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Pattern formation in the diffusive Fisher equation†

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Abstract

In this paper, numerical simulations of nonlinear Fisher's equation in one- and two-dimensions have been considered. The derivatives and integrals are replaced by the necessary matrices, and the resulting algebraic system of equations was advanced by the popular fourth-order exponential time differencing Runge-Kutta (ETDRK4) schemes proposed by Cox and Matthew [Exponential time differencing for stiff systems, *Journal of Computational Physics* **176** (2002) 430-455], and later in a modified version modified by Kassam and Trefethen [Fourth-order time-stepping for stiff PDEs, *SIAM Journal on Scientific Computing* **26** (2005) 1214-1233]. Numerical results obtained in this paper have further granted an insight to the understanding of pattern formation in both one- and two- dimensional systems. Computations are carried out on a large spatial domain size l to actually give enough room for waves propagation. Some initial data and parameter values were chosen to mimic some of the existing patterns.

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1 Introduction

Mathematically, reaction-diffusion systems take the form of semi-linear parabolic differential equations presented in the form

$$u_t = D\nabla^2 u + F(u) \tag{1.1}$$

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where $u(x,t)$ is the concentration of a substance or vector dependent variables, D is the diagonal matrix of diffusion coefficients, $\nabla^2 = \nabla \cdot \nabla$ represents the Laplacian operator and $F(u)$ is a nonlinear term that accounts for all the local reactions. Reaction-diffusion system of the form (1.1) mathematically explains how the concentration of one or more substances distributed in space changes under the influence of two processes; the local chemical reactions in which the substances are transformed into each other and diffusion (D) which causes the substances to spread out over a surface in space (x). The solution of this class of equation has displayed a wide range of behaviours, such as the formation of travelling waves, self-organised patterns like sports and stripes, or more intricate structure like dissipative solitons.

In the present paper, we are interested in the numerical simulations of the prototype reaction-diffusion equation

$$u_t = D\nabla^2 u + \beta u\left(1 - \frac{u^\alpha}{\kappa}\right), \quad t > 0, \tag{1.2}$$

where D remains as the diffusion coefficient, β is consider to be the growth rate, u is the species density, κ is carrying capacity and ∇^2 is the Laplacian operator in one and two dimensions, equation (1.2) has a long attention span history in the study of propagation phenomena such as flames propagation, migration of biological species or tumor growth, heat and mass transfer, ecology among many others. Reaction-diffusion equation of the form (1.2) was earlier suggested by Fisher [12] in one-dimension as a deterministic model for the spatial spread of a favoured gene in a population and it has since then applied to other fields like population dynamics, combustion theory and chemical kinetics. Applications of the Fisher equation have been extended to various fields of research. For instance, the logistic population growth models [6, 23, 29], flame propagation [19, 39] and the Brownian motion processes [4]. Analytical representations of traveling wave solutions for Fisher equation has been investigated in one dimension [2, 10, 13, 24, 27] and in two space dimensions [5]. The seminal and classical paper is that by Kolmogorov, Petrovskii and Piskunov [22], with extensive discussion and references in the books by Britton [6], Fife [11], Kot [25] and Murray [30, 31, 32, 33] among others.

The rest of this paper is structured as follows. In Section 2, local stability analysis of equation (1.2) is discussed. In Section 3, numerical methods of attack in both space and time is considered. Numerical experiments in one and two dimensions is presented in Section 4. Section 5 concludes the paper.

2 Local stability analysis

We present briefly here some information on the local dynamics of the reaction-diffusion equation (1.2). This idea enhances conditions on the parameters necessary for the solutions to have biologically meaningful equilibria as well as a perfect guide for the choice of parameters in the numerical simulations. We consider in this section the solutions of the popular Fisher equation (1.2) that contains logistic growth and simple Fickian diffusion for both steady states and the travelling wave solutions.

2.1 Steady state solution

In one-dimension with $\alpha = 1$, equation (1.2) reduces to the form

$$\begin{aligned} u_t &= Du_{xx} + \beta u\left[1 - \frac{u}{\kappa}\right], \quad 0 \le x \le L, \\ u(0,t) &= u(L,t) = 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{2.3}$$

where β, κ and D remain the growth rate, the carrying capacity and the diffusivity respectively. We shall investigate the numerical study of dynamic (2.3) with a view in the context of biology, that is the main reason for setting out the domain here to include just only the positive half plane, that is $u > 0$, actually corresponds to biologically meaningful solutions. For simplicity, the three parameters r, κ and D can be eliminated by rescaling the dependent and independent variables, this can be achieved by setting the parameters as $\hat{u} \equiv \frac{u}{\beta}$, $\hat{t} \equiv \tau t$, $\hat{x} \equiv \sqrt{\frac{\beta}{D}}x$, after dropping the hats and without loss of generality especially in the context of biology, we only rescale the density as $u \equiv u/\kappa$ but leave both space and time alone. So, the system take the form

$$\begin{aligned} u_t &= Du_{xx} + \beta u(1-u), \quad 0 \le x \le L, \\ u(0,t) &= u(L,t) = 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{2.4}$$

System (2.4) is steady if it is independent of time, so at steady-state, the solution satisfy $\partial u/\partial t = 0$, and hence

$$\begin{aligned} \tau u(1-u) + Du'' &= 0, \\ u(0) = 0, \quad u(L) &= 0. \end{aligned} \tag{2.5}$$

We sought solutions around equation (2.5) that are equilibrium in time for which $u(x) \geq= 0$. At $u = 0$ is the trivial case that satisfy the system for all values of L. For the nontrivial state, with a new variable $v = du/dx$, we reduce equation (2.5) to system of first order ordinary differential equations

$$u' = v,\ v' = -\frac{\beta}{D}u(1-u), \tag{2.6}$$

with Dirichlet boundary conditions $u(0) = 0$, $u(L) = 0$, $v(0) = 0$, $v(L) = 0$. It is noticeable that this ODEs system has two phase-plane equilibria at say, $Q_1 = (0, 0)$, $Q_2 = (1, 0)$. This system has purely imaginary eigenvalues $\lambda_{a,b} = \pm i\sqrt{\frac{\beta}{D}}$. So, $Q_1 = (0, 0)$ is the center - for the linear system. Again, linearization about $Q_2 = (1, 0)$ yields

$$Df(1, 0) = \begin{pmatrix} 0 & 1 \\ \frac{\beta}{D} & 0 \end{pmatrix}, \tag{2.7}$$

with $\lambda_{a,b} = \pm\frac{\beta}{D}$. Hence, $Q_2 = (1, 0)$ which is called a saddle point. Since (1,0) is a saddle for the linearization, it is also regarded as a saddle for the original nonlinear system.

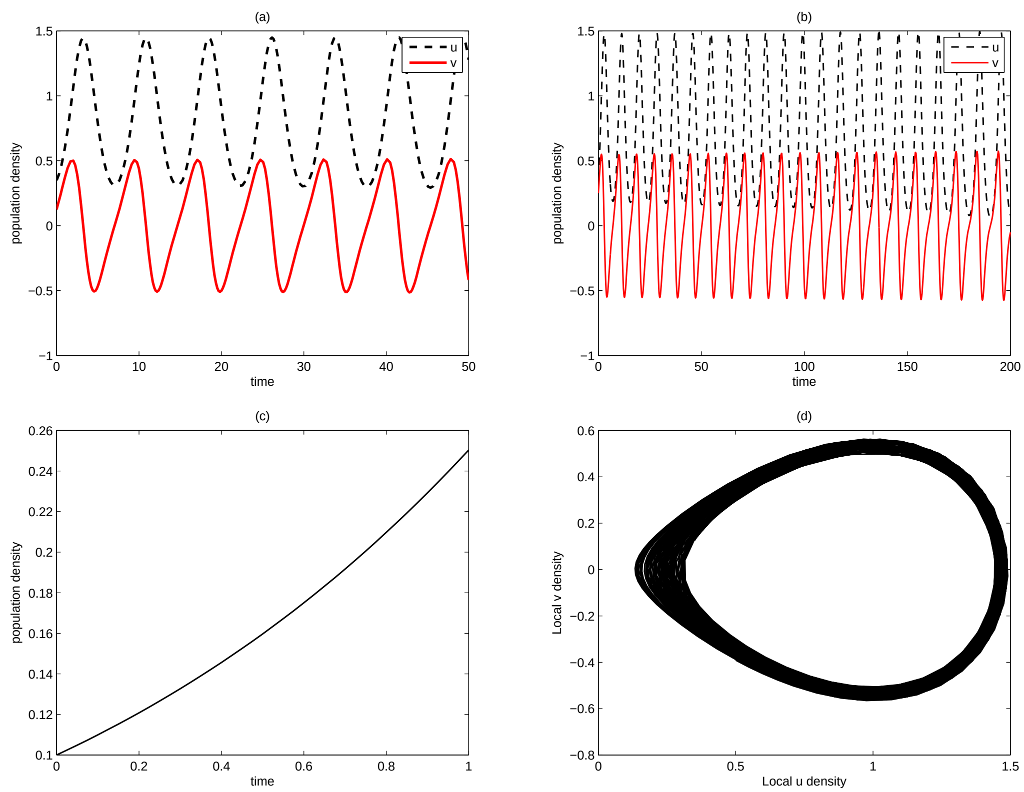

Figure 1: Time series solution of equation (2.6): (a) obtained at $t = 50$ and (b) obtained at $t = 200$. Plot (c) represents the logistic population growth at $t = 1$, $\tau = 0.9$, $K = 2$. Plot (d) is the phase plane limit circle obtained at $t = 400$, with ratio $\frac{\tau}{D} = 1$.

2.2 Traveling wave solution

The Fisher equation of the form

$$u_t = Du_{xx} + \beta u(1-u) \tag{2.8}$$

is considered, where β, κ and D are all positive parameters. The travelling wave solutions of this equation have been studied broadly, the book by Fife [11], Britton [6], Kot [25] and Murray [29, 32, 33] give an extensive report of this equation. The discovery, investigation and analysis of traveling waves of (2.8) was earlier reported by Luther [28] in 1906 for the modeling of chemical reaction. In his work, he stated that the wave-speed is a consequence of the differential equations and obtained the waves speed in terms of the parameters associated with the reactions he was studying. The first explicit analytic form of a cline solution for the Fisher equation were obtained by Albowitz and Zeppetella in making use of the painleve analysis [1] The work of Murray [32] has shown that the analytical form of (2.8) is the same as that found by Kolmogoroff et al. [22] and Fisher [12].

Again, we shall rescale the variables in (2.8) by applying $\hat{u} \equiv \frac{u}{K}$, $\hat{t} \equiv \beta t$, $\hat{x} \equiv \sqrt{\frac{\beta}{D}}x$, it reduces to

$$u_t = u_{xx} + u(1-u) \tag{2.9}$$

after dropping the hats. We had previously considered the steady-state solutions but in the context of biology, we need study the spread of population dynamics (2.8), it is obvious that we can not keep a particular specie say an animal in a particular spot or position without exhibiting a kind of movement. The best way to tackle this is by introducing another new variable, say $\xi = x - ct$. We equally consider the positive movement since it is unrealistic to have negative speed, the moving wave is let to be $u(x,t) = u(x-ct) = u(\xi)$, then $u(x,t)$ is a traveling wave, and it moves at constant speed c in the positive $x-$direction. Obviously, if $x - ct$ is constant, so also is u. This also implies that the coordinate of the system moves with speed c. Our interest is to determine the wave-speed c, The dependent variable ξ is called sometimes the wave variable. To be physically realistic, $u(\xi)$ has to be bounded for all ξ and nonnegative with the quantities with which we are concerned, such as chemicals and populations.

With this information in place, it is convenient to reduce the partial differential equation (2.9) to an equivalence system of ordinary differential equations

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = -cu', \tag{2.10}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = u', \tag{2.11}$$

so that equation (2.9) reduces to $u'' + cu' + u(1-u) = 0$, which we can write further to give a first order system of equations

$$\begin{aligned} u' &= v, \\ v' &= -cv - u(1-u). \end{aligned} \tag{2.12}$$

The pair of equation (2.12) possess two equilibria at (1, 0) and (0, 0), so that $\lim_{\xi \to -\infty} (u, v) \to (1, 0)$ and $\lim_{\xi \to +\infty} (u, v) \to (0, 0)$.

Next, there is need to find a heteroclinic connection between the two equilibria points. First, we start with the determination of the nature of the two phase-plane equilibria. At point (1, 0), the Jacobian becomes

$$J = \begin{pmatrix} 0 & 1 \\ -1+2u & -c \end{pmatrix}_{(1,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \tag{2.13}$$

whose characteristic equation is $\lambda^2 + c\lambda - 1 = 0$. Obviously, the equilibrium here is a saddle point for $c > 0$. The stable and unstable saddle eigenvectors that are compatible with the heteroclinic connection can also be determined at this point. At the origin, the equilibrium points satisfy the characteristics equation $\lambda^2 + c\lambda + 1 = 0$, whose solutions yield

$$\lambda_{1,2} = -\frac{c}{2} \pm \frac{\sqrt{c^2 - 4}}{2}, \tag{2.14}$$

which implies by following the Routh-Hurwitz criterion [18] which gives the necessary and sufficient conditions for the roots of the characteristic equation (2.14) to be asymptotically stable at the origin. We can see that the origin is a stable focus for $0 < c < 2$, all orbits close to the origin in this range oscillate to mean that finding a nonnegative heteroclinic connection between the points (1, 0) and (0, 0) is becoming impossible. For $c \geq 2$, the origin is no longer a stable focus, it is now a stable node and a nonnegative heteroclinic connection may be possible. Therefore, $c \geq 2$ is the necessary condition of a traveling wave.

3 Numerical method

We now consider briefly the numerical integration of equation (1.2) in one and two dimensions. We adopt the concept of method of lines [14, 15, 34, 35, 36] for the spatial approximation of the derivatives in (1.2) using finite difference scheme. In one dimension, we first discretize the spatial domain with step size $h = x/(N-1)$ and approximate the second -order spatial derivative at the right hand side of (1.2) with fourth order central finite difference scheme, which results to a system of nonlinear ordinary differential equations

$$\frac{du_{i,j}}{dt} = D\left(\frac{-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j}}{12h^2}\right) + \beta u_{i,j}\left(1 - \frac{(u_{i,j})^{\alpha}}{\kappa}\right) \tag{3.1}$$

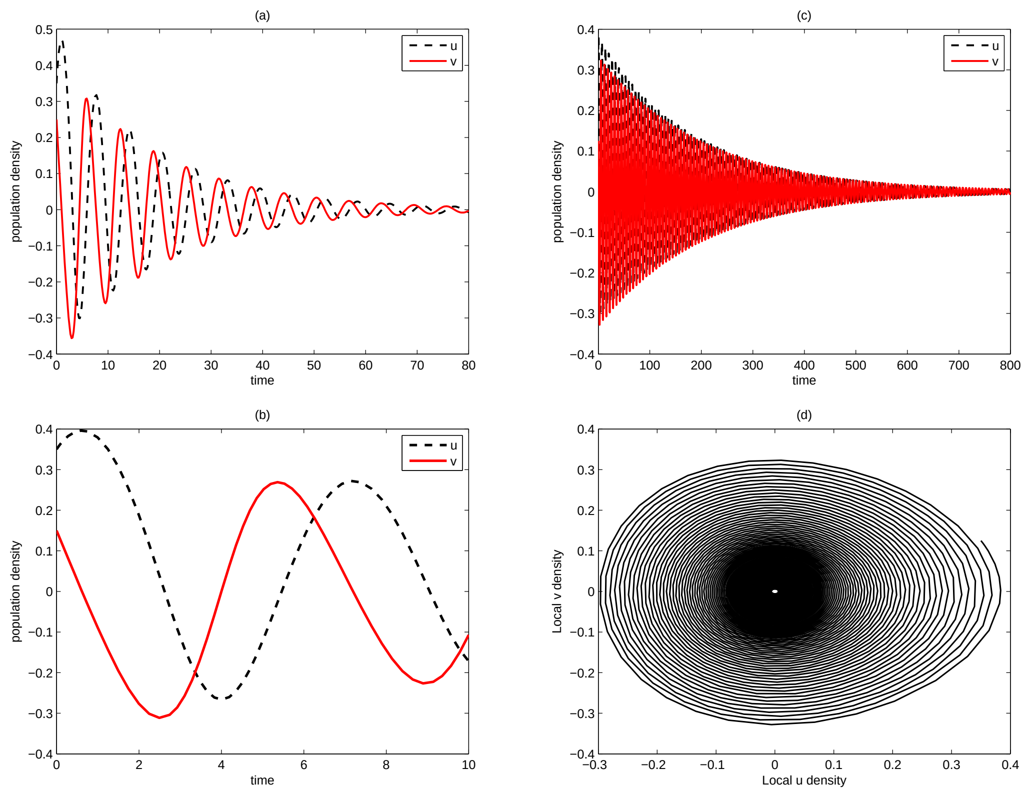

Figure 2: Time series solutions and phase trajectory plane of equation (2.12)

for $i = 1, 2, \ldots, N$ and $u = [u_1^j, u_2^j, \ldots, u_N^j]^T$. Equations (3.1) are then integrated subject to initial condition $u(x, t = 0) = u_0(x)$ for $x \in [a, b]$, and the zero flux boundary conditions $u_a = u_b = 0$.

In two dimensions, the initial and boundary Fisher equation becomes

$$
\begin{aligned}
u_t &= D(u_{xx} + u_{yy}) + \beta u \left(1 - \frac{u^\alpha}{\kappa}\right), \quad (x, y) \in \Omega = (a \le x, y \le b), \quad t > 0, \\
u(x, y, 0) &= u_0(x, y), \quad a \le x, y \le b, \\
u(a, t) &= u(b, t) = 0, \quad t > 0,
\end{aligned} \tag{3.2}
$$

following our previous approach [35], we first discretize the spatial domain by mesh $(x_i, y_j) = (a + i \times h_x, a + j \times h_y)$ where $h_x = (b - a)/(N_x + 1)$, $h_y = (b - a)/(N_y + 1)$ and $0 \le i \le N_x + 1$ and $0 \le j \le N_y + 1$. Using fourth order central difference discretization on the diffusion, we obtain a system of nonlinear ODEs

$$\begin{aligned}
\frac{du_{i,j}}{dt} &= \frac{D}{12}\left[\frac{-u_{i+2,j}+16u_{i+1,j}-30u_{i,j}+16u_{i-1,j}-u_{i-2,j}}{h_x^2}\right]+ \\
&\quad \frac{D}{12}\left[\frac{-u_{i,j+2}+16u_{i,j+1}-30u_{i,j}+16u_{i,j-1}-u_{i,j-2}}{h_y^2}\right]+ \\
&\quad \beta u_{i,j}\left(1-\frac{(u_{i,j})^\alpha}{\kappa}\right), \qquad (3.3)
\end{aligned}$$

which in compact form could be written as

$$u_t = \mathbf{L}u + \mathbf{F}u \qquad (3.4)$$

where $\mathbf{L}$ is the toeplitz matrix representing the linear part and $\mathbf{F}$ is a vector containing the nonlinear part and

$$u = \begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,N_y} & u_{1,N_y+1} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,N_y} & u_{2,N_y+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{N_x,1} & u_{N_x,2} & \cdots & u_{N_x,N_y} & u_{N_x,N_y+1} \end{pmatrix}_{N_x\times N_y+1} \qquad (3.5)$$

Another popular numerical spatial discretization method that has been widely used in conjunction with time stepping methods especially, the exponential time differencing Runge-Kutta [7, 16, 20, 21, 36] is the Fourier spectral method [3, 37, 38]. We shall briefly discuss Fourier spectral method with description adapted from [37] and applied it to (3.2). Given a periodic function u on the spatial grid x_j, we can define the discrete Fourier transform (DFT) as

$$\widehat{u}_k = h\sum_{j=i}^{N} e^{-ikx_j}u_j, \quad k = -\frac{N}{2}+1,\ldots,\frac{N}{2} \qquad (3.6)$$

and the inverse DFT is given by

$$u_j = \frac{1}{2\pi}\sum_{k=-N/2}^{N/2} e^{-ikx_j}\widehat{u}_k, \quad j = 1,2,\ldots,N, \qquad (3.7)$$

where k is regarded as the wavenumber. On applying this method to (3.2) with all the time stepping in Fourier space, gives the following system of ordinary differential equations

$$\widehat{u_t} = -Dk^2\widehat{u} + \widehat{F(u)}, \qquad (3.8)$$

so that the linear term of (3.2) now becomes a diagonal. After spatial discretization, we can now advance the resulting ODEs with time stepping method as presented below.

The Cox and Matthews fourth-order exponential time differencing Runge-Kutta scheme [8] that was used to advance the resulting ODEs are:

$$\begin{aligned} u_{n+1} &= u_n e^{\mathbf{L}h} + \mathbf{F}(u_n, t_n)[-4 - h\mathbf{L} + e^{\mathbf{L}h}(4 - 3h\mathbf{L} + h^2\mathbf{L}^2)] \\ &\quad +2((F)(a_n, t_n + h/2) + \mathbf{F}(b_n, t_n + h/2))[2 + h\mathbf{L} + e^{\mathbf{L}h}(-2 + h\mathbf{L})] \\ &\quad +\mathbf{F}(c_n, t_n + h)[-4 - 3h\mathbf{L} - h^2\mathbf{L}^2 + e^{\mathbf{L}h}(4 - h\mathbf{L})]/h^2\mathbf{L}^3. \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} a_n &= u_n e^{\mathbf{L}h/2} + (e^{\mathbf{L}h/2} - \mathbf{I})\mathbf{F}(u_n, t_n)/\mathbf{L}, \\ b_n &= u_n e^{\mathbf{L}h/2} + (e^{\mathbf{L}h/2} - \mathbf{I})\mathbf{F}(a_n, t_n + h/2)/\mathbf{L}, \\ c_n &= u_n e^{\mathbf{L}h/2} + (e^{\mathbf{L}h/2} - \mathbf{I})(2\mathbf{F}(b_n, t_n + h/2) - \mathbf{F}(u_n, t_n))/\mathbf{L}. \end{aligned} \tag{3.10}$$

If (3.9) is used in conjunction with either of ODEs (3.4) or (3.8), $\mathbf{L}$ is the linear diffusion term, while $\mathbf{F}$ represents the nonlinear term. For detail analysis on derivation and stability of this scheme, readers are referred to our recent paper [36] and references [8, 9, 16, 21, 26, 36] therein.

4 Illustrative examples

We consider a range of illustrative examples based on Fisher equations that are of current and recurrent interest in one and two dimensions.

Example 4.1. One-dimensional Fisher equation. We begin our numerical experiment with one dimensional habitat Fisher-Kolmogoroff equation

$$u_t = u_{xx} + \beta u \left(1 - \frac{u^\alpha}{\kappa}\right), \quad \alpha > 0, \quad t > 0, \ |x| < l \tag{4.1}$$

where D is the diffusion coefficient, β is the growth rate and κ is known as the carrying capacity. This equation is well known to have the uniform steady states at the points $u = 0$ and $u = 1$ with traveling wave solutions given in the form $u(x,t) = U(z)$, for $z = x - ct$, and that $U(-\infty) = 1$ and $U(\infty) = 0$, which satisfy the boundary conditions. We subject (4.1) to initial condition

$$u(x, 0) = u_0(x),$$

on an infinite domain $(-\infty, \infty)$, which we truncate with large finite value l. The first numerical simulation is performed with initial condition [17]

$$u_0(x) = \exp(-20(x)^2) + \frac{5}{2}\exp(-10(x-4)^2) + 3\exp(-20(x+4)^2). \tag{4.2}$$

Series of features emerge from this problem as a result of variation in the choices of parameter values in (4.1), we present some of these travelling wave patterns in Figure 3. The space scale l is adjusted to ensure that there is sufficient space for waves to propagate.

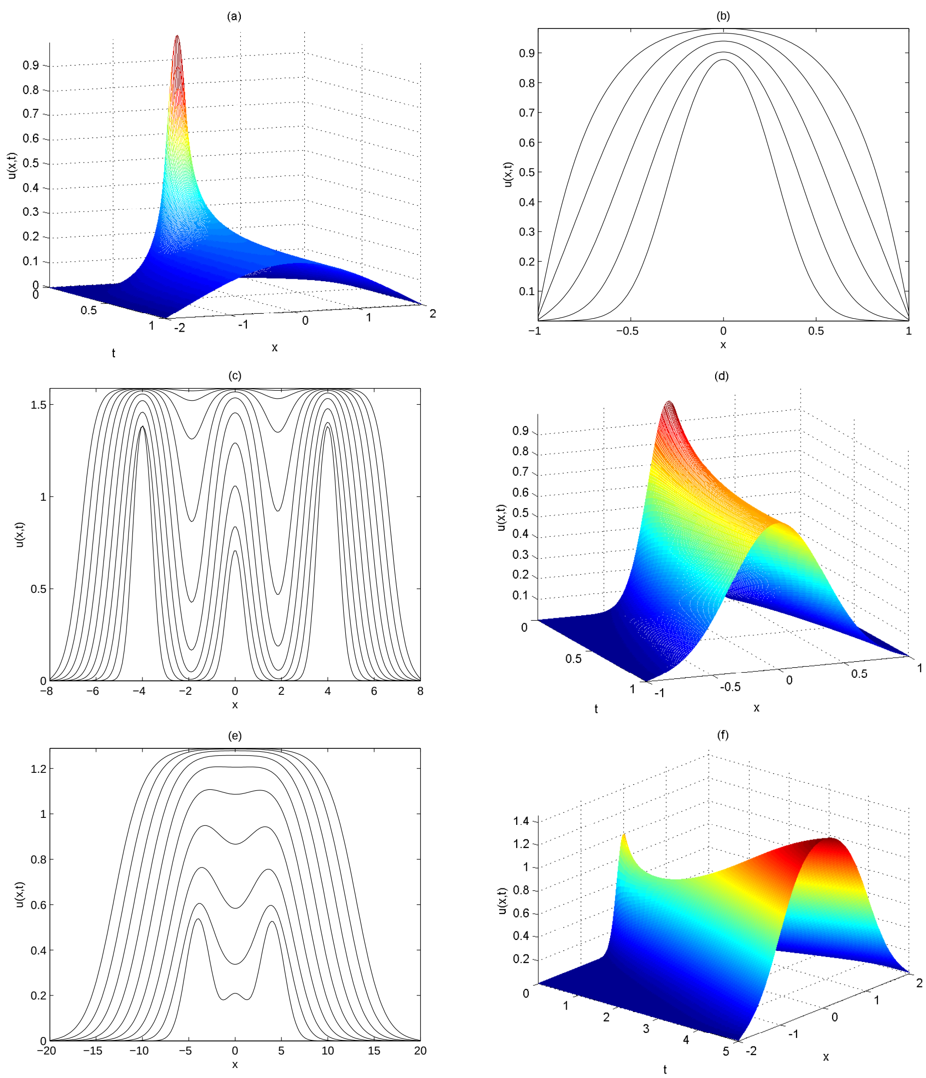

Figure 3: Solutions to problem (4.1) at different parameter values. Take note of variations in both space and amplitude.

In Figure 3, it is evident that each solution results to sinusoidal pattern with initial sharp peaks in the middle which disappear and get flatter as time is increased. The parameter values are: (a) $D = 0.8, \kappa = 1.667, \beta = 1, \alpha = 2$ at final time $t = 1$; (b) $D = 0.1, \kappa = 1, \beta = 0.5, \alpha = 1$ for $t = 1(1)5$; (c) $D = 0.05, \kappa = 4, \beta = 0.5, \alpha = 3$ for $t = 1(1)10$, we obtain surface plot (d) with the seme parameter values in (c) except at $t = 1$. Panels (e) is obtained with $D = 0.8, \kappa = 10/6, \beta = 1, \alpha = 2$ for $t = 1(1)10$, while (f) is obtained with the same parameter values in (d), at $t = 5$.

We examine further by considering the case when $\beta = \alpha = \kappa = 1$, D=0.5 in equation (4.1) subject to the initial condition

$$u_0(x) = \left(1 + \exp\left(\frac{\sqrt{6}}{6}x\right)\right)^{-2} \tag{4.3}$$

with traveling wave solution

$$u(x,t) = \left(1 + \exp\left(-\frac{5}{6}t\frac{\sqrt{6}}{6}x\right)\right)^{-2} \tag{4.4}$$

Example 4.2. Two-dimensional Fisher equation. The one-dimensional Fisher equation considered above is relatively simple and, it has been the major focus of mathematical studies over few decades which almost completely investigated. However, due to the task involving in the two-dimensional case that remains less studied and is still poorly understand with just a handful of research papers. We are motivated in this paper to examine the behaviours of Fisher equation in two-dimension. To examine the variability in the two-dimensional dynamic, our numerical simulations are subjected to some initial conditions taking to replicate some features, and zero-flux boundary conditions.

Mathematically, this situation is described by the following equations:

$$\begin{aligned} u_t &= D(u_{xx} + u_{yy}) + \beta u\left(1 - \frac{u^\alpha}{\kappa}\right), \quad (x,y) \in \Omega = (a \le x, y \le b), \quad t > 0, \\ u(x,y,0) &= u_0(x,y), \quad a \le x, y \le b, \\ u(a,t) &= u(b,t) = 0, \quad t > 0, \end{aligned} \tag{4.5}$$

where D is the diffusion, *beta* is the natural growth rate, $\alpha > 0$ is real exponent and κ is termed the carrying capacity. Our choice of using variables $u = u/\beta,\ t = \beta t$, $x = (\beta/D)^{1/2}x$ and $y = (\beta/D)^{1/2}y$ when $\alpha = 1$ leads to a dimensionless form

$$u_t = u_{xx} + u_{yy} + u(1-u), \quad t > 0, (x,y) \in \mathbb{R}^2, \tag{4.6}$$

we begin our simulations with initial condition

$$u_0(x,y) = \sqrt{|1 - x^2 - y^2|}, \quad x^2 + y^2 \le 1 \tag{4.7}$$

on an infinite domain $(x,y) \in (-\infty, \infty)$ that we truncate with interval of length l.

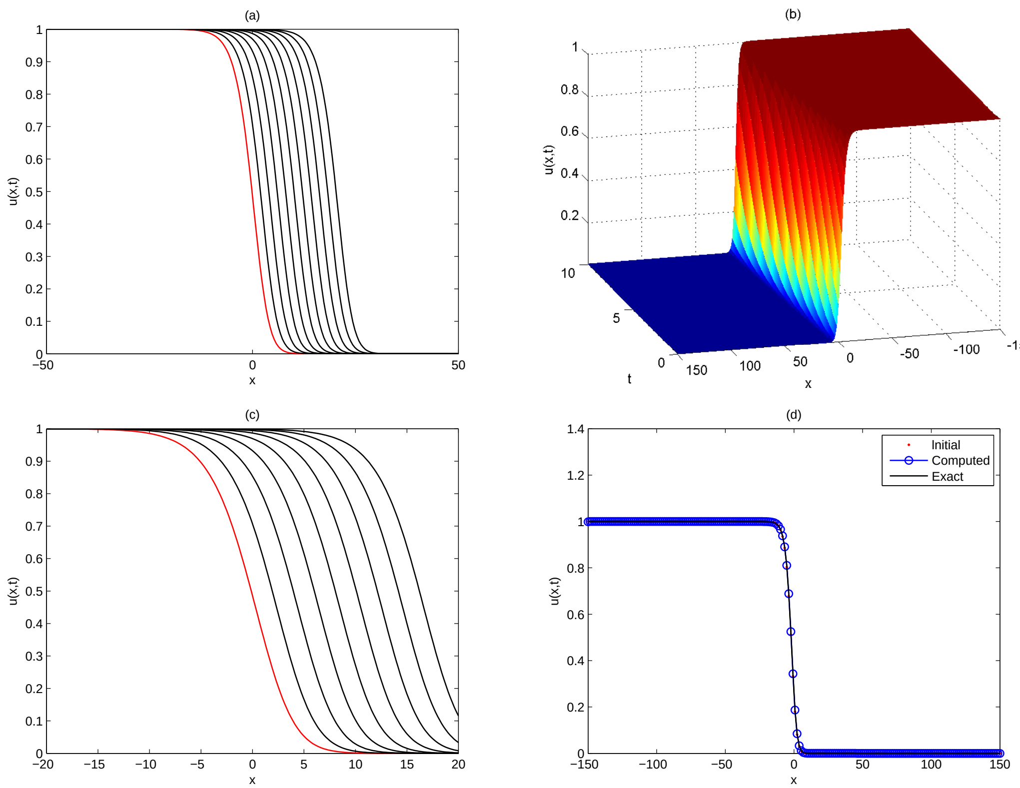

Figure 4: Numerical solutions of example (4.1) at different parameter values. Panels (a) and (c) show profiles of u at unit time intervals until $t = 10$. Panel (b) is the surface plot of u at final time $t = 10$. Plot (d) justify the accuracy of ETDRK4 method on comparison with exact computation. Take note of variations in both space and amplitude.

In our last experiment, we subject problem (4.6) to two different types of initial conditions that evolve a similar waves in Figure 7, panels (a)-(e) are obtained with the initial condition

$$u_0(x,y) = \left[\frac{1}{2}\tanh\left[-\frac{\alpha}{2\sqrt{2\alpha+4}}\left(x - \frac{\alpha+4}{2\sqrt{2\alpha+4}}y\right)\right] + \frac{1}{2}\right]^{2/\alpha} \tag{4.8}$$

where $\alpha > 0$. Similarly, we obtained the results in Figure 7 (f)-(h) with the initial data

$$u_0(x,t) = \frac{1}{2}(1+\beta) + 0.5(1-\beta)\tanh\left[\sqrt{2}(1-\beta)\frac{x}{4} + \frac{(1-\beta^2)}{4}y\right]. \tag{4.9}$$

Different shapes emerged, ranging from sharp peak, to sharp slope, gentle slope and smooth flat roof.

5 Conclusions

In this paper, efficiency and compatibility of fourth-order exponential time differencing Runge-Kutta (ETDRK4) method in conjunction with spatial discretisation schemes as suggested by Cox and Matthews [8], Krogstad [26] and Kassam and L.N. Trefethen [21] have again been justified. This method is tested on nonlinear reaction-diffusion Fisher's equation in one and two dimensions, our numerical simulations was carried out on a large spatial interval of length l that is chosen sufficiently enough to allow the waves to propagate. Some of the initial conditions are taken to actually replicate some existing patterns. The method is stable and reliable, it can extended to solve problems involving the convection-reaction-diffusion terms of any dimension.

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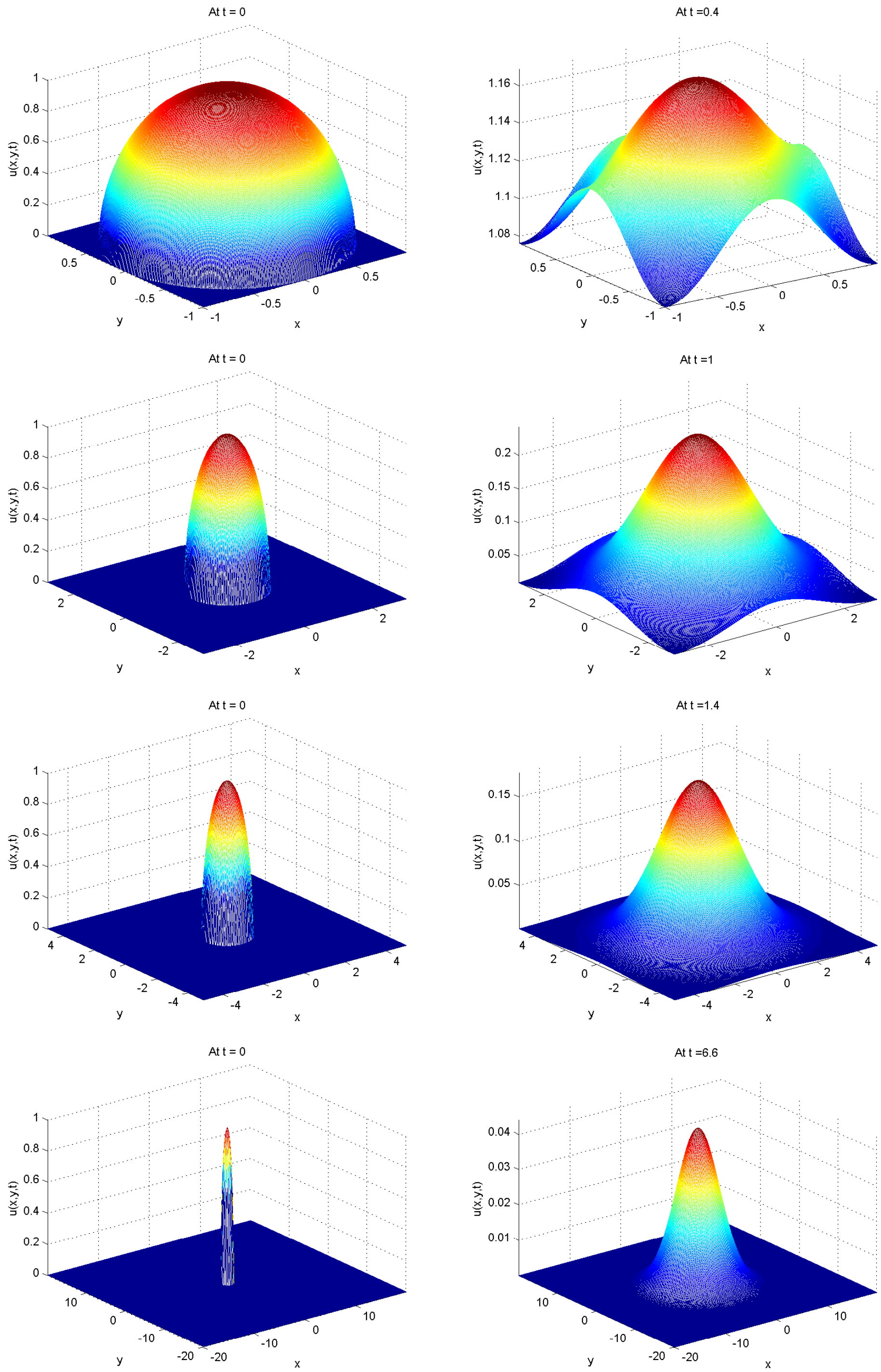

Figure 5: Solutions to problem (4.5) with variation in space, time and amplitudes. Here, as time progresses, there is decrease in solution profiles, this justify the effect of growth with time in the spatial domains.

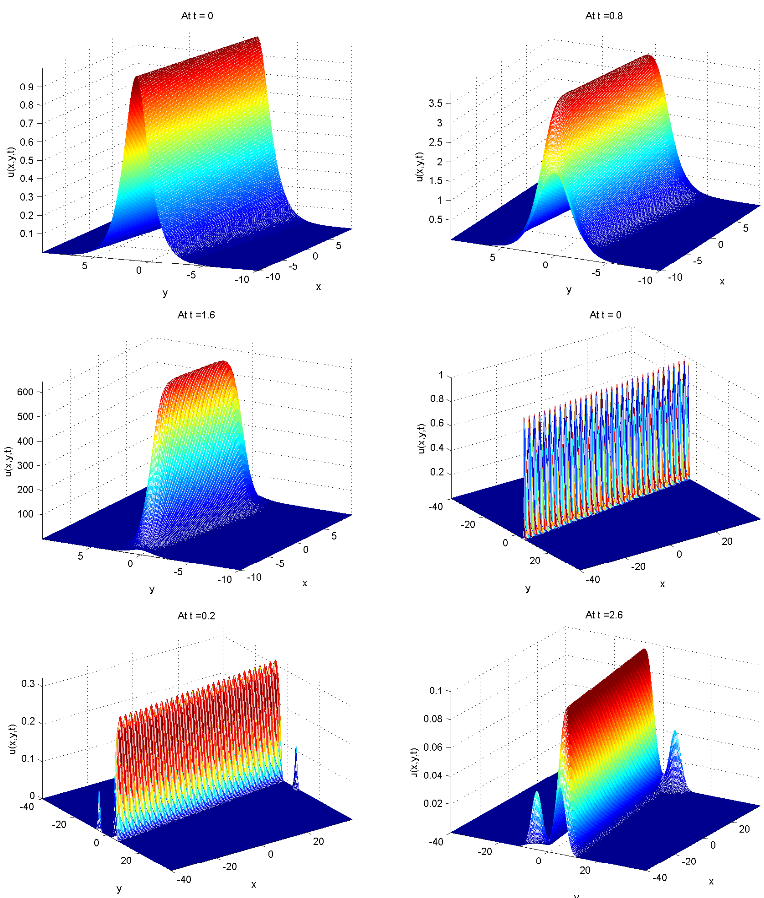

Figure 6: Numerical solutions of example (4.5) with initial condition $u_0(x, y) = 1/\cosh(\delta x + y)$. The first three panels are obtained with $\delta = 1/8$ where the density $u(x, y, t)$ is increase with time. The last three panels are obtained with $\delta = 8$, as time is increasing, the amplitude of $u(x, y, t)$ is decreasing.

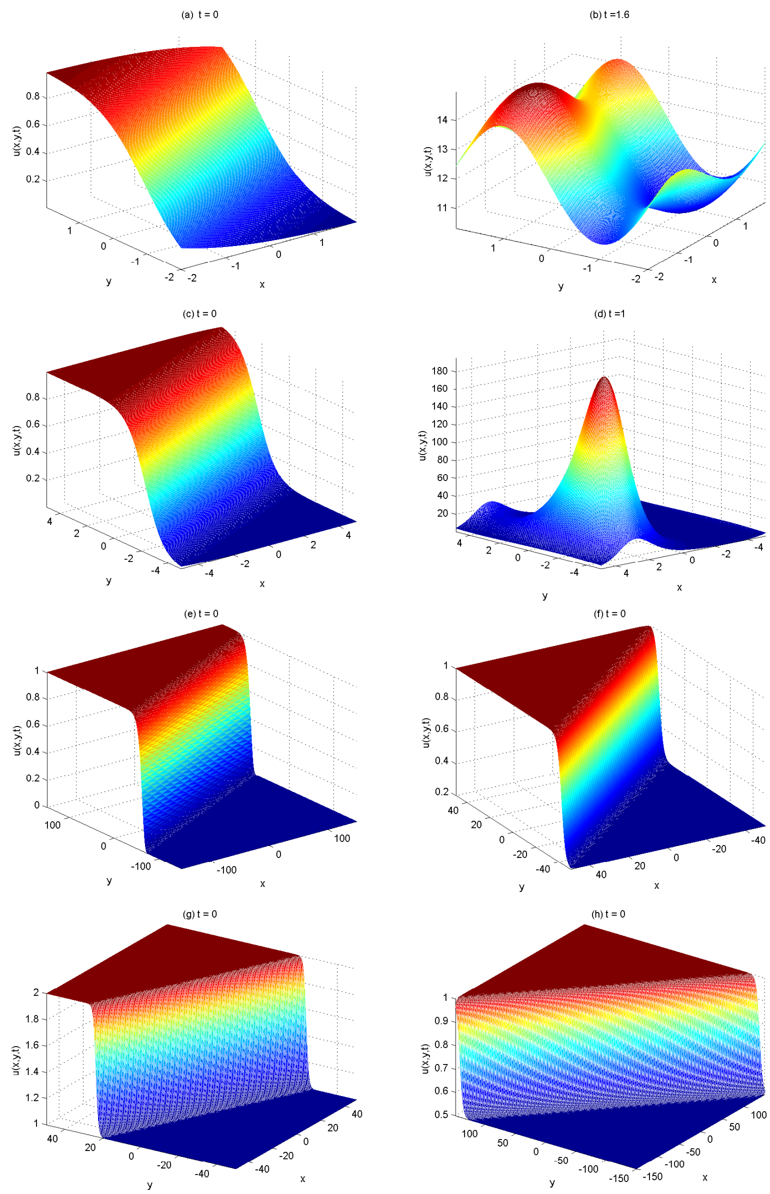

Figure 7: Solutions to problem (4.5) with initial conditions (4.8) for panels (a)-(e) and (4.9), for surface plots (e)-(g). The solutions evolve different travelling waves.