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On Solving Fuzzy Knapsack Problem by Multistage Decision Making using Dynamic Programming

Debjani Chakraborty and V. P. Singh
Department of Mathematics, Indian Institute of Technology Kharagpur, India
e-mail: debjani@maths.iitkgp.ernet.in, vishnupratapsingh56@gmail.com

Abstract

This paper considers knapsack problem with fuzzy weights, says fuzzy knapsack problem (FKP). Here we introduce possibility index which gives the possibility of choosing the items with fuzzy weights for knapsack with crisp capacity. In this paper a dynamic programming technique has been introduced to optimize the utility value of objective function. An algorithm has been proposed which gives the optimal solution for FKP with some possibility. The possibility index gives an idea to choose the solution according to decision maker's choice. An illustrative example is given to demonstrate the methodology.

Key words: Knapsack problem, Triangular fuzzy number, Dynamic programming, Possibility index.

1 Introduction

Knapsack problem is one of the most relevant mathematical programming problem with numerous applications in different areas. The knapsack problem [Martello and Toth (1990)] is a problem where a tramper is searching for a combination of different items for filling the knapsack. The objective is to optimize the total utility value of all chosen items by the tramper subject to the capacity of knapsack. The knapsack may correspond to a ship, truck or a resource. There are varieties of applications available for fuzzy knapsack problem such as various packing problem, cargo loading, cutting stock or economic planning. For example the problem of making investment decisions in which the size of an investment is based on the amount of money required, the knapsack capacity is the amount of available money to invest, the investment profit is the expected return. Knapsack problem has a simple structure which permits to study combinatorial optimization problems.
In the real world the utility value used for knapsack problem is imprecise in nature because of the presence of inherent subjectivity. Some researcher used fuzzy theory to solve this type of problem. Zadeh [Zadeh (1965)] in 1965 proposed fuzzy set theory, using this theory Okada and Gen [Okada and Gen (1994)] described multiple choice knapsack problem with fuzzy coefficients. Kasperski and Kulej [Kasperski and Kulej (2007)] solved the 0-1 knapsack problem with fuzzy data. Lin and Yao [Lin and Yao (2001)] described FKP by taking each weight $w_i, i = 1, 2...n$ as imprecise value. They consider $\tilde{w_i} = (w_i - \Delta_{i1}, w_i, w + \Delta_{i2})$ as fuzzy number such that the decision maker should determine an acceptable range of values for each $\tilde{w_i}$, which is the interval $[w_i - \Delta_{i1}, w_i + \Delta_{i2}], 0 \leq \Delta_{i1} < w_i$ and $0 \leq \Delta_{i2}$. Then the decision maker chooses a value from the interval $[w_i - \Delta_{i1}, w_i + \Delta_{i2}]$ as an estimate of each weight. Estimate is exactly w_i if the acceptable grade is 1, otherwise, the acceptable grade will get smaller when the estimate approaches either $w_i - \Delta_{i1}$ or $w_i + \Delta_{i2}$. To calculate an estimate of the fuzzy weight defuzzification of the fuzzy number $\tilde{w_i}$ from the interval $[w_i - \Delta_{i1}, w_i + \Delta_{i2}]$ has been used. The main idea behind this paper is to solve fuzzy knapsack problem in multi-stage decision making. In this paper, we choose the weight as triangular fuzzy number and solved it without defuzzification. Defuzzification of fuzzy number gives a real value corresponding to that fuzzy number with some loss of information. Defuzzification of fuzzy number, converts the fuzzy knapsack problem into crisp knapsack problem. Since the weights are fuzzy in nature we can fill the weights with some possibility, having any value between

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[0, 1]. Sengupta and Pal [Sengupta and Pal (2000)] introduced acceptability index to order two intervals in terms of value. Similarly We introduced a possibility index for calculating the possibility [Dubois and Prade (1988)] of putting fuzzy weight within a knapsack. Proposed possibility index provides the measure whether the knapsack can hold fuzzy weight. There are three types of decision makers [North (1968)] who want to get the solution. If the possibility index is 1 we can fill that weight completely in the knapsack and if it is zero we can not fill that weight. If the possibility index lies between [0,1] we can fill the weight with this much possibility. Possibility index may be near 1 and also may be closer to zero. It depends on the decision maker how he chooses the weight.There are Pessimistic decision maker, Optimistic decision maker and Moderate decision maker. An optimistic decision maker can take the worst case for optimizing the solution i.e. he tolerates the less possibility index for expected higher utility value and on the other hand pessimistic decision maker always chooses the highest possibility index even for total low utility value. A moderate decision maker can choose the middle value of the possibility. The two notable features of our approach are as follows:

- A new possibility index for calculating the possibility of putting fuzzy weight into a knapsack of crisp capacity has been developed. Possibility index gives an opportunities to the decision maker's to select the fuzzy weight according to their choice.

- A recursion based dynamic programming algorithm has been introduced to solve fuzzy knapsack problem which gives the optimal solution with some possibility index. The selection of possibility index may vary as the choice of decision maker's.

Due to possibility index, it is possible to solve fuzzy knapsack problem without doing defuzzification of its weight. possibility index also gives an opportunities to decision maker's for selecting the items according to their choice of possibility.

The remainder of the paper is organized as follows. Section 2 explains the preliminaries i.e definitions of fuzzy set, LR-type fuzzy number, Triangular fuzzy number. The concept of possibility index has been introduced in Section 3. Fuzzy knapsack problem in multi-stage decision making and an algorithm is given in section 4. Numerical example is given in section 5. Section 6 concludes the work.

2 Preliminaries: concepts and definitions

Definition 2.1. *A fuzzy number $\widetilde{M}$ is of $LR-type$ if there exist reference functions L(for left), R(for right), and scalars $\alpha > 0, \beta > 0$ with*

$$\mu_{\widetilde{M}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & \text{for } x \leq m \\ R(\frac{x-m}{\beta}), & \text{for } x \geq m \end{cases} \tag{1}$$

m, called the mean value of $\widetilde{M}$, is a real number and α and β are called the left and right spreads, respectively. Symbolically $\widetilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$.

For reference function L, different functions can be chosen. Dubois and Prade in 1988 mentioned, for instance, $L(x) = \max(0, 1-x)^p$, $L(x) = \max(0, 1-x^p)$, with $p > 0$, $L(x) = e^{-x}$ or $L(x) = e^{-x^2}$. If m is not a real number but an interval $[\underline{m}, \overline{m}]$ then the fuzzy set $\widetilde{M}$ is not a fuzzy number but a fuzzy interval. For LR fuzzy number the computations necessary for the arithmetic operations are considerably simplified: Dubois and Prade[1979] showed that the exact formulas can be given for $\oplus$ and $\ominus$. Let $\widetilde{A} = (a, \alpha, \beta)_{LR}, \widetilde{B} = (b, \gamma, \delta)_{LR}$ be two fuzzy number of LR-type. Then,

1. $(a, \alpha, \beta)_{LR} \oplus (b, \gamma, \delta)_{LR} = (a+b, \alpha+\gamma, \beta+\delta)_{LR}$

2. $-(a, \alpha, \beta)_{LR} = (-a, \beta, \alpha)_{LR}$

3. $(a, \alpha, \beta)_{LR} \ominus (b, \gamma, \delta)_{LR} = (a-b, \alpha+\delta, \beta+\gamma)_{LR}$

2.1 Triangular Fuzzy Number

Definition 2.2 (Kaufmann et al. (1985)). *It is a fuzzy number represented with three points as follows :*

$$\widetilde{A} = (a_1, a_2, a_3)$$

this representation is interpreted as membership function:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & x > a_3 \end{cases} \tag{2}$$

2.2 Operations of triangular fuzzy number

Some important properties of operations on triangular fuzzy number are summarized Kaufmann et al. (1985)

1. The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.
2. The results from multiplication or division are not triangular fuzzy numbers.
3. Max or min operation does not give triangular fuzzy number.

2.2.1 Arithmetic Operations

First, consider addition and subtraction. Here we need not use membership function. Suppose triangular fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are defined as,

$$\widetilde{A} = (a_1, a_2, a_3), \widetilde{B} = (b_1, b_2, b_3)$$

1. Addition:
$$\begin{aligned}\widetilde{A}(+)\widetilde{B} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3)\end{aligned}$$
2. Subtraction:
$$\begin{aligned}\widetilde{A}(-)\widetilde{B} &= (a_1, a_2, a_3) - (b_1, b_2, b_3) \\ &= (a_1 - b_3, a_2 - b_2, a_3 - b_1)\end{aligned}$$
3. Symmetric image:
$$-(\widetilde{A}) = (-a_3, -a_2, -a_1)$$

3 The Possibility Index

Let us consider two triangular type fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$. Now if we take a knapsack of imprecise capacity $\widetilde{B}$ and we want to fill the weight $\widetilde{A}$ in to the knapsack of capacity $\widetilde{B}$ then we have three possibilities for filling the weight in the knapsack which are classified as follows.

1. $\widetilde{A}$ can be completely filled in to the knapsack of capacity $\widetilde{B}$ i.e. possibility is one.
2. $\widetilde{A}$ cannot be filled in to the knapsack of capacity $\widetilde{B}$ i.e. possibility is zero.
3. $\widetilde{A}$ can be filled with some possibility in to the knapsack of capacity $\widetilde{B}$ i.e. possibility lies between zero and one.

If $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ are two fuzzy numbers then the possibility index for filling fuzzy weight in given capacity is denoted by $PI(\widetilde{A} \blacktriangle \widetilde{B})$ i.e. the possibility of filling $\widetilde{A}$ in $\widetilde{B}$ and given by

$$PI(\widetilde{A}\blacktriangle\widetilde{B}) = \begin{cases} 1 - y_1\frac{(a_3-b_3)}{(a_3-a_1)}, & \text{if } b_3 < a_3 \text{ and } a_2 \leq b_2 \\ y_2\frac{(b_3-a_1)}{(a_3-a_1)}, & \text{if } b_3 < a_3 \text{ and } a_2 > b_2 \\ 1, & \text{if } b_3 \geq a_3 \\ 0, & \text{if } b_3 \leq a_1 \end{cases} \tag{3}$$

Where $y_1 = \{\mu_{\widetilde{D}}(x)|\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)\ for\ x \geq b_2\}$, $y_2 = \{\max \mu_{\widetilde{D}}(x)|\mu_{\widetilde{D}}(x) = \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}$ and $\mu_{\widetilde{D}}(x)$ represents the membership value of fuzzy set $\widetilde{D} = \widetilde{A} \cap \widetilde{B}$. Figure 1 and 2 shows some sets which define above conditions for calculating the possibility index.

When $b_3 < a_3$ and $a_2 < b_2$, the possibility index can be calculated as:

$$PI(\widetilde{A}\blacktriangle\widetilde{B}) = \frac{\text{Area occupied by } \widetilde{A} \text{ in the knapsack of capacity } \widetilde{B}}{\text{Total area of } \widetilde{A}} = 1 - y_1\frac{(a_3 - b_3)}{(a_3 - a_1)}$$

Where $y_1 = \{\mu_{\widetilde{D}}(x)|\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)\ for\ x \geq b_2\}$ and $\mu_{\widetilde{D}}(x)$ represents the membership value of fuzzy set $\widetilde{D} = \widetilde{A} \cap \widetilde{B}$.

When $b_3 < a_3$ and $a_2 > b_2$ the possibility index can be calculated as:

$$PI(\widetilde{A}\blacktriangle\widetilde{B}) = \frac{\text{Area occupied by } \widetilde{A} \text{ in the knapsack of capacity } \widetilde{B}}{\text{Total area of } \widetilde{A}} = y_2\frac{(b_3 - a_1)}{(a_3 - a_1)}$$

Where $y_2 = \{\max \mu_{\widetilde{D}}(x)|\mu_{\widetilde{D}}(x) = \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}$ and $\mu_{\widetilde{D}}(x)$ represents the membership value of fuzzy set $\widetilde{D} = \widetilde{A} \cap \widetilde{B}$.

When $b_3 \geq a_3$ the possibility index can be calculated as:

$$PI(\widetilde{A}\blacktriangle\widetilde{B}) = \frac{\text{Area occupied by } \widetilde{A} \text{ in the knapsack of capacity } \widetilde{B}}{\text{Total area of } \widetilde{A}} = 1$$

When $a_1 \geq b_3$ the possibility index can be calculated as:

$$PI(\widetilde{A}\blacktriangle\widetilde{B}) = \frac{\text{Area occupied by } \widetilde{A} \text{ in the knapsack of capacity } \widetilde{B}}{\text{Total area of } \widetilde{A}} = 0$$

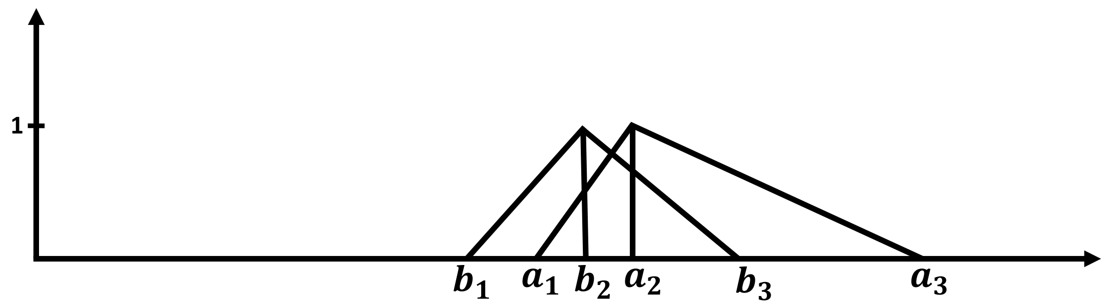

Figure 1: Both are fuzzy numbers.

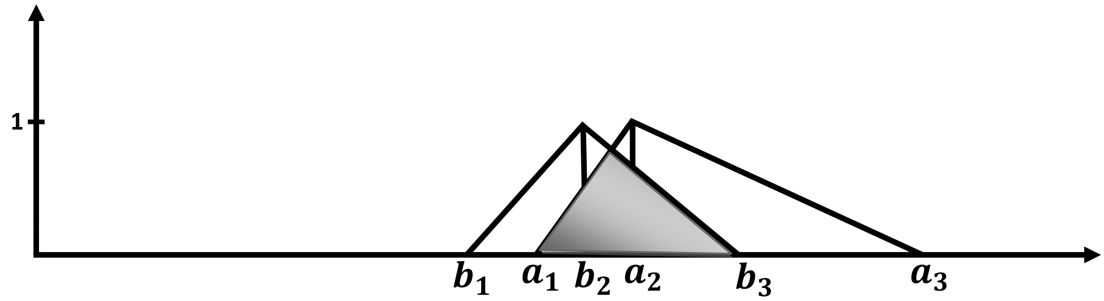

Figure 2: Both are fuzzy numbers.

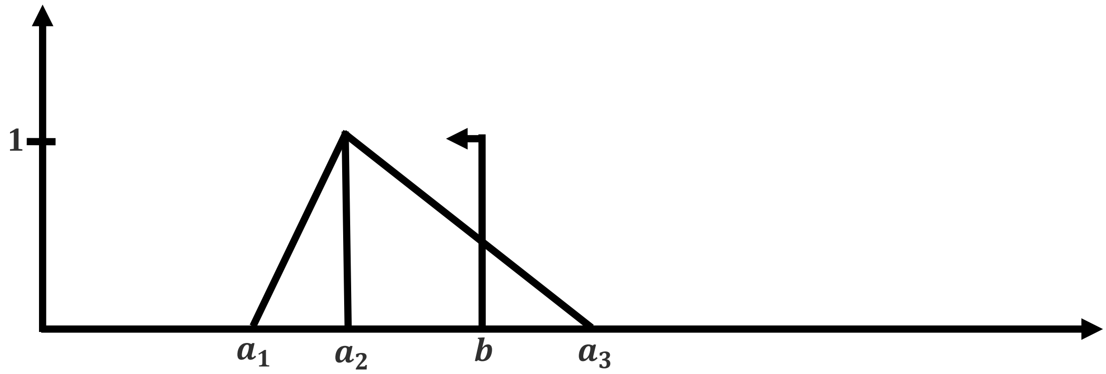

Figure 3: PI of $\widetilde{A} = (a_1, a_2, a_3)$ for different value of b(crisp).

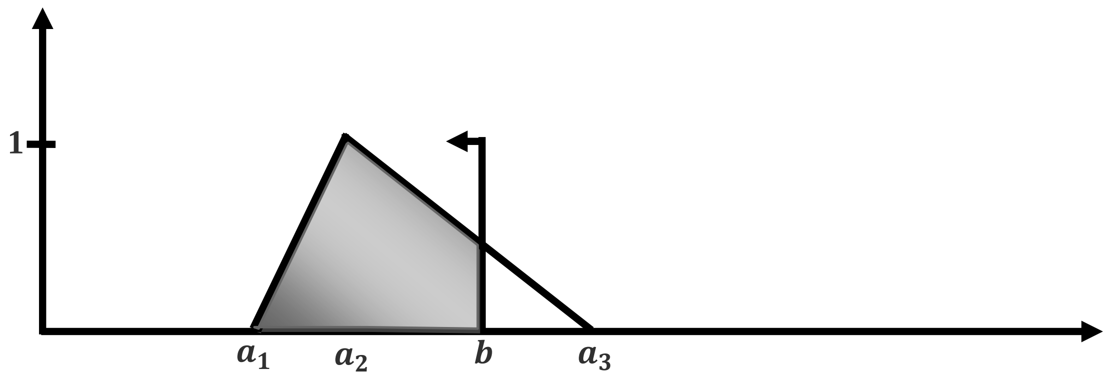

Figure 4: PI of $\widetilde{A} = (a_1, a_2, a_3)$ for different value of b(crisp).

If the knapsack capacity is crisp value say b (shown in figure 3 and 4) then the possibility index can be deduced as:

$$PI(\widetilde{A}\blacktriangle b) = \begin{cases} 1, & \text{if } a_3 \leq b \\ 1 - \mu_{\widetilde{A}}(b)\frac{(a_3-b)}{(a_3-a_1)}, & \text{if } a_2 \leq b < a_3 \\ \mu_{\widetilde{A}}(b)\frac{(b-a_1)}{(a_3-a_1)}, & \text{if } a_1 < b < a_2 \\ 0, & \text{if } b \leq a_1 \end{cases} \tag{4}$$

4 Fuzzy knapsack problem by multi-stage decision process

In the classical Knapsack problem all the weights and item value are assumed to be crisp in nature. Mathematically, it is defined as,

$$\max \sum_{i=1}^{n} u_i x_i$$
$$s.t \quad \sum_{i=1}^{n} w_i x_i \leq W, \quad i = 1, \ldots, n,$$

If the items are n in number then u_i represents the utility value of each item for $i = 1, 2...n$ and w_i represents the crisp weight of each item with knapsack capacity W. In practice, we see many knapsack problems with items whose weights or price value are imprecise. Here we consider the problem in which weights of the items are triangular fuzzy number $\widetilde{w_i} = (w_{1i}, w_{2i}, w_{3i})$, knapsack capacity W and utility value are crisp or also may be considered as fuzzy. Now the fuzzy knapsack problem as a linear programming model is described by

$$\max \sum_{i=1}^{n} u_i x_i$$
$$s.t \quad \sum_{i=1}^{n} \widetilde{w_i} x_i \preceq W, i = 1, \ldots, n,$$

After first stage when we have filled some weight in knapsack the capacity of the knapsack will be fuzzy and the linear programming model is now described by

$$\max \sum_{i=1}^{n} u_i x_i$$
$$s.t \quad \sum_{i=1}^{n} \widetilde{w_i} x_i \preceq \widetilde{W'}, i = 1, \ldots, n,$$

Where $\widetilde{W'}$ is fuzzy weight.

A fuzzy knapsack problem may be viewed as n stage decision process where the fuzzy stage transformation equation unite all the stages. In a dynamic programming structure of fuzzy knapsack problem, the stage transformation equation transforms input state variable and decision variable to an output state which works as an input state variable for its next stage and this process continues up to n^{th} stage. If S^0 is the input state variable for 1^{st} stage and d^1 is the decision variable then stage 1 will consume some part of input and decision variable and it will give an immediate return in the form of utility value (f_1) and possibility index(PI_1) at first stage. At first stage decision variable has a crisp value while from 2^{nd} stage onwards it becomes fuzzy. If we have S^j as input state fuzzy variable at j^{th} stage which is output from the $(j-1)$ stage and $\widetilde{d^j}$ for $(j > 1)$ is decision variable then the immediate return at stage j is given by (f_j, PI_j). Similarly at n^{th} stage we get an optimal return f_n with possibility d_n. This optimal value will be our solution and by moving backward direction with respect to the corresponding decision variable we calculate the non-dominated set of items. Here possibility index plays an important role at each stage. Since the decision makers have values of possibility index at each stage so that they can select the optimal value according to their tolerance limits. The selection of possibility index will change the solution according to DM's choices.

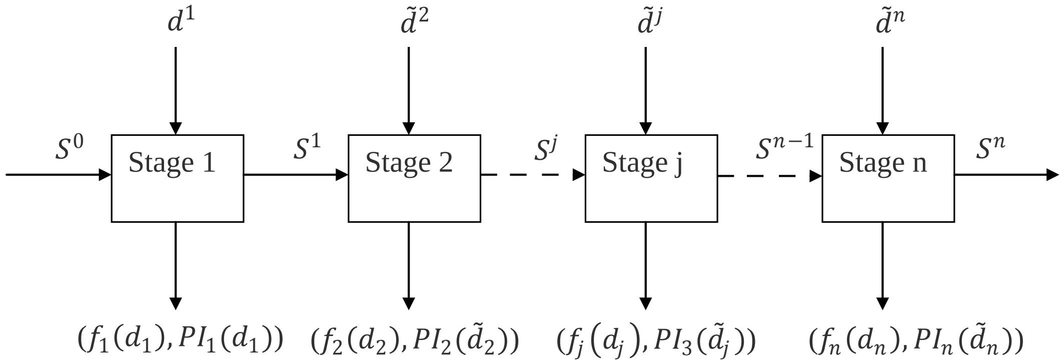

Figure 5: Multistage Decision Process

Theorem 1. *In a fuzzy knapsack problem, a non-dominated set of items selected by DM's choice has a property that it contains non-dominated subsets of items.*

Proof: Let N_1, N_2 and N_3 are the tolerance limits for optimistic, moderate and pessimistic decision makers respectively. If X is the set of all feasible sets of the knapsack problem. Then, for a set $X_k \in X$ where $X_k = \{x_1, x_2 \ldots x_N\}$,let the possibility index for X_k is given by PI_{X_k}, total utility value from stage 1 to N can be defined as

$$f_{1-N}(X_k) = \sum_{i=1}^{N} x_i u_i \tag{5}$$

A set X_k is said to be optimal set if, $f_{1-N}(X_i) \leq f^*_{1-N}(X_k)$ for $i = 1, 2 \ldots p, 1 \leq k \leq p$,
here, $f^*_{1-N}(X_k) = \max_{1 \leq k \leq p}(f_{1-N}(X_i) | \max(PI_{X_1} \wedge PI_{X_2} \wedge \cdots \wedge PI_{X_p}) \in N_t)$ and N_t is the selected tolerance limits for any value of $t = 1, 2, 3$.

A set $X_k = \{x_1, x_2 \ldots x_N\}$ with possibility PI_{X_k} is said to be a non-dominated set if each $x_i \in X_k$ gives an optimal value at stage i. Let us consider $X_k = \{x_1, x_2 \ldots x_N\}$ represents a non-dominated set of items with possibility PI_{X_k} where each x_i is an optimal at stage i. Then a set $X_l = \{y_1, y_2 \ldots y_N\}$ with possibility PI_{X_l} is said to be dominated by set X_k if $f^*_{1-N}(X_l) < f^*_{1-N}(X_k)$ for $1 \leq k, l \leq p$. Let $X_k = X_{k_1}X_{k_2}$, X_{k_1} be a non-dominated set considering all sub optimal from state 1 to S and X_{k_2} represents non-dominated set from state S to N. Our claim is that X_{k_1} is a non-dominated subset. Now let us consider that X_{k_1} is not a non-dominated set and B_i represents all the feasible set from stage 1 to S for $i = 1, 2 \ldots k$. Since X_{k_1} is not a non-dominated set then there exists at least one feasible set say for $(i = m)B_m$ such that

$$f^*_{1-S}(X_{k_1}) < f^*_{1-S}(B_m) \tag{6}$$

From (6) it is clear that

$$\begin{aligned} f^*_{1-S}(X_{k_1}) + f^*_{S-N}(X_{K_2}) &< f^*_{1-S}(B_m) + f^*_{S-N}(X_{k_2}) \\ \implies f^*_{1-N}(X_k) &< f^*_{1-N}(C) \end{aligned}$$

Where $C = B_m X_{K_2}$. This shows the contradiction that X_k is a non-dominated set.

4.1 Methodology

A dynamic programming technique of decision making in fuzzy environment Bellman and Zadeh (1970) is given to solve FKP using the possibility index introduced in the previous section. The solution obtained by this method depends upon the DM who chooses the profit with respect to possibility index in each stage. Since in the dynamic programming we divide an n-stage problem into n single stage problem and then used backward recursive approach to get the solution. Following steps are given to solve fuzzy knapsack problem.

Step1: First we formulate the problem by defining the symbol given below-
x_i: Number of copies of an item i selected for knapsack.
d_i: State variable (Available weight in each stage i).
u_i: Value of an item i selected for knapsack.
$F_i(x_i)$: Value in stage i given x_i number of copies.
$f_i^{N_t}(d_i)$: Maximum possible value chooses for stage i to n according to the decision makers tolerance limit N_t for $t = 1, 2, 3$.
$PI_i(x_i)$: Possibility index in stage i for weights selected in the knapsack.

Step2: We start from n^{th} item and calculate the optimal value and possibility index of weight for this item. In the next stage we take n^{th} and $(n-1)^{th}$ item and again calculate the optimal value and possibility index of weights. Continuing in this manner at i^{th} stage we have i number of item, for calculating the optimal value and possibility index which is selected by the DM in each stage we require optimal value and possibility index from previous stage. So we defined stage transformation equations for profit and the possibility index -

$$F_i(x_i, d_i) = x_i * u_i + f_{i+1}^{N_t}(d_i) \tag{7}$$

$$PI_i(x_i, d_i) = \frac{1}{2}(PI((x_i * \widetilde{w}_i)\blacktriangle d_i) + PI(\widetilde{ow}_i \blacktriangle \widetilde{rw}_i)) \tag{8}$$

$$f_i^{N_t}(d_i) = \max\{F_i(x_i, d_i) | \max\{PI_i(x_i, d_i)\} \in N_t \quad \text{for} \quad t = 1, 2, 3\} \tag{9}$$

Here $d' = \lceil \widetilde{rw}_{i3} \rceil$, $\widetilde{rw}_i = d_i - (x_i * \widetilde{w}_i)$ is the remaining fuzzy weight at stage i and $\widetilde{ow}_i = \sum_{j=i+1}^{n} x_j^* \widetilde{w}_j$ is the optimal weight at stage i due to its all previous stages where x_i^* is the optimal value at stage i. $PI_i(\widetilde{ow}_i \blacktriangle \widetilde{rw}_i)$ represent the possibility index of two fuzzy number which is calculated by equation 3. Initial values are given by the equations.

$$f_{n+1}^{N_t}(d_i) = 0 \tag{10}$$

$$PI_{n+1}(\widetilde{ow}_n \blacktriangle \widetilde{rw}_n) = PI((x_i * \widetilde{w}_i)\blacktriangle d_i) \tag{11}$$

Step3: Once calculate the value of $f_i^{N_t}(d_i)$, $PI_i(x_i)$ at first stage it depends upon the decision maker to choose the optimal value in next stage. Let N_1, N_2, N_3 are the tolerance limit for the optimistic, moderate and pessimistic decision makers respectively.

Step4: Now we have profit values and possibility index for all stages. Moving backward by considering the optimal value (chooses by DM) corresponding to remaining weight from first stage to n^{th} stage will give the solution for that decision maker. Similarly for other decision makers they can select there tolerance limit.

4.2 Algorithm based on dynamic programming for Fuzzy knapsack problem

Require: $\widetilde{w}_i = (w_{1i}, w_{2i}, w_{3i}), W, u_i, n, N_t$selected by DM
Ensure: $f_i(d_i), PI_i^{'}(d_i)$
$\widetilde{w}_i, i = 1, 2, ...n$
for $i := n \to 1$ **do**
  $k := 0$
  while $w_{2i} * k \leq W$ **do**
    $k := k + 1$
  end while
  for $x_i := 0 \to k$ **do**
    for $d_i := 0 \to W$ **do**
      $x^*_{n+1}(d_i) := 0$
      if $(d_i < w_{1i})$ **then**
        $f_n(d_i) := 0$
        $PI_n(d_i) := 0$
      end if
      if $(w_{3i} < d_i < x_i * w_{3i})$**or**$(d_i \geq x_i * w_{3i})$ **then**
        $f_{n+1}^{N_t}(d_i) := 0$ for all decision makers $i.e. t = 1, 2, 3$
        $\widetilde{rw}_i := d_i - (x_i * \widetilde{w}_i)$
        $\widetilde{ow}_i := \sum_{j=i+1}^{n} x_j^* \widetilde{w}_j$
        $PI(\widetilde{ow}_n \blacktriangle \widetilde{rw}_n) := PI((x_i * \widetilde{w}_i) \blacktriangle d_i)$
        $F_i(x_i, d_i) := x_i * u_i + f_{i+1}^{N_t}(d_i)$
        $PI_i(x_i, d_i) := \frac{1}{2}(PI((x_i * \widetilde{w}_i) \blacktriangle d_i) + PI(\widetilde{ow}_i \blacktriangle \widetilde{rw}_i))$
        $f_i^{N_t}(d_i) := \max\{F_i(x_i, d_i) | \max\{PI_i(x_i, d_i)\} \in N_t$ for $t = 1, 2, 3\}$
        $x_i^*(d_i) := x_i$ for which $\max\{F_i(x_i, d_i) | \max\{PI_i(x_i, d_i)\} \in N_t\}$ is selected.
        {Here N_1, N_2, N_3 are the tolerance limits for optimistic, moderate, pessimistic decision maker respectively.}
      end if
    end for
  end for
end for

5 Numerical Example

Consider the fuzzy knapsack model given in Table 1. Our objective is to optimize the utility value subject to knapsack capacity (10 unit) and find number of copies per item.

Table 1: Data for knapsack problem

Weight	$\widetilde{1} = (0.5, 1, 2)$	$\widetilde{2} = (1.5, 2, 3)$	$\widetilde{2} = (1.5, 2, 3)$
Profit	20	50	60
Type	A	B	C

Mathematical model as defined in section 4 can be written as

$$\max 20x_1 + 50x_2 + 60x_3$$
$$s.t \ \widetilde{1}x_1 + \widetilde{2}x_2 + \widetilde{2}x_3 \preceq \widetilde{10}, i = 1, 2, 3.$$

These tables represent the solution of fuzzy knapsack problem given above. We can solve this problem for three decision makers' i.e. pessimistic DM, moderate DM and optimistic decision maker. The given tables show the solution for moderate DM who chooses the maximum profit with the maximum possibility which lies in the interval [0.5, 1].

Table 2:

d	$x_3 = 0$	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$x_3 = 4$	$x_3 = 5$	max	x_3^*
1	-	-	-	-	-	-	-	-
2	-	(60, 0.34)	-	-	-	-	(60, 0.34)	1
3	-	(60, 1)	-	-	-	-	(60, 1)	1
4	-	(60, 1)	(120, 0.34)	-	-	-	(60, 1)	1
5	-	(60, 1)	(120, 0.83)	(180, 0.03)	-	-	(120, 0.83)	2
6	-	(60, 1)	(120, 1)	(180, 0.34)	-	-	(120, 1)	2
7	-	(60, 1)	(120, 1)	(180, 0.7)	(240, 0.08)	-	(180, 0.7)	3
8	-	(60, 1)	(120, 1)	(180, 0.92)	(240, 0.34)	(300, 0.01)	(180, 0.92)	3
9	-	(60, 1)	(120, 1)	(180, 1)	(240, 0.62)	(300, 0.12)	(240, 0.62)	4
10	-	(60, 1)	(120, 1)	(180, 1)	(240, 0.83)	(300, 0.34)	(240, 0.83)	4

In table 3 calculating the profit value when $d = 7, x_2 = 1$, we have available weight 7 unit in which we can fill 1 copy of item B with possibility index 1. Now we have remaining weight $\widetilde{rw}_i = (4, 5, 5.5)$ this implies $d^{'} = 6$. From the previous stage at $d = 6$ we have optimal profit is 120 with $x_3^* = 2$. So the optimal profit becomes 170 with possibility index 0.97.

Table 3:

d	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$	max	x_2^*	x_3^*
1	-	-	-	-	-	-	-		
2	(60, 0.34)	(50, 0.34)	-	-	-	-	(60, 0.34)	0	1
3	(60, 1)	(50, 1)	-	-	-	-	(60, 1)	0	1
4	(60, 1)	(110, 0.84)	(100, 0.34)	-	-	-	(110, 0.84)	1	1
5	(120, 0.83)	(110, 1)	(160, 0.55)	(150, 0.03)	-	-	(160, 0.55)	2	1
6	(120, 1)	(170, 0.75)	(160, 1)	(150, 0.34)	-	-	(170, 0.75)	1	2
7	(180, 0.7)	(170, 0.97)	(160, 1)	(210, 0.66)	(200, 0.08)	-	(210, 0.66)	3	1
8	(180, 0.92)	(230, 0.72)	(220, 0.83)	(210, 0.96)	(260, 0.2)	(250, 0.01)	(230, 0.72)	1	3
9	(240, 0.62)	(230, 0.9)	(220, 1)	(270, 0.65)	(260, 0.81)	(250, 0.12)	(270, 0.65)	3	2
10	(240, 0.83)	(290, 0.78)	(280, 0.77)	(270, 0.96)	(260, 0.91)	(310, 0.11)	(290, 0.78)	1	4

Table 4:

d	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$x_1 = 6$	$x_1 = 7$	$x_1 = 8$	$x_1 = 9$	$x_1 = 10$	max
10	(290,0.78)	(310,0.58)	(310,0.54)	(330,0.50)	(310,0.58)	(330,0.50)	(330,0.47)	(350,0.43)	(330,0.48)	(350,0.26)	(360,0.17)	(330,0.50)

From tables 2,3 and 4 it is clear that a moderate decision maker can choose three copies of type C, one copy of type B and five copies of type A so that the optimal solution is 330 with possibility 0.5, another solution with the same possibility is given by two copies of type C, three copies of type B and three copies of type A.

6 Conclusion

From the crisp knapsack problem when we extend in to fuzzy knapsack problem and solve it without defuzzification the resulting value of optimal profit varies with the selection of possibility index by DM. The possibility index of selecting fuzzy weights in available fuzzy weights gives the opportunity to DM to select the optimal value. The proposed dynamic programming technique provides us an useful way to deal with knapsack problem in fuzzy environment.

References

Bellman, R. E. and Zadeh, L. A. (1970). Decision-making in a fuzzy environment, *Management science.* **17**(4), B–141.

Dubois, D. and Prade, H. (1988). *Possibility theory*, Springer.

Kasperski, A. and Kulej, M. (2007). The 0-1 knapsack problem with fuzzy data, *Fuzzy Optimization and Decision Making.* **6**(2), 163–172.

Kaufmann, A., Gupta, M. M. and Kaufmann, A. (1985). *Introduction to fuzzy arithmetic: theory and applications*, Van Nostrand Reinhold Company New York.

Lin, F.-T. and Yao, J.-S. (2001). Using fuzzy numbers in knapsack problems, *European Journal of Operational Research.* **135**(1), 158–176.

Martello, S. and Toth, P. (1990). *Knapsack Problems*, Wiley New York.

North, D. W. (1968). A tutorial introduction to decision theory, *Systems Science and Cybernetics, IEEE Transactions on.* **4**(3), 200–210.

Okada, S. and Gen, M. (1994). Fuzzy multiple choice knapsack problem, *Fuzzy Sets and Systems.* **67**(1), 71–80.

Sengupta, A. and Pal, T. K. (2000). On comparing interval numbers, *European Journal of Operational Research*. **127**(1), 28–43.

Zadeh, L. A. (1965). Fuzzy sets, *Information and control*. **8**(3), 338–353.