Modeling of a real life problem on production network through conservation laws

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Abstract: In this paper, a real life problem on the production network has been considered. The common approach is to model the problem through the discrete event simulations (DES). The limitations on DES encourage us to model a production system through the continuous models. The impressive success of gas network and traffic flow network motivate us to approach in the direction of conservation laws. Several challenges, encountered in DES, can be incorporated through the continuous models without facing much difficulties. The scopes and limitations of the continuous models have been pointed out in details.

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1. Introduction

A production network may involve different stages: suppliers, manufacturers, retailers, customers to move certain product from suppliers to customers. In this paper, a production network is characterized by a forward flow of materials from suppliers to customers. Overview on the theory of supply chain has been analyzed in [4].

Let us study a real supply chain network as in Figure 1 [5]. The raw materials are acquired from a forest. Then two producers receive the raw materials and produce semifinished goods. They supply the semifinished goods to two different factories. The factories produce final products. Some of the products are same and some are different. The products are placed in the market to make it available for the customers.

One can model a supply chain network in various ways. The described problem can be viewed as a operational research problem. In Figure 2, P1 and P2 represent plants, W1 and W2 perform as warehouses and C1, C2 are retailers or customers. In

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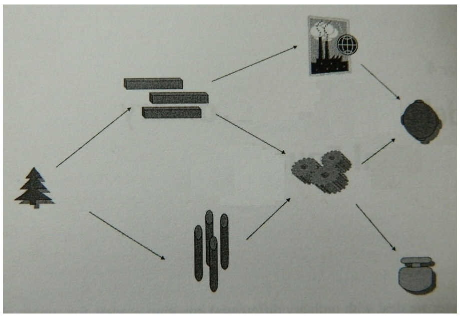

Figure 1: A real production network

operational research based approach, detailed discussions on modeling, applications and research directions can be found in [6] and [11]. Beamon et al. [2] provide a complete view on the design and management of supply chain network. For more details on strategy, planning and operation, we refer to [3].

The limitations on discrete event approach motivate us to think for the continuous models. Continuous models using differential equations have been investigated by the researchers during recent years (see [7]-[8] and the references therein). Re-entrant supply chain model and subsequent analysis can be found in [9]. Unlike the re-entrant models, our presented continuous models do not have differentiable flux function.

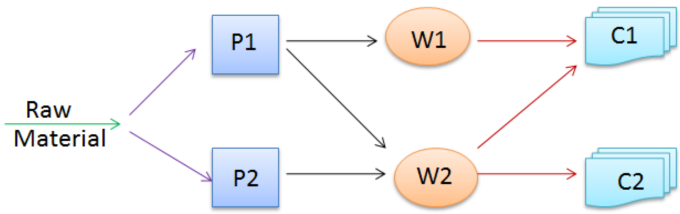

Figure 2: Viewed as a operational research problem

In this paper, our goal is to model this real life problem on production network in a continuous set up. We have represented the supplier dynamics through the conservation laws and ordinary differential equations (ODE). The motivation behind this is the availability of well developed theory on conservation laws and ODE. We have suggested several challenging features which can be handled much better way in the continuous approach.

The paper is organized as follows: we precisely describe our goal in section 2. The details of the mathematical model of the above described problem from discrete to continuous approach are described rigorously in section 3. In section 4, we shall suggest the applicability and limitation of continuous models. Concluding remarks and further research prospects are mentioned in section 5.

2. Problem Description

One can consider the path of DES to model the production network. If the quantity of the goods are very high, simulation of discrete model will be complicated and computationally expensive. It is sensible and efficient to represent the supply chain network as a model of material flows, i.e. through continuous models using the partial differential equations (PDE). Our goal in this paper is to model such a network through the conservation laws. We also focus on the development of the model by treating various aspects to analyze the production system in a better way.

3. Mathematical Model

3.1. Discrete Model

We assume FIFO (First In First Out) policy in the production system. Before we proceed in the direction of continuous models (CM), we require the fundamental information from DES. Each supplier contains some amount of goods which are measured in unit of parts. DES is based on the consideration of individual parts in a production system. We track each item in the supplier, i.e. in mathematical sense, we compute the arrival time of each part in the supplier.

Each supplier is characterized by its throughput time (T) and its maximal processing rate (μ). In DES, modeling of the queues are essential. State of the queue will be either empty or non-empty. Whenever the queue is non-empty, the part has to wait and the waiting time is inversely proportional with the processing rate. Let m denotes the corresponding supplier. If $\tau(m,n)$ denotes the arrival time of part n in supplier m, then it leads to the following recursive relation [1]:

$$\tau(m+1,n) = \max\left\{\tau(m,n) + T(m), \tau(m+1,n-1) + \frac{1}{\mu(m,n-1)}\right\}. \qquad (3.1)$$

In the problem of consideration, each supplier will satisfy the above recursive relation with the prescribed initial and influx conditions. If the number of suppliers or the number of goods becomes very high, the computation in DES will be a daunting task. Another point is that no experiment can be performed with a whole factory. Moreover, DES is not scalable to a whole production network. These are the drawbacks of DES. Inevitably, a continuous approach is necessary.

3.2. Continuous model

We associate each supplier onto one grid point in space. By performing the following steps: asymptotic analysis, scaling, dimensionless formulation, interpolation and weak formulation and introducing virtual processors (to validate in finite suppliers case), under appropriate assumptions we can represent the supply chain dynamics through conservation laws (for more details, see [1]).

The materials flow from one supplier to next supplier. Materials or goods will not enter into the subsequent supplier if the supplier is already saturated with the materials, i.e. it reaches up to its maximal capacity. The CM should incorporate this important feature. It is quite evident that if we contemplate on supply chain like a fluid flow models, it does not reflect the actual scenario. So, how does one overcome this fact?

One way is to model the queue separately and make it coupled with the governing PDE's. Every supplier j can be parameterized by an interval $[a_j, b_j]$. It is reasonable to choose x as a continuous variable representing the completion of the product within the supplier. Raw materials entered into the supplier are represented by the parts at $x = a_j$. The finished products are going out of the supplier at $x = b_j$. Let $\rho_j(x,t)$ be the density of parts at stage x and time t. We consider $u(t)$ as arrival rate of the raw materials into the production network.

Yield loss is an unavoidable feature in a production system. Continuous model should incorporate this fact. Let $(y_l)_j(x,t)$ denote the yield-loss at stage x and time t at supplier j, which can be considered as a function of parts density: $(y_l)_j = y_l(\rho_j(x,t))$. Assume that $\mu_j(x,t)$ is the maximal processing rate.

The velocity of the product moving in the supplier is represented by $v_j(x,t)$. Velocity or speed of the material flow may not be constant throughout. Hence several velocity forms are pointed out in [12]. The velocity of materials basically depends on the work in progress (WIP) of the supplier [10].

The initial situation of the suppliers is described by $\rho_{j,0}(x)$. This basically represent the amount of materials present in the suppliers at the start of the process. We need to be careful regarding the influx condition. The first node need to be treated separately. We impose an influx which is eventually influx of the supply chain for the first node. Influx for other suppliers will be described through the equation of the queues.

We consider a queue in front of each supplier to buffer the demand of the supplier. Each queue is a time dependent function. Depending on the demand of the supplier and supply of the previous suppliers, the queue increases or decreases. Mathematically, dynamics of the queue can be represented by the ordinary differential equation. Initial value of the queue will be prescribed, which describes the parts present in the queue at the start of the process.

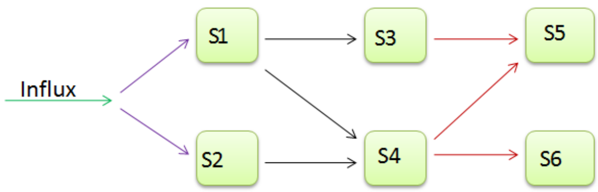

Figure 3: Viewed as a material flow problem

In our real life problem, introduced above, we assume that there are no queue for supplier S1 and S2 since raw material comes directly to these suppliers. Now, we concentrate on the queue equations for supplier S3 and S4. Some of semifinished products from S1 transported to S3 and remaining products to S4. Let r_1 be the percentage of goods received by S3 from S1. Queue equations in front S3 and S4 will be

$$\frac{dq_3}{dt} = r_1 f_1(\rho_1(b_1,t)) - (y_l)_3(a_3,t) - f_3(\rho_3(a_3,t)),$$

where

$$f_3(\rho_3(a_3,t)) = \begin{cases} \min\{r_1 f_1(\rho_1(b_1,t)) - (y_l)_3(a_3,t), \mu_3(t)\} & \text{if } q_3(t) = 0 \\ \mu_3(t) & \text{if } q_3(t) \neq 0. \end{cases}$$

$$\frac{dq_4}{dt} = (1-r_1) f_1(\rho_1(b_1,t)) + f_2(\rho_2(b_2,t)) - (y_l)_4(a_4,t) - f_4(\rho_4(a_4,t)),$$

where $f_4(\rho_4(a_4,t)) =$

$$\begin{cases} \min\{(1-r_1)f_1(\rho_1(b_1,t)) + f_2(\rho_2(b_2,t)) - (y_l)_4(a_4,t), \mu_4(t)\} & \text{if } q_4(t) = 0 \\ \mu_4(t) & \text{if } q_4(t) \neq 0. \end{cases}$$

Let r_2 be the percentage of goods received by S5 from S4. Similarly, we can model the queues for S5 and S6. The dynamics of every supplier is represented by the nonlinear conservation laws.

Let us assume that $\lambda_j(t)$ be the additional materials fed into supplier j from outside of the chain. This is sensible if one observe a real supply chain carefully. Several ingredients need to be provided form outside the chain to make the final product more desirable to the customers. To make the model more adoptable to any complex supply chain network, we assign A as a connectivity matrix which was introduced in [12]. This helps us to treat the model in more efficient way. Finally, we formulate the real supply chain problem using the conservation laws.
Governing equation:

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial f_j}{\partial x} + y_l(\rho_j) = 0, \quad t > 0, \quad x \in (a_j, b_j], \qquad j = 1, 2, ..., 6,$$

where the flux function f is defined as

$$f_j(x,t) := \min\{\mu_j(x,t), v_j(x,t)\rho_j(x,t)\}.$$

Initial condition:

$$\rho_j(x,0) = \rho_{j,0}(x), \quad \forall x \in [a_j, b_j], \quad j = 1, 2, ..., 6.$$

Influx condition :

$$f_1(0,t) = u(t), \quad t > 0.$$

Queue equation: For supplier $j = \ 2, \ 3, ..., \ 6,$

$$\frac{dq_j}{dt} = \lambda_j(t) + \sum_{k=1}^{6} A_{jk} f_k(b_j,t) - (y_l)_j(a_j,t) - f_j(a_j,t),$$
$$q_j(0) = q_{j,0},$$

where $f_j(a_j,t)$ is defined as follows

$$f_j(a_j,t) = \begin{cases} \min\left\{\lambda_j(t) + \sum_{k=1}^{6} A_{jk} f_k(b_j,t) - (y_l)_j(a_j,t), \mu_j(t)\right\} & \text{if } q_j(t) = 0 \\ \mu_j(t) & \text{if } q_j(t) \neq 0. \end{cases}$$

Velocity form: $v_j = v(W_j(t))$, where $W_j(t)$ denotes work in progress in supplier j at time t. $W_j(t) = \displaystyle\int_0^1 \rho_j(x,t)dx$.

The connectivity matrix A is of order 6×6. A_{jk} denotes the fraction of products flowing from supplier k to supplier j. The matrix A has the following properties.

(i) $A_{kk} = 0, \ \forall \ k = 1, 2, ..., 6.$

(ii) $A_{jk} = 0$, for $j = 1, 2$ and $k = 1, 2, ..., 6$ since S1 and S2 are entrant suppliers in the chain.

(iii) $A_{jk} \geq 0, \ \forall \ j, k$ since we have not considered any backward flow in the chain.

(iv) $\sum_{j=1}^{6} A_{jk} \leq 1, \ \forall \ k$ since outflux is non-negative.

Explicitly, the matrix A will be

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & 0 & 0 & 0 \\ 1 - r_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_2 & 0 & 0 \\ 0 & 0 & 0 & 1 - r_2 & 0 & 0 \end{pmatrix}$$

4. Scopes and Limitations of Continuous Model

In this section, we focus on the features of DES which can be treated through continuous models (CM). Also we pointed out the limitations of CM.

- A supply chain is characterized by two integrated processes. One is production planning and inventory control process; other one is the distribution and logistics process.

  (i) Production planning describes the design and management of entire manufacturing process which includes material scheduling and control. In CM, we deal with this issue by analyzing the density distribution and velocity control of material flow.

  (ii) Inventory control describes the design and management of storage policies. In CM, work in progress (WIP) and the bottlenecks in the supply chain apprise the same.

  (iii) Distribution and logistic process determines the way products are transported to retailers. In CM, we introduce r_k (like in above model) as the percentage of products transported to the retailers and subsequently it is used to control the process.

- The time varying or dynamic behavior of a supply chain can be described very efficiently through the CM.

- Since the CM is independent of the amount of individual goods, they allow for fast computing times even for a large number of parts or a large number of suppliers.

- Incorporating the yield loss in CM is quite efficient than in DES.

- In several supply chain problems, extra materials need to be provided from the outside. This feature can be easily taken into account in CM.
- In CM, optimal control problems are difficult to solve due to the continuous constraints.
- In DES, various aspects like the bounded buffers, location of facilities can be included. But these are very challenging to incorporate in the CM.

5. Conclusion

We have presented a continuous model for a real life problem on production network incorporating various features observed in a production system. A comparison study has been carried out between the continuous approach and discrete approach. Several challenges faced in the DES can be treated much efficiently through the continuous models.

An important future extension of the current work is to incorporate the random variation components into the modeling system for treating the associated stochastic partial differential equations. A suitable numerical scheme in terms of efficiency and accuracy is also desirable. Several optimization aspects can be analyzed by considering the presented conservation law model.

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