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Dirichlet Time-Optimal Control of Co-operative Hyperbolic Systems

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Abstract.

In this communication, we introduce and study the various time-optimal control problems for Dirichlet co-operative hyperbolic linear system involving Laplace operator with distributed or boundary controls and with observations belong to different spaces. For each problem, we have answer to three question (controllability, existence of control, properties of this control if it exist) arise naturally in connection with this problems.

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1. **Introduction.**

The " *time optimal* " control problem is one of the most important problems in the field of control theory. The simple version is that steering an initial state y_0 in a Hilbert space H to hitting a target set $K \subset H$ in minimum time , with *control* subject to constraints ($u \in U \subset H$) .

In this work, we will focus our attention on some special aspects of minimum time problems for co-operative parabolic system involving Laplace operator with distributed or boundary control. In order to explain the results we have in mind, it is convenient to consider the abstract form:

Let V and H be two real Hilbert spaces such that V is a dense subspace of H. Identifying the dual of H with H, we may consider $V \subset H \subset V'$, where the embedding is dense in the following space. Let $A(t)$ ($t \in]0, T[$) be a family of continuous operators associated with a symmetry bilinear forms $\pi(t; ., .)$ defined on $V \times V$ which are satisfied Gårding's inequality

$$\pi(t; y, y) + c_0\|y\|_H^2 \geq c_1\|y\|_V^2, \quad c_0 \geq 0,\ c_1 > 0, \quad \text{for } y \in V,\ t \in [0, T]. \tag{1}$$

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Then, from [1] and [2], for given f, y_0, y_1 and for a bounded linear operator B from a Hilbert space U to $L^2(0,T;H)$ the following abstract systems:

$$\left.\begin{aligned} &\frac{d^2}{dt^2}y(t) + A(t)y(t) = f + Bu,\ t \in]0,T[,\\ &y(0) = y_0, \quad y'(0) = y_1 \end{aligned}\right\} \tag{2}$$

have a unique solution.

Let

$$\left.\begin{aligned} &y(t;u) := \text{be the unique solution of (2)},\\ &z(t;u) := D(y(t;u), y'(t;u)),\\ &K := \text{be a given target set}, \end{aligned}\right\} \tag{3}$$

where y' denote to $\frac{dy}{dt}$ and D is a given bounded linear operator.

The time optimal control problem we shall concern reads:

$$min\{t:\ \ z(t;u) \in K,\ \ u \in U\} \tag{4}$$

where K is a given target set.

A control u^0 is called a time optimal control if $u^0 \in U$ and if there exists a number $\tau^0 > 0$ such that $z(\tau^0;u^0) \in K$ where $\tau^0 = \min\{\tau :\ z(\tau;u) \in K,\ u \in U\}$, we call the number τ^0 as the optimal time.

Three questions (problems) arise naturally in connection with this problem:

(a) Does there exist a control u, and $\tau > 0$ such that $z(\tau;u) \in K$? (this is an approximate controllability problem).

(b) Assume that the answer to (a) is in the affirmative. Does there exist a control u^0 which steering $z(\tau^0;u^0)$ to hitting a target set K in minimum time?

(c) If u^0 exists, is it unique? what additional properties does it have?

A typical application of time-optimal control governed by a system of hyperbolic equations is that of stabilizing a vibrating system by means of the application of suitable forces during a certain time interval;

$$\left.\begin{aligned} \frac{\partial^2 y}{\partial t^2} = \Delta y + u \qquad & \text{in } Q = \Omega\times]0,T[,\\ y(x,0) = y_0(x) \qquad & \text{in } \Omega,\\ y'(x,0) = y_1(x) \qquad & \text{in } \Omega,\\ y(x,t) = 0 \qquad & \text{on } \Gamma\times]0,T[, \end{aligned}\right\} \tag{5}$$

where $\Omega \subset R^N$ is a bounded open domain with smooth boundary Γ, and $\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

In [3] and [4] the following time-optimal control problem was investigated: Let y_0, y_1, $\bar{y_0}, \bar{y_1} \in L^2(\Omega)$, $M > 0$. Does there exist a time $T > 0$ and a control function $u \in L^2(Q)$ with $\int_0^T \|u(t)\|_{L^2(\Omega)} \geq M$ such that the corresponding solution of (5) with $y(0) = y_0$, $y'(0) = y_1$ satisfies $y(T) = \bar{y_0}$, $y'(T) = \bar{y_1}$. The results in [3] partly overlap with results in [4] and they were shown that : For every $T > 0$ there exists exactly one control function of the above problem and this control is bang-bang i.e $\|u(t)\|_{L^2(\Omega)} = M$.

More early, in the works [5] and [6] the time optimal controls problem of the wave equations are studied.

The bang-bang principle for minimum-time control of time-invariant finite-dimensional linear systems was presented in [7]. It asserts that, if a minimum time control exists for a given system, then there exists a bang-bang optimal control in which the control inputs are almost always at a vertex of the polyhedron defined by the input bounds. The bang-bang principle has been extended to distributed-parameter systems in [8] - [10] among others.

In this paper, we extend the above results to time-optimal control problems for $n \times n$ Dirichlet co-operative linear hyperbolic system with distributed or boundary controls as well as we will take various cases of observations. We will consider various time-optimal control problems for the following system (here and everywhere below the vectors are denoted by bold letters and the index $i = 1, 2, ..., n$):

$$\left.\begin{aligned} \frac{\partial^2 y_i}{\partial t^2} &= (A(t)\mathbf{y})_i + u_i(x,t) && \text{in } Q, \\ y_i(x,0) &= y_{i,0}(x) && \text{in } \Omega, \\ y_i'(x,0) &= y_{i,1}(x) && \text{in } \Omega, \\ y_i(x,t) &= v_i(x,t) && \text{on } \Sigma, \end{aligned}\right\} \tag{6}$$

where $y_{i,0}$, $y_{i,1}$ are given functions, u_i represents either a distributed control or a given function defined in Q, v_i represents either a Dirichlet boundary control or a given function defined in Σ and $A(t)$ ($t \in]0, T[$) are a family of $n \times n$ continuous matrix operators;

$$A(t)y = \begin{pmatrix} \Delta + a_1 & a_{12} & \dots & a_{1n} \\ a_{21} & \Delta + a_2 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

with co-operative coefficient functions a_i, a_{ij} satisfying the following conditions:

$$\left.\begin{aligned} & a_i,\ a_{ij} \text{ are positive functions in } L^\infty(Q), \\ & a_{ij}(x,t) \le \sqrt{a_i(x,t)a_j(x,t)}, \\ & a_{ij}(x,t) = a_{ji}(x,t) \\ & \lambda_1(a) \ge n. \end{aligned}\right\} \tag{7}$$

A classical time-optimal control problem consists in steering an initial vector state $(\mathbf{y}_0, \mathbf{y}_1) = \left(\begin{pmatrix} y_{1,0} \\ y_{2,0} \\ \vdots \\ y_{n,0} \end{pmatrix}, \begin{pmatrix} y_{1,1} \\ y_{2,1} \\ \vdots \\ y_{n,1} \end{pmatrix} \right)$ for system (6), with a distributed control $\mathbf{u} = (u_1, u_2, ..., u_n)^T$ belonging to a given control set U_Q or with a Dirichlet boundary control $\mathbf{v} = (v_1, v_2, ..., v_n)^T$, belonging to a given control set U_Σ so that an observation $\mathbf{z}(t) = (z_1(t), z_2(t), ..., z_n(t))^T$ hitting a given target set K in minimum time.

2. **Abstract form and Solutions of the state.**

This section is devoted to the analysis of the existence and uniqueness of solutions of system (6). We distinguish two classes of solutions: weak and ultra-weak solutions defined by transposition(cf [11]-[12]).

Let $H^{\ell}(\Omega)$, be the usual Sobolev space of order ℓ which consists of all $\phi \in L^2(\Omega)$ whose distributional derivatives $D^q\phi \in L^2(\Omega)$, $|q| \leq \ell$, with the scalar product

$$\langle y, \phi \rangle_{H^{\ell}(\Omega)} = \sum_{|q|\leq \ell} \langle D^q y, D^q \phi \rangle_{L^2(\Omega)},$$

$q = \{q_1, ..., q_N\}$, $|q| = q_1 + ... + q_N$, $D^q = D_1^{q_1} ... D_N^{q_N}$, $D_i = \frac{\partial}{\partial x_i}$.

We define $H_0^{\ell}(\Omega)$ by

$$H_0^{\ell}(\Omega) := \left\{ \phi \in H^{\ell}(\Omega) : \quad \frac{\partial^k}{\partial \nu^k} = 0, \quad 0 \leq k \leq \ell \right\}$$

which endowed with the same scalar product defined on $H^{\ell}(\Omega)$. We have the following dense embedding chain [13]

$$H_0^{\ell}(\Omega) \subset L^2(\Omega) \subset H^{-\ell}(\Omega),$$

where $H^{-\ell}(\Omega)$ is the dual of $H_0^{\ell}(\Omega)$ with the usual norm;

$$||\phi||_{H^{-\ell}(\Omega)} = \sup_{\psi \in H_0^{\ell}(\Omega)} \frac{\langle \phi, \psi \rangle}{||\psi||_{H_0^{\ell}(\Omega)}}$$

Here $\langle ., . \rangle$ denotes the duality paring between $H_0^{\ell}(\Omega)$ and $H^{-\ell}(\Omega)$.

We choose $H = (L^2(\Omega))^n$ with its usual norm and $V = (H^1(\Omega))^n$ endowed with the norm

$$||\mathbf{y}||_V = \sum_{i=1}^{n} \int_{\Omega} \left[|\nabla y_i|^2 - a_i(x,t) y_i^2 \right] dx - \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x,t) y_i y_j dx$$

which is equivalent to the usual norm induced by $(H^1(\Omega))^n$.

For $\mathbf{y} = (y_1, y_2, ..., y_n)^T$, $\phi = (\phi_1, \phi_2, ..., \phi_n)^T \in V$ and $t \in]0, T[$, let us define a family of continuous bilinear forms

$$\pi(t; ., .) : V \times V \to \Re \quad \text{by}$$

$$\pi(t; \mathbf{y}, \phi) = \sum_{i=1}^{n} \int_{\Omega} \left[(\nabla y_i)(\nabla \phi_i) - a_i(x,t) y_i \phi_i \right] dx - \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x,t) y_j \phi_i dx \tag{8}$$

.

In [14]-[16], we proved that the bilinear form (8) under the conditions (7) satisfy the Gårding inequality (1), then using Lax-Milgram Lemma, (cf [17]), the solution of (6) can be defined as the solution of the abstract problem

$$\left\langle \frac{d^2 \mathbf{y}(t)}{dt^2}, \phi \right\rangle_{V', V} + \pi(t; \mathbf{y}, \phi) = M(\phi) \quad \forall \phi \in V, \tag{9}$$

with initial conditions

$$\mathbf{y}(0) = \mathbf{y}_0(x), \ \ \mathbf{y}'(0) = \mathbf{y}_1(x) \quad \text{in } \Omega$$

where $\phi \to M(\phi)$ is continuous linear form on V

We define the following Hilbert space $W(0, T)$:

$$W(0,T) = \left\{ f : \ f \in L^2(0, T; (H^1(\Omega))^n), \ \frac{d^2 f}{dt^2} \in L^2(0, T; (H^{-1}(\Omega))^n) \right\}$$

endowed with the norm

$$\Big\{\int_0^T ||y(t)||^2_{(H^1(\Omega))^n}dt + \int_0^T ||\frac{d^2 f}{dt^2}||^2_{(H^{-1}(\Omega))^n}dt\Big\}^{\frac{1}{2}}$$

For any pair of real numbers $r,\, s \geq 0,$ the sobolev space $H^{r,s}(Q)$ is defined by

$$H^{r,s}(Q) = H^0\left(0,T;(H^r(\Omega))^n\right)\bigcap H^s(0,T;(H^0(\Omega))^n)$$

which is a Hilbert space normed by

$$\Big\{\int_0^T ||y(t)||^2_{(H^r(\Omega))^n}dt + ||y||^2_{H^s(0,T;(H^0(\Omega))^n)}\Big\}^{\frac{1}{2}}$$

where $H^s(0,T;X)$ denotes the sobolev space of order s of functions defined on $[0,T]$ and taking values in X. (see Chapter 1 of [2]).

Definition 1. We say that the function $(\mathbf{y},\mathbf{y}')$ is a weak solution for system (6) if

$$(\mathbf{y},\mathbf{y}') \in C\Big([0,T];(H^1(\Omega))^n\Big) \times C\Big([0,T];(L^2(\Omega))^n\Big) \cap W(0,T)$$

and

$$\begin{aligned}\int_0^T\int_\Omega y_i f_i dxdt = &\int_0^T\int_\Omega u_i\phi_i dxdt + \int_\Omega y_{i,1}\phi_i(0)dx - \int_\Omega y_{i,0}\phi_i'(0)dx\\ &+\int_0^T\int_\Gamma v_i\frac{\partial\phi_i}{\partial\nu}d\Gamma dt \quad \text{for all}\, \phi=(\phi_1,\phi_2,...,\phi_n)^T\ \in\Phi,\end{aligned}\tag{10}$$

$$\Phi = \left\{\phi:\begin{array}{ll}\dfrac{\partial^2\phi}{\partial t^2}+(A(t)\phi)_i = f_i \in L^2(0,T;L^2(\Omega)) & \text{in}\, Q,\\ \phi(T)=\phi'(T)=0 & \text{in}\,\Omega,\\ \phi=0 & \text{on}\,\Sigma,\end{array}\right\}$$

where ν is the outward normal

To justify this definition we can apply Theorem 1.1 p.273 in [1] and Theorem 3.1 p.19 in [2] to obtain the following theorem:

Theorem 1. *For every* $(\mathbf{y}_0,\mathbf{y}_1,\mathbf{u},\mathbf{v}) \in (H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(0,T;(L^2(\Omega)^n)) \times H^{\frac{1}{2},\frac{3}{2}}(\Sigma),$ *the problem* (6) *has a unique weak solution* $(\mathbf{y},\mathbf{y}')$. *Furthermore, the linear maps* $(\mathbf{y}_0,\mathbf{y}_1,\mathbf{u},\mathbf{v}) \to (\mathbf{y},\mathbf{y}')$ *is continuous of* $(H^1(\Omega))^n\times(L^2(\Omega))^n\times L^2(0,T;(L^2(\Omega))^n)\times H^{\frac{1}{2},\frac{3}{2}}(\Sigma) \to C([0,T];(H^1(\Omega))^n)\times C([0,T];(L^2(\Omega))^n)$.

Definition 2. We say that the function $(\mathbf{y},\mathbf{y}')$ is an ultra- weak solution for system (6) if

$$(\mathbf{y},\mathbf{y}') \in C\Big([0,T];(L^2(\Omega))^n\Big)\times C\Big([0,T];(H^{-1}(\Omega))^n\Big)\cap L^2\Big(0,T;(L^2(\Omega))^n\Big)\times L^2\Big(0,T;(H^{-1}(\Omega))^n\Big)$$

and

$$\begin{aligned}\int_0^T\int_\Omega y_i f_i dxdt = &\int_0^T\int_\Omega u_i\phi_i dxdt + \big\langle y_{i,1},\phi_i(0)\big\rangle - \int_\Omega y_{i,0}\phi_i'(0)dx\\ &+\int_0^T\int_\Gamma v_i\frac{\partial\phi_i}{\partial\nu}d\Gamma dt \quad \text{for all}\, \phi=(\phi_1,\phi_2,...,\phi_n)^T\ \in\bar\Phi,\end{aligned}\tag{11}$$

$$\bar\Phi = \left\{\phi:\begin{array}{ll}\dfrac{\partial^2\phi}{\partial t^2}+(A(t)\phi)_i = f_i \in L^1(0,T;L^2(\Omega)) & \text{in}\, Q,\\ \phi(T)=\phi'(T)=0 & \text{in}\,\Omega,\\ \phi=0 & \text{on}\,\Sigma.\end{array}\right\}$$

To justify this definition we can apply transposition theorem, Theorem 3.1 p.292 in [1] or Theorem 3.2 p.150 in [12] to obtain the following theorem:

Theorem 2. *For every* $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{u}, \mathbf{v}) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times L^1(0,T;(L^2(\Omega))^n) \times L^2(0,T;(L^2(\Gamma))^n$, *the problem* (6) *has a unique weak solution* $(\mathbf{y}, \mathbf{y}')$. *Furthermore, the linear maps* $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{u}, \mathbf{v}) \to (\mathbf{y}, \mathbf{y}')$ *is continuous of* $(L^2(\Omega))^n \times (H^{-1}(\Omega))^n \times \ L^1(0,T;(L^2(\Omega))^n) \times L^2(0,T;(L^2(\Gamma))^n \to C([0,T];(L^2(\Omega))^n) \times C([0,T];(H^{-1}(\Omega))^n)$.

Based on the above theorems, we may state our results more explicitly by choosing the observation in a less abstract fashion. We may consider the following problems:

(I) distributed control problem with observation $\mathbf{y} \in C([0,T];(L^2(\Omega))^n)$
(II) distributed control problem with observation $\mathbf{y} \in C([0,T];(H^1(\Omega))^n)$
(III) distributed control problem with observation $\mathbf{y}' \in C([0,T];(L^2(\Omega))^n)$
(IV) boundary control problem with observation $\mathbf{y} \in C([0,T];(L^2(\Omega))^n)$
(V) boundary control problem with observation $\mathbf{y}' \in C([0,T];(H^{-1}(\Omega))^n)$

In the next sections, we will denote by $\mathbf{y}(t;\mathbf{u})$ to the unique weak of (6) at time t corresponding to a given functions $\mathbf{y}_0, \mathbf{y}_1, \mathbf{v}$ and distributed control $\mathbf{u} \in U_Q$ satisfying the hypothesis of Theorem 1 with U_Q given by;

$$U_Q = \text{ closed convex subset of } L^2(0,T;(L^2(\Omega)^n)). \tag{12}$$

Similarly, we will denote by $\mathbf{y}(t;\mathbf{v})$ to the unique ultra-weak solution of (6) at time t corresponding to a given functions $\mathbf{y}_0, \mathbf{y}_1, \mathbf{u}$ and boundary control $\mathbf{v} \in U_\Sigma$ satisfying the hypothesis of Theorem 2 with U_Σ given by;

$$U_\Sigma = \text{ closed convex subset of } L^2(0,T;(L^2(\Gamma)^n)). \tag{13}$$

Occasionally, we write $\mathbf{y}(t;\mathbf{u},x)$ or $\mathbf{y}(t;\mathbf{v},x)$ when the explicit dependence on x is required.

Also, we will denote by K_H, K_V, and $K_{V'}$ to the following target sets:

$$K_H = \Big\{\mathbf{z} = (z_1, z_2, ..., z_n)^T \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} \le \varepsilon\Big\},$$
$$\varepsilon > 0 \quad \text{and } z_{id} \in L^2(\Omega) \text{ are given.}$$

$$K_V = \Big\{\mathbf{z} \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} + \sum_{j=1}^{N} \|\frac{\partial z_i}{\partial x_j} - z_{id}\|_{L^2(\Omega)} \le \epsilon\Big\},$$
$$\varepsilon > 0 \quad \text{and } z_{id} \in L^2(\Omega) \text{ are given.}$$

$$K_{V'} = \Big\{\mathbf{z} \in (H^{-1}(\Omega))^n : \|z_i - z_{id}\|_{H^{-1}(\Omega)} \le \varepsilon\Big\},$$
$$\varepsilon > 0 \quad \text{and } z_{id} \in H^{-1}(\Omega) \text{ are given.}$$

3. Distributed control - position observation problem.

We consider the following first time-optimal control problem with distributed control $\mathbf{u}$ and position observation $\mathbf{z} = \mathbf{y}(t;\mathbf{u}) \in C([0,T];(L^2(\Omega))^n)$:

$$(TOP1): \qquad \min\Big\{t: \ \mathbf{y}(x,t;\mathbf{u}) \in K_H, \quad \mathbf{u} \in U_Q\Big\}$$

Theorem 3. *If* T *is large enough , then there exists a* $\tau \in]0,T]$ *and* $\mathbf{u} \in U_Q$ *with* $\mathbf{y}(\tau;\mathbf{u}) \in K_H$.

Proof. let us first remark that by translation we may always reduce the problem of controllability to the case were the system (6) with $y_{i,0} = y_{i,1} = v_i = 0$. We can show quit easily that (6) is approximately controllable in $(L^2(\Omega))^n$ if and only if the set of reachable states $R(\tau)$ at any finite time $\tau > 0$ is dense in $(L^2(\Omega))^n$;

$$R(\tau) = \Big\{\mathbf{y}(\tau; \mathbf{u}) : \mathbf{u} \in L^2(0, T; (L^2(\Omega)^n))\Big\}.$$

By the Hahn-Banach theorem, this will be the case if

$$\int_\Omega \psi_i(x) y_i(x, \tau; \mathbf{u}) dx = 0, \quad \psi_i \in L^2(\Omega), \tag{14}$$

for all $\mathbf{u} \in L^2(0, T; (L^2(\Omega)^n))$, implies that $\psi_i(x) = 0$.

We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left.\begin{array}{ll} \dfrac{\partial^2 \xi_i}{\partial t^2}(t; \mathbf{u}) - (A(t)\xi(t; \mathbf{u}))_i = 0 & \text{in}\, \Omega \times]0, \tau[, \\ \xi_i(x, \tau) = 0 & \text{in}\, \Omega, \\ \xi_i'(x, \tau) = -\psi_i(x) \in L^2(\Omega) & \text{in}\, \Omega, \\ \xi_i(x, t) = 0. & \text{on}\, \Gamma \times]0, \tau[. \end{array}\right\} \tag{15}$$

The problem (15) can be solved in the sense of Theorem 1 and Definition 1 (with an obvious change of variables $t \to \tau - t$ and $\mathbf{y} \to \xi, \quad \phi \to \mathbf{y}$); Problem (15) have a unique weak solution such that

$$\int_\Omega \psi_i(x) y_i(x, \tau; \mathbf{u}) dx = \int_0^\tau \int_\Omega \xi_i(x, t; \mathbf{u})\, u_i dx dt.$$

and so, if (14) holds, then

$$\int_0^\tau \int_\Omega \xi_i(t; \mathbf{u})\, u_i dx dt = 0 \quad \forall u_i \in L^2(Q)$$

hence $\xi_i(t; \mathbf{u}) = 0$, But from the continuity property, $\xi_i(\tau; \mathbf{u}) \equiv 0$ and hence $\psi_i(x) = 0$. □

Now, set

$$\tau_1^0 = \inf\{\tau : \mathbf{y}(\tau; \mathbf{u}) \in K_H \text{ for some } \mathbf{u} \in U_Q\}. \tag{16}$$

The following result holds .

Theorem 4. *There exists an admissible control $\mathbf{u}^0$ to the problem (TOP1), which steering $\mathbf{y}(t; \mathbf{u}^0)$ to hitting a target set K_H in minimum time τ_1^0 (defined by (16)). Moreover*

$$\sum_{i=1}^n \int_\Omega (y_i(\tau_1^0; \mathbf{u}^0) - z_{id}) \left(y_i(\tau_1^0; \mathbf{u}) - y_i(\tau_1^0; \mathbf{u}^0)\right) dx \geq 0 \quad \forall \mathbf{u} \in U_Q. \tag{17}$$

Proof. We can choose $\tau^m \to \tau_1^0$ and admissible controls $\{\mathbf{u}^m\}$ such that

$$\mathbf{y}(\tau^m; \mathbf{u}^m) \in K_H, \quad m = 1, 2, \ldots.$$

Set $\mathbf{y}^m = \mathbf{y}(\mathbf{u}^m)$. Since U_Q is bounded, we may verify that $\mathbf{y}^m$ (respectively $\frac{d\mathbf{y}}{dt}$) ranges in a bounded set in $L^2(0, T; (H^1(\Omega))^n)$ (respectively $L^2(0, T; (L^2(\Omega))^n$).

We may then extract a subsequence, again denoted by $\{\mathbf{u}^m, \mathbf{y}^m\}$ such that

$$\left.\begin{aligned} &\mathbf{u}^m \to \mathbf{u}^0 \quad \text{weakly in } L^2(0,T;(L^2(\Omega))^n), \quad u^0 \in U_Q, \\ &\mathbf{y}^m \to \mathbf{y} \quad \text{weakly in } L^2\left(0,T;\left(H^1(\Omega)\right)^n\right) \\ &\frac{d\mathbf{y}^m}{dt} \to \quad \text{weakly in } L^2(0,T;(L^2(\Omega))^n). \end{aligned}\right\} \tag{18}$$

We deduce from the equality

$$\frac{d^2\mathbf{y}^m}{dt^2} = \mathbf{u}^m + A(t)\mathbf{y}^m$$

that

$$\frac{d^2\mathbf{y}^m}{dt^2} \to \frac{d^2\mathbf{y}}{dt^2} = \mathbf{u}^0 + A(t)\mathbf{y} \quad \text{in } L^2\left(0,T;\left(H^{-1}(\Omega)\right)^n\right),$$

and

$$\mathbf{y}(0) = \mathbf{y}_0, \quad \frac{d\mathbf{y}}{dt}(0) = \mathbf{y}_1.$$

But

$$\mathbf{y}(\tau^m;\mathbf{u}^m) - \mathbf{y}(\tau_1^0;\mathbf{u}^0) = \mathbf{y}(\tau^m;\mathbf{u}^m) - \mathbf{y}(\tau_1^0;\mathbf{u}^m) + \mathbf{y}(\tau_1^0;\mathbf{u}^m) - \mathbf{y}(\tau_1^0;\mathbf{u}^0)$$

then, from (18) we have

$$\mathbf{y}(\tau_1^0;\mathbf{u}^m) \to \mathbf{y}(\tau_1^0;\mathbf{u}^0) \quad \text{weakly in } (H^1(\Omega))^n \tag{19}$$

and

$$\begin{aligned} \|\mathbf{y}(\tau^m;\mathbf{u}^m) - \mathbf{y}(\tau_1^0;\mathbf{u}^m)\|_{(L^2(\Omega))^n} &= \left\| \int_{\tau^0}^{\tau^m} \frac{d}{dt}\mathbf{y}(t;\mathbf{u}^m)dt \right\|_{(L^2(\Omega))^n} \\ &\leq \sqrt{\tau^m - \tau_1^0} \left(\int_{\tau_1^0}^{\tau^m} \left\| \frac{d}{dt}\mathbf{y}(t;\mathbf{u}^m)dt \right\|^2_{(L^2(\Omega))^n} dt \right)^{\frac{1}{2}} \\ &\leq c\sqrt{\tau_n - \tau_1^0} \end{aligned} \tag{20}$$

Combine(19) and (20) show that

$$\mathbf{y}(\tau^m;\mathbf{u}^m) - \mathbf{y}(\tau_1^0;\mathbf{u}^0) \to 0 \quad \text{weakly in } \left(L^2(\Omega)\right)^n. \tag{21}$$

Similarly, we can verify that

$$\mathbf{y}'(\tau^m;\mathbf{u}^m) - \mathbf{y}'(\tau_1^0;\mathbf{u}^0) \to 0 \quad \text{weakly in } \left(H^{-1}(\Omega)\right)^n. \tag{22}$$

And so, $\mathbf{y}(\tau_1^0;\mathbf{u}^0) \in K_H$ as K_H is closed and convex, hence weakly closed. This shows that K_H is reached in time τ_1^0 by admissible control $\mathbf{u}^0$.

For the second part of the theorem, really, from Theorem 1, the mapping $t \to \mathbf{y}(t;\mathbf{u})$ and $t \to \mathbf{y}'(t;\mathbf{u})$ from $[0,T] \to (H^1(\Omega))^n$ and $[0,T] \to (L^2(\Omega))^n$, respectively, are continuous for each fixed $\mathbf{u}$ and so $\mathbf{y}(\tau_1^0;\mathbf{u}) \notin int\, K_H$, for any $\mathbf{u} \in U_Q$, by minimality of τ_1^0.

Using Theorem 1 it is easy to verify that the mapping $\mathbf{u} \to \mathbf{y}(\tau_1^0;\mathbf{u})$, from $L^2(0,T;(L^2(\Omega)^n)) \to (L^2(\Omega))^n$, is continuous and linear. Then, the set

$$\mathcal{A}(\tau_1^0) = \{\mathbf{y}(\tau_1^0;\mathbf{u}) \,:\, \mathbf{u} \in U_Q\}$$

is the image under a linear mapping of a convex set hence $\mathcal{A}(\tau_1^0)$ is convex. Thus we have $\mathcal{A}(\tau_1^0) \cap intK_H = \emptyset$ and $\mathbf{y}(\tau_1^0;\mathbf{u}^0) \in \partial K_H$ (boundary of K_H) . Since $intK_H \neq \emptyset$ (from Theorem 7) so there exists a closed hyperplane separating $A(\tau_1^0)$ and K_H containing $\mathbf{y}(\tau_1^0;\mathbf{u}^0)$, i.e there is a nonzero $\mathbf{g} \in (L^2(\Omega))^n$ such as

$$\sup_{\mathbf{y}\in A(\tau_1^0)} \Big\langle \mathbf{g}, \mathbf{y}(\tau^0;\mathbf{u}) \Big\rangle_{(L^2(\Omega))^n} \le \Big\langle \mathbf{g}, \mathbf{y}(\tau^0;\mathbf{u}^0) \Big\rangle_{(L^2(\Omega))^n} \le \inf_{\mathbf{y}\in K_H} \Big\langle \mathbf{g}, \mathbf{y}(\tau_1^0;\mathbf{u}) \Big\rangle_{(L^2(\Omega))^n}. \tag{23}$$

From the second inequality in (23), $\mathbf{g}$ must support the set K_H at $\mathbf{y}(\tau^0;\mathbf{u}^0)$ i.e

$$\big\langle \mathbf{g}\,, (\mathbf{y}(\tau_1^0;\mathbf{u}) - \mathbf{y}(\tau_1^0;\mathbf{u}^0)) \big\rangle_{(L^2(\Omega))^n} \ge 0 \quad \forall \mathbf{u} \in U_Q$$

and since $(L^2(\Omega))^n$ is a Hilbert space , $\mathbf{g}$ must be of the form

$$\mathbf{g} = \lambda(\mathbf{y}(\tau_1^0;\mathbf{u}^0) - z_{id}) \quad \text{for some } \lambda > 0.$$

Dividing the inequality (23) by λ gives the desired result. □

The condition (17) can be simplified by introducing the following adjoint equation. For each $\mathbf{u}^0 \in U_Q$, we define $p(t;\mathbf{u}^0)$ as the solution of the following system

$$\left.\begin{array}{ll} \dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{u}^0) - \big(A(t)\mathbf{p}(t;\mathbf{u}^0)\big)_i = 0 & \text{in } \Omega\times]0,\tau_1^0[, \\ p_i(x,\tau_1^0;\mathbf{u}^0) = 0 & \text{in } \Omega, \\ p_i'(x,\tau_1^0;\mathbf{u}^0) = -(y_i(x,\tau_1^0;\mathbf{u}^0) - z_{id}) & \text{in } \Omega, \\ p_i(x,t;\mathbf{u}^0) = 0 & \text{on } \Gamma\times]0,\tau_1^0[. \end{array}\right\} \tag{24}$$

The problem (24) can be solved in the sense of Theorem 1 and Definition 1 (with an obvious change of variables $t \to \tau_0^1 - t$ and $\mathbf{y} \to \mathbf{p}$, $\quad \phi \to \mathbf{y}(t;\mathbf{u}) - \mathbf{y}(t;\mathbf{u}^0)$); Problem (24) have a unique weak solution such that

$$\int_\Omega (y_i(x,\tau_1^0;\mathbf{u}^0) - z_{id})(y_i(x,\tau_1^0;\mathbf{u}) - y_i(x,\tau_1^0;\mathbf{u}^0))dx = \int_0^{\tau_1^0}\int_\Omega p_i(u_i - u_i^0)dxdt.$$

Condition (17) then becomes

$$\sum_{i=1}^n \int_0^{\tau_1^0}\int_\Omega p_i(u_i - u_i^0)dxdt \ge 0 \quad \forall \mathbf{u} \in U_Q. \tag{25}$$

This result can be summarized as:

Theorem 5. *The optimal control* $\mathbf{u^0}$ *of problem (TOP1) is characterized by* (24),(25) *together with* (6) *(with* $u_i = u_i^0$ *).*

The maximum conditions (25) of the optimal control leads to the following result :

Theorem 6. *(Bang-bang theorem) We assume that*

$$\begin{aligned} U_Q &= \Big\{\mathbf{u} \in (L^2(0,T;(L^2(\Omega))^n), \quad \mathbf{u}(t) \in E_\Omega\Big\}, \\ E_\Omega &= \textit{ closed, bounded, convex subset of} (L^2(\Omega))^n. \end{aligned} \tag{26}$$

Then the optimal control of (TOP1) is bang-bang, i.e.

$$\mathbf{u}(t) \in \partial E_\Omega = \textit{ boundary of } E_\Omega. \tag{27}$$

Proof. According to Theorem 9, In $]0,\tau_1^0[$, $\mathbf{p}(t;\mathbf{u}^0)$, satisfies

$$\left.\begin{array}{ll}\dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{u}^0)-\big(A(t)\mathbf{p}(t;\mathbf{u}^0)\big)_i=0 & \text{in}\,\Omega\times]0,\tau_1^0[,\\ p_i(x,\tau_1^0;\mathbf{u}^0)=0 & \text{in}\,\Omega,\\ p_i'(x,\tau_1^0;\mathbf{u}^0)=-(y_i(x,\tau_1^0;\mathbf{u}^0)-z_{id}) & \text{in}\,\Omega,\\ p_i(x,t;\mathbf{u}^0)=0 & \text{on}\,\Gamma\times]0,\tau_1^0[.\end{array}\right\}$$

and the optimality of $\mathbf{u}^0$ being characterized by

$$\sum_{i=1}^{n}\int_0^{\tau_1^0}\int_\Omega p_i(t)(u_i-u_i^0)dxdt\geq 0\quad\forall\mathbf{u}\in U_Q.$$

From Theorem 2.1 Chapter 2 in [1] this condition is equivalent to

$$\sum_{i=1}^{n}\int_\Omega p_i(t)(e_i-u_i^0)dxdt\geq 0\quad\forall\mathbf{e}=(e_1,e_2,...,e_n)^T\in E_\Omega. \tag{28}$$

Then $p_i(t)\neq 0$. This is true, since if $p_i(s)=0$ hence the backward uniqueness property implies $p_i=0$ in $]s,\tau_1^0[$. Hence $p_i(t)=0$ and from Theorem 1, the mapping $t\to p_i(t;\mathbf{u})$ is continuous from $[0,T]\to H^1(\Omega)$, and $t\to p_i'(t;\mathbf{u})$ is continuous from $[0,T]\to L^2(\Omega)$, and so

$$p_i'(\tau_1^0;\mathbf{u}^0)=-(y_i(\tau_1^0;\mathbf{u}^0)-z_{id})=0,$$

which contradicts the fact that $y_i(\tau_1^0;\mathbf{u}^0)\neq z_{id}$.

Finally from (28) we obtain (27). □

Corollary 1. *Let the hypotheses of Theorem 6 hold. If further E_Ω is strictly convex, the control of (TOP1) is unique.*

Proof. If $\mathbf{u}^0$ and $\hat{\mathbf{u}}^0$ are two optimal controls, $\frac{\mathbf{u}^0+\hat{\mathbf{u}}^0}{2}$ is also optimal control (since U_Q is convex) and hence from (27) and the strict convexity of E_Ω we obtain $\mathbf{u}^0=\hat{\mathbf{u}}^0$. □

Remark 1. In (TOP1), if we take the observation $\mathbf{y}(t;\mathbf{u})\in C([0,T];(H^1(\Omega))^n)$ and replace the target set K_H by the target set K_V then the necessary optimality conditions coincide with (24),(25), (6) (with $u_i=u_i^0$,) and $(y_i(x,\tau_1^0;\mathbf{u}^0)-z_{id})$ in (24) is replaced by $(-\Delta+I)(y_i(x,\tau_1^0;\mathbf{u}^0)-z_{id})$

4. **Distributed control - velocity observation problem.**

In this section, We consider the following time-optimal control problem with distributed control $\mathbf{u}$ and velocity observation $\mathbf{z}=\mathbf{y}'(t;\mathbf{u})\in C([0,T];(L^2(\Omega))^n)$:

$$(TOP2):\qquad \min\Big\{t:\ \mathbf{y}'(x,t;\mathbf{u})\in K_H,\quad \mathbf{u}\in U_Q\Big\}$$

As in the above section, we can prove the following controllability theorem:

Theorem 7. *If T is large enough , then there exists a $\tau\in]0,T]$ and $\mathbf{u}\in U_Q$ with $\mathbf{y}'(\tau;\mathbf{u})\in K_H$.*

Set

$$\tau_2^0=\inf\{\tau:\mathbf{y}'(\tau;\mathbf{u})\in K_H\ \text{ for some }\ \mathbf{u}\in U_Q\}. \tag{29}$$

then similar to (TOP1), we can also prove the following theorem:

Theorem 8. *There exists an admissible control* $\mathbf{u}^0$ *to the problem (TOP2), which steering* $\mathbf{y}'(t;\mathbf{u}^0)$ *to hitting a target set* K_H *in minimum time* τ_2^0 *(defined by* (29) *). Moreover*

$$\sum_{i=1}^{n}\int_{\Omega}(y_i'(\tau_2^0;\mathbf{u}^0)-z_{id})\left(y_i'(\tau_2^0;\mathbf{u})-y_i'(\tau_2^0;\mathbf{u}^0)\right)dx\geq 0\quad\forall\mathbf{u}\in U_Q. \tag{30}$$

The condition (30) can be simplified by introducing the following adjoint equation. For each $\mathbf{u}^0\in U_Q$, we define $p(t;\mathbf{u}^0)$ as the solution of the following system

$$\left.\begin{array}{ll}\dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{u}^0)-\left(A(t)\mathbf{p}(t;\mathbf{u}^0)\right)_i=0 & \text{in }\Omega\times]0,\tau_2^0[,\\ p_i(x,\tau_2^0;\mathbf{u}^0)=(y_i'(x,\tau_2^0;\mathbf{u}^0)-z_{id}) & \text{in }\Omega,\\ p_i'(x,\tau_2^0;\mathbf{u}^0)=0 & \text{in }\Omega,\\ p_i(x,t;\mathbf{u}^0)=0 & \text{on }\Gamma\times]0,\tau_2^0[.\end{array}\right\} \tag{31}$$

Since $(y_i'(x,\tau_2^0;\mathbf{u}^0)-z_{id})\in L^2(\Omega)$, then the problem (31) can be solved in the sense of Theorem 2 and Definition 2 (with an obvious change of variables $t\to\tau_2^0-t$ and $\mathbf{y}\to\mathbf{p}$, $\phi\to\mathbf{y}(t;\mathbf{u})-\mathbf{y}(t;\mathbf{u}^0)$); Problem (31) have a unique ultra-weak solution such that

$$\int_{\Omega}(y_i'(x,\tau_2^0;\mathbf{u}^0)-z_{id})(y_i'(x,\tau_2^0;\mathbf{u})-y_i'(x,\tau_2^0;\mathbf{u}^0))dx=\int_0^{\tau_2^0}\int_{\Omega}p_i(u_i-u_i^0)dxdt.$$

Condition (30) then becomes

$$\sum_{i=1}^{n}\int_0^{\tau_2^0}\int_{\Omega}p_i(u_i-u_i^0)dxdt\geq 0\quad\forall\mathbf{u}\in U_Q. \tag{32}$$

Thus, the results in this section can be summarized as:

Theorem 9. *The optimal control* $\mathbf{u^0}$ *of problem (TOP2) is characterized by* (31),(32) *together with* (6) *(with* $u_i=u_i^0$*.) Moreover, if* U_Q *is given by* (26) *then* $\mathbf{u^0}$ *is bang-bang. If further* E_Ω *is strictly convex,* $\mathbf{u^0}$ *is unique.*

5. **Boundary control - position observation problem.** In this section, we consider the following time-optimal control problem with boundary control $\mathbf{v}\in U_\Sigma$ and observation $\mathbf{z}=\mathbf{y}(t;\mathbf{v})\in C([0,T];(L^2(\Omega))^n)$:

$$(TOP3):\qquad \min\Big\{t:\ \mathbf{y}(x,t;\mathbf{v})\in K_H,\quad \mathbf{v}\in U_\Sigma\Big\}$$

Theorem 10. *If* T *is large enough, then there exists a* $\tau\in]0,T]$ *and* $\mathbf{v}\in U_\Sigma$ *with* $\mathbf{y}(\tau;\mathbf{v})\in K_H$.

Proof. Here $\mathbf{y}(\tau;\mathbf{v})\in(L^2(\Omega))^n$. To show the system is controllable let $\psi_i(x)\in L^2(\Omega)$ such that

$$\int_{\Omega}\psi_i(x)y_i(x,\tau;\mathbf{v})dxdt=0\quad\forall\mathbf{v}\in L^2\left(0,T;\left(L^2(\Gamma)\right)^n\right).$$

We introduce $\xi=(\xi_1,\xi_2,...,\xi_n)^T$ as the solution of the following system

$$\left.\begin{array}{ll}\dfrac{\partial^2\xi_i}{\partial t^2}(t;\mathbf{v})-\left(A(t)\xi(t;\mathbf{v})\right)_i=0 & \text{in }\Omega\times]0,\tau[,\\ \xi_i(x,\tau)=0 & \text{in }\Omega,\\ \xi_i'(x,\tau)=-\psi_i(x)\in L^2(\Omega) & \text{in }\Omega,\\ \xi_i(x,t)=0. & \text{on }\Gamma\times]0,\tau[.\end{array}\right\} \tag{33}$$

Since $\psi_i(x) \in L^2(\Omega)$, then according to Theorem 2 and Definition 2 , system (33) admits an unique ultra-weak solution ξ. (after reversing sense of time), and we obtain the following identity:

$$\int_0^\tau \int_\Omega \xi_i \frac{\partial y_i}{\partial \nu} d\Gamma dt = 0;$$

hence $\xi_i = 0$ on Σ. The Cauchy data of $\xi(t;\mathbf{v})$ on Σ being zero, we conclude (see [18]) $\xi = 0$ and hence $\psi = 0$. □

Now, set

$$\tau_3^0 = \inf\{\tau : \mathbf{y}(\tau;\mathbf{v}) \in K_H \quad \text{for some} \quad \mathbf{v} \in U_\Sigma\}. \tag{34}$$

Then the following result holds (proof as in the above section).

Theorem 11. *There exists an admissible control* $\mathbf{v}^0$ *to the problem (TOP3), which steering* $\mathbf{y}(t;\mathbf{v}^0)$ *to hitting a target set* K_H *in minimum time* τ_3^0 *(defined by* (34) *). Moreover*

$$\sum_{i=1}^n \int_\Omega (y_i(\tau_3^0;\mathbf{v}^0) - z_{id}) \left(y_i(\tau_3^0;\mathbf{v}) - y_i(\tau_3^0;\mathbf{v}^0)\right) dx \geq 0 \quad \forall \mathbf{v} \in U_\Sigma \tag{35}$$

which can be interpreted as the above sections to obtaining the following theorem:

Theorem 12. *The optimal control* $\mathbf{v^0}$ *of problem (TOP3) is characterized by* (6) *(with* $v_i = v_i^0$ *) and the following system of equations and inequalities*

$$\left.\begin{aligned}
&\frac{\partial^2 p_i}{\partial t^2}(t;\mathbf{v}^0) - \left(A(t)\mathbf{p}(t;\mathbf{v}^0)\right)_i = 0 && \text{in}\,\Omega\times]0,\tau_3^0[,\\
&p_i(x,\tau_3^0;\mathbf{v}^0) = 0 && \text{in}\,\Omega,\\
&p_i'(x,\tau_3^0;\mathbf{v}^0) = -(y_i(x,\tau_3^0;\mathbf{v}^0) - z_{id}) && \text{in}\,\Omega,\\
&p_i(x,t;\mathbf{v}^0) = 0 && \text{on}\,\Gamma\times]0,\tau_3^0[,
\end{aligned}\right\} \tag{36}$$

$$\sum_{i=1}^n \int_0^{\tau_3^0} \int_\Gamma \frac{\partial p_i}{\partial \nu}(v_i - v_i^0) d\Gamma dt \geq 0 \quad \forall \mathbf{u} \in U_\Sigma. \tag{37}$$

If U_Σ is given by

$$\begin{aligned}
&U_\Sigma = \left\{\mathbf{v} \in L^2\left(0,T;\left(L^2(\Gamma)\right)^n\right), \quad \mathbf{v}(t) \in E_\Gamma\right\},\\
&E_\Gamma = \text{closed, bounded, convex subset of}\,(L^2(\Gamma))^n.
\end{aligned} \tag{38}$$

Then $\mathbf{v^0}$ is bang-bang. If further E_Γ is strictly convex, $\mathbf{v^0}$ is unique.

6. **Boundary control - velocity observation problem.** In this section, we consider the following time-optimal control problem with boundary control $\mathbf{v} \in U_\Sigma$ and observation $\mathbf{z} = \mathbf{y}'(t;\mathbf{v}) \in C([0,T];(H^{-1}(\Omega))^n)$:

$$(TOP4): \qquad \min\left\{t : \ \mathbf{y}'(x,t;\mathbf{v}) \in K_{V'}, \quad \mathbf{v} \in U_\Sigma\right\}$$

and here the norm (which is equivalent to the usual norm) in $H^{-1}(\Omega)$ is defined by

$$\|f\|_{H^{-1}(\Omega)} = \left(\int_\Omega ((-\Delta)^{-1}f)f dx\right)^{\frac{1}{2}}$$

where $(-\Delta)^{-1} f = \phi$ satisfies $-\Delta\phi = f \quad \text{in}\, \Omega, \quad \phi = 0 \quad \text{on}\, \Gamma.$

Theorem 13. *If T is large enough, then there exists a $\tau \in]0,T]$ and $\mathbf{v} \in U_\Sigma$ with $\mathbf{y}(\tau;\mathbf{v}) \in K_{V'}$.*

Proof. Here $\mathbf{y}(\tau;\mathbf{v}) \in (H^{-1}(\Omega))^n$. To show the system is controllable let $\psi_i(x) \in H^1(\Omega)$ such that

$$\Big\langle \psi_i(x), y_i(x,\tau;\mathbf{v}) \Big\rangle = 0 \quad \forall \mathbf{v} \in L^2\left(0,T;\left(L^2(\Gamma)\right)^n\right).$$

We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left.\begin{aligned}
&\frac{\partial^2 \xi_i}{\partial t^2}(t;\mathbf{v}) - \left(A(t)\xi(t;\mathbf{v})\right)_i = 0 && \text{in}\, \Omega\times]0,\tau[,\\
&\xi_i(x,\tau) = \psi_i(x) \in H^1(\Omega) && \text{in}\, \Omega,\\
&\xi_i'(x,\tau) = 0 && \text{in}\, \Omega,\\
&\xi_i(x,t) = 0. && \text{on}\, \Gamma\times]0,\tau[.
\end{aligned}\right\} \tag{39}$$

Since $\psi_i(x) \in H^1(\Omega)$, then according to Theorem 1 and Definition 1 , system (39) admits an unique weak solution ξ. (after reversing sense of time), and we obtain the following identity:

$$\int_0^\tau \int_\Omega \xi_i \frac{\partial y_i}{\partial \nu} d\Gamma dt = 0;$$

hence $\xi_i = 0$ on Σ. The Cauchy data of $\xi(t;\mathbf{v})$ on Σ being zero, we conclude (see [18]) $\xi = 0$ and hence $\psi = 0$. □

Now, set

$$\tau_4^0 = \inf\{\tau : \mathbf{y}'(\tau;\mathbf{v}) \in K_{V'} \ \text{ for some } \ \mathbf{v} \in U_\Sigma\}. \tag{40}$$

Then the following result holds (proof as in the above section).

Theorem 14. *There exists an admissible control $\mathbf{v}^0$ to the problem (TOP4), which steering $\mathbf{y}'(t;\mathbf{v}^0)$ to hitting a target set $K_{V'}$ in minimum time τ_4^0 (defined by (40)). Moreover*

$$\sum_{i=1}^n \int_\Omega -(-\Delta)^{-1}(y_i'(\tau_4^0;\mathbf{v}^0) - z_{id})\left(y_i'(\tau_4^0;\mathbf{v}) - y_i(\tau_4^0;\mathbf{v}^0)\right) dx \geq 0 \quad \forall \mathbf{v} \in U_\Sigma \tag{41}$$

which can be interpreted as the above sections to obtaining the following theorem:

Theorem 15. *The optimal control $\mathbf{v^0}$ of problem (TOP4) is characterized by (6) (with $v_i = v_i^0$) and the following system of equations and inequalities*

$$\left.\begin{aligned}
&\frac{\partial^2 p_i}{\partial t^2}(t;\mathbf{v}^0) - \left(A(t)\mathbf{p}(t;\mathbf{v}^0)\right)_i = 0 && \text{in}\, \Omega\times]0,\tau_4^0[,\\
&p_i(x,\tau_4^0;\mathbf{v}^0) = -(-\Delta)^{-1}(y_i(x,\tau_4^0;\mathbf{v}^0) - z_{id}) && \text{in}\, \Omega,\\
&p_i'(x,\tau_4^0;\mathbf{v}^0) = && \text{in}\, \Omega,\\
&p_i(x,t;\mathbf{v}^0) = 0 && \text{on}\, \Gamma\times]0,\tau_4^0[,
\end{aligned}\right\} \tag{42}$$

$$\sum_{i=1}^n \int_0^{\tau_4^0} \int_\Gamma \frac{\partial p_i}{\partial \nu}(v_i - v_i^0) d\Gamma dt \geq 0 \quad \forall \mathbf{u} \in U_\Sigma. \tag{43}$$

If U_Σ is given by (38) then $\mathbf{v^0}$ is bang-bang. If further E_Γ is strictly convex, $\mathbf{v^0}$ is unique.

Conclusions.

- Here, some time-optimal control problems for $n \times n$ Dirichlet co-operative hyperbolic linear system involving Laplace operator with distributed or boundary control have been studied. For each problem, the optimal controls are characterized in terms of an adjoint system and (for special cases of control) shown to be unique and bang-bang.
- We note that, in this paper, we have chosen to treat a special systems involving Laplace operator, just for simplicity. Most of the results we described in this paper apply, without any change on the results, to more general parabolic systems involving the following second order operator :

$$L(x,.) = \sum_{i,j=1}^{n} b_{ij}(x,.)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,.)\frac{\partial}{\partial x_j} + b_0(x,.)$$

  with sufficiently smooth coefficients (in particular, $b_{ij},\, b_j,\, b_0 \in L^{\infty}(Q),\;\; b_j,\, b_0 > 0$) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^{n} \eta_i \eta_j \geq \sigma \sum_{i=1}^{n} \eta_i \quad \forall (x,t) \in Q,$$

  for all $\eta_i \in \Re$ and some constant $\sigma > 0$.

  In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator L (see [19]).
- The results in this paper, carry over to the optimal control problems with fixed -time ([1] chapter 3), for example, the results of (TOP1) carry over to the fixed -time problem

$$\text{minimize} \sum_{i=1}^{n} \int_{\Omega} |y_i(x,T;\mathbf{u}) - z_{id}(x)|^2 dx, \quad T \text{ fixed },$$

  subject to (6) [except in the trivial case where $z_{id}(x) = y_i(x,T;\mathbf{v})$ for some admissible control $\mathbf{u}$.] This can proven in an analogous manner, as the necessary and sufficient conditions for optimality for this problem coincide with (24),(25) and (6) (with $u_i = u_i^0$).

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