are computed by solving this system with nonsmooth optimization approach in reference [11]. In the following section we present the Matlab code for solving the problem (3.3) using generalized Newton method [3, 19].

% The Mathlab Code 1 For Solving The problem (3.3)

$pl = inline('(abs(x) + x)/2')$; $[m, n] = size(A)$; $yy = ones(n, 1)$;
$tol = 1e-10; delta = 1e-4; itmax = 50; i = 0; zz = 0;$
$While(i < itmax \quad \& \quad norm(yy - zz, inf) > tol)$
$df = A' * pl(A * yy - b);$
$d2f = A' * spdiags(sign(pl(A * yy - b)), 0, m, m) * A + (delta) * speye(n);$
$f1 = 0.5 * norm(pl(A * yy - b))^2;$
% Armijo Stepsize
$sigma = 0.5; betaa = 0.5; kk = 0; kkmax = 10;\ dd = -df/d2f;$
$tt = 1; pp = [\quad];\ z = yy + tt * dd; f2 = 0.5 * norm(pl(A * z - b))^2;$
$ss = f2 - f1 - sigma * tt * (df)' * dd;$
$While(ss > 0. \quad \& \quad kk < kkmax)$
$tt = tt * betaa;\ z = yy + tt * dd;\ f2 = 0.5 * norm(pl(A * z - b))^2;\ ss = f2 - f1 - sigma * tt * (df)' * dd;\ kk = kk + 1;$
end
$j = j + 1;\ i = i + 1;\ zz = yy; yy = yy + tt * dd;$
end

4 Numerical Experiments

The efficiency of the approximate subgradient algorithm when descent directions are computed by solving a system with *Matlab code 1* method was verified by applying it to some problems with nonsmooth objective functions.
Problems 2 and 3 from [14] have been used in numerical experiments as P_1 and P_2. We also provide the others problems with nonsmooth and nonconvex objective function by ourselves.

Problem 1

$f(x) = |x_1 - 1| + 100|x_2 - |x_1|.$

Problem 2

$f(x) = |x_1 - 1| + 100|x_2 - |x_1|| + 90|x_4 - |x_3|| + |x_3 - 1| + 10.1(|x_2 - 1| + |x_4 - 1|) + 4.95(|x_2 + x_4 - 2| - |x_2 - x_4|).$

Problem 3

$f(x) = |x_1 - x_2 - x_3 + 1| + |x_1 - |x_3|| + |x_2 - 1|.$

Problem 4

$f(x) = |2x_1 - 2| + 2|x_2 - |x_3|| + |3x_2 - 3x_4|.$

Problem 5

$f(x) = 25|x_1 - x_2| + |2x_2 - x_3 + x_1 - 2| + 50|x_3 - |x_4|| + |2|x_2| - 2x_5|.$

Problem 6

$f(x) = |5|x_1| - 5| + ||x_2| - x_1 + 7x_3 - 7| + 9|x_4 - |x_5|| + 2|3x_3 - x_6 - 3x_5 + 1|.$

Problem 7

$f(x) = t(i)|x_1 - x_2| + (i+3)|1 - |x_3|| + y(i)|x_3 - 2x_4 - x_5 + 2| + i|x_5 - |x_6|| +$
$u(i)|ix_6 - i + x_7 - 1| + y(i)|(i+2)x_8 - (i+2)|,$
$t(i) = 2(i-1)/4,$
$y(i) = i|sin(2t(i)) + 4cos(2t(i))|,$
$u(i) = 2e^{2t(i)}.$

We have different problems with $i = 1, 2, 3, ...$ Here we use $i = 2$ and $i = 3$ for P_7 and P_8.

Problem 8

$f(x) = |ix_1 - i| + ||x_2| - x_1 + (i+2)x_3 - (i+2)| + t(i)|x_5 - |x_6|| +$
$y(i)|ix_6 - i + x_7 - 1| + 17|x_2 - x_4|,$

$t(i) = 2(i-1)/4,$
$y(i) = 3e^{-2t(i)}|3sin(2t(i)) + 11cos(2t(i))|.$

We have different problems with $i = 1, 2, 3, ...$ Here we use $i = 1$ and $i = 2$ for P_9 and P_{10}.

In Table 1, ASM stands for the approximate subgradient method when descent directions are computed by solving a system with nonsmooth optimization approach in reference [11] and SKM stands for the approximate subgradient method when descent directions are computed by solving the system with *Matlab code 1*.

Table 1: Results of numerical experiments

Problem	Method	$\bar{f}$	Time (s)
$P1$	ASM	$4.000 * 10^{-1}$	0.04
	SKM	$4.792 * 10^{-5}$	2.21
P_2	ASM	$9.000 * 10^{-1}$	0.22
	SKM	$4.410 * 10^{-2}$	4.10
P_3	ASM	$4.786 * 10^{-3}$	1.17
	SKM	$5.762 * 10^{-2}$	14.96
P_4	ASM	$6.334 * 10^{-2}$	2.10
	SKM	$3.394 * 10^{-5}$	6.86
P_5	ASM	$6.147 * 10^{-1}$	0.44
	SKM	$3.954 * 10^{-2}$	3.39
P_6	ASM	$2.991 * 10^{-1}$	0.81
	SKM	$1.388 * 10^{-2}$	3.05
P_7	ASM	$3.235 * 10^{0}$	0.41
	SKM	$8.685 * 10^{-2}$	10.58
P_8	ASM	$1.169 * 10^{0}$	5.51
	SKM	$2.574 * 10^{-1}$	5.31
P_9	ASM	$4.682 * 10^{0}$	1.89
	SKM	$1.780 * 10^{-2}$	5.82
P_{10}	ASM	$3.152 * 10^{0}$	1.23
	SKM	$7.671 * 10^{-3}$	7.51

The value $\bar{f}$ is the values of the objective function found by algorithms and the exact value of the objective function is equal to zero in all problems.

5 Conclusion

In this paper, we used an approximate subgradient algorithm for solving unconstrained nonsmooth convex and nonconvex optimization problems. Descent directions in this algorithm are computed by solving a system of linear inequalities. In this paper, we present a new and efficient method for solving the system of linear inequalities.

We compare the proposed method for solving the system of linear inequalities with nonsmooth optimization approach in reference [11] using the results of numerical experiments. Comparing these results, one can see that the approximate subgradient method when descent directions are computed by solving this system with the proposed method is more efficient than approximate subgradient method when descent directions are computed by solving this system with nonsmooth optimization approach in reference [11].

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