AMO - Advanced Modeling and Optimization, Volume 16, Number 2, 2014

PRODUCT CORDIAL LABELING FOR SOME BISTAR RELATED GRAPHS

S K Vaidya[1]
Department of Mathematics, Saurashtra University, Rajkot-360 005, GUJARAT (INDIA).
E-mail: *samirkvaidya@yahoo.co.in*

N B Vyas
Atmiya Institute of Technology and Science, Rajkot-360 005, GUJARAT (INDIA).
E-mail: *niravbvyas@gmail.com*

Abstract. For the graph $G = (V(G), E(G))$, a function $f : V(G) \to \{0, 1\}$ is called a product cordial labeling of G if the induced edge labeling function defined by the product of end vertex labels be such that the edges with label 1 and label 0 differ by at most 1 and the vertices with label 1 and label 0 also differ by at most 1. Here we investigate product cordial labeling for some bistar related graphs.

Key words: Cordial labeling, Porduct cordial labeling, Bistar.

AMS Subject classification: 05C78.

1. Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$. For standard terminology and notations we follow Chartrand and Lesniak[2].

The graph theory is mainly evolved with the rise of computer age. This theory has rigorous applications in diversified fields like chemistry, physics, computer science, communication engineering, electrical engineering and social sciences.

One of the emerging fields in graph theory is the labeling of discrete structures. If the vertices are assigned values subject to certain condition(s) then it is known as *graph labeling*. If the domain of mapping is the set of vertices (edges) then the labeling is called a vertex (an edge) labeling.

[1]Corresponding author.

An extensive survey on various graph labeling problems with bibliographic references can be found in Gallian[3].

A function f is called *graceful labeling* of graph $G = (V(G), E(G))$ if $f : V(G) \to \{0, 1, 2, \ldots, |E(G)|\}$ is injective and for each edge $e = uv$ the induced function $f^* : E(G) \to \{1, 2, \ldots, |E(G)|\}$ defined by $f^*(e = uv) = |f(u)-f(v)|$ is bijective. A graph G is *graceful* if it admits a graceful labeling.

The famous Ringel-Kotzig [5] tree conjecture and many illustrious works on graceful graphs brought a tide of different ways of labeling the elements of graph such as odd graceful labeling, harmonious labeling etc. Graham and Sloane[4] have introduced the concept of harmonious labeling during their study on modular versions of additive bases problems stemming from error correcting codes.

A function f is called *harmonious labeling* of a graph $G = (V(G), E(G))$ if $f : V(G) \to Z_q$ is injective and for each edge $e = uv$ the induced function $f^* : E(G) \to Z_q$ defined by $f^*(e = uv) = (f(u) + f(v))(mod\, q)$ is bijective. A graph G is *harmonious* if it admits harmonious labeling.

A mapping $f : V(G) \to \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of vertex v of G under f.
Notations: For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$ then $v_f(i) =$ the number of vertices of G having label i under f and $e_f(i) =$ the number of edges of G having label i under f^* for $i = 0, 1$.

The concept of cordial labeling was introduced by Cahit[1] as a weaker version of graceful and harmonious labeling.

A binary vertex labeling of graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *cordial* if it admits cordial labeling.

Motivated through the concept of cordial labeling, Sundaram *et al.* [6] have introduced a labeling which has the flavour of cordial lableing but absolute difference of vertex labels is replaced by product of vertex labels.

A binary vertex labeling of graph G which induces edge labeling $f^* : E(G) \to \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *product cordial* if it admits product cordial labeling.

Many researchers have explored this concept. The graph structures like trees, unicyclic graphs of odd order, triangular snakes, dragons, helms and union of two path graphs are product cordial as reported in the work of Sundaram *et al.* [6]. In the same paper it has been proved that a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < \frac{p^2-1}{4} + 1$. Vaidya and Dani[8] have proved that the graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are product cordial while Vaidya and Kanani[9] have reported the product cordial labeling for some cycle related graphs and investigated product cordial labeling for the shadow graph of cycle C_n. The same authors have investigated some new product cordial graphs in [10]. Vaidya and Vyas [11] have investigated product cordial labeling in the context of tensor product of some standard graphs. The product cordial labelings for closed helm, web graph, flower graph, double triangular snake and gear graph are reported in Vaidya and Barasara [7].

Now we will give brief summary of definitions which are useful for the present paper. For a graph G the splitting graph $S'(G)$ is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighbourhood sets of v and v' respectively. The *Duplication of a vertex* v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G' such that $N(v'_k) \cap N(v''_k) = \{v_k\}$. The *Duplication of an edge* $e = v_i v_{i+1}$ by a vertex v' in a graph G produces a new graph G' such that $N(v') = \{v_i, v_{i+1}\}$. For a simple graph G the *square of a graph* G is denoted by G^2 and defined as the graph obtained with the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance at most 2 apart in G. For a connected graph G, let G' be the copy of G then shadow graph $D_2(G)$ is obtained by joining each vertex u in G to the neighbours of the corresponding vertex u' in G'.

2. Main Results

Theorem 2.1: $S'(B_{n,n})$ is a product cordial graph.

Proof: Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i and v_i are pendant vertices. In order to obtain $S'(B_{n,n})$, add vertices u', v', u'_i, v'_i corresponding to u, v, u_i, v_i, respectively where $1 \leq i \leq n$. Let $G = S'(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 6n+3$. We define $f : V(G) \to \{0, 1\}$ as follows:

$f(u) = 1,\ f(v) = 1;$

$f(u') = 0,\ f(v') = 1;$

For $1 \leq i \leq n$:
$f(u_i) = 0;$
$f(u'_i) = 0;$

$f(v_i) = 1$;

For $1 \leq i \leq n-1$:
$f(v_i') = 1$;
$f(v_n') = 0$.

In view of above defined labeling patterns we have

$v_f(0) = v_f(1) = 2n+2$, $e_f(0) = e_f(1) + 1 = 3n+2$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, $S'(B_{n,n})$ is a product cordial graph.

Illustration 2.2: $S'(B_{6,6})$ and its product cordial labeling is shown in *Figure 1*.

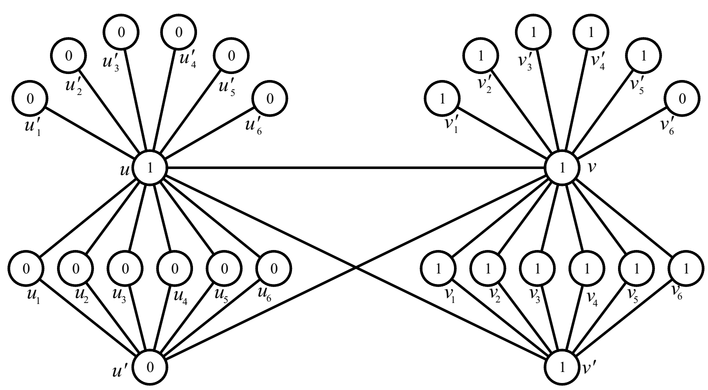

Figure 1

Theorem 2.3: Duplicating each edge by a vertex in $B_{n,n}$ is a product cordial graph.

Proof: Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i and v_i are pendant vertices. Let G be the graph obtained from $B_{n,n}$ by duplicating each edge uu_i by a vertex u_i', vv_i by a vertex v_i' and uv by a vertex w, where $1 \leq i \leq n$. We note that $V(G) = \{u, v, w, u_i, v_i, u_i', v_i'/1 \leq i \leq n\}$ such that $|V(G)| = 4n+3$ and $|E(G)| = 6n+3$. We define $f : V(G) \to \{0, 1\}$ as follows:

$f(u) = 1$; $f(v) = 1$; $f(w) = 0$;

For $1 \leq i \leq n$:

$f(u_i) = 0$;
$f(u'_i) = 0$;
$f(v_i) = 1$;
$f(v'_i) = 1$;

In view of above defined labeling patterns we have

$v_f(0) + 1 = v_f(1) = 2n + 2$, $e_f(0) = e_f(1) + 1 = 3n + 2$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, G is a product cordial graph.

Illustration 2.4: The graph obtained by duplicating each edge of $B_{3,3}$ by a vertex and its product cordial labeling is shown in *Figure 2*.

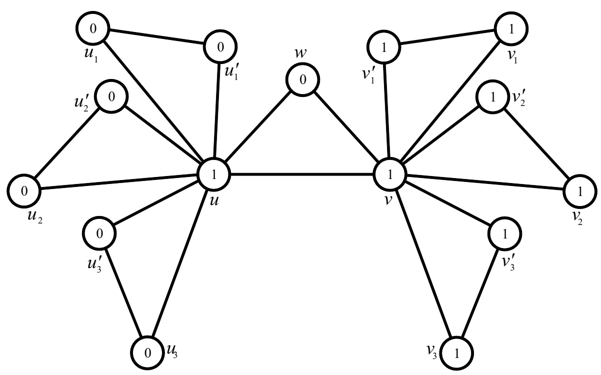

Figure 2

Theorem 2.5: Duplicating each vertex by an edge in $B_{n,n}$ is a product cordial graph.

Proof: Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i and v_i are pendant vertices. Let G be the graph obtained from $B_{n,n}$ by duplicating each vertex u_i by an edge $u'_i u''_i$, v_i by an edge $v'_i v''_i$, u by an edge $w'_1 w''_1$ and v by an edge $w'_2 w''_2$, where $1 \leq i \leq n$. We note that $V(G) = \{u, v, u_i, v_i, u'_i, v'_i, u''_i, v''_i, w'_j, w''_j / 1 \leq i \leq n; j = 1, 2\}$ such that $|V(G)| = 6n + 6$ and $|E(G)| = 8n + 7$. We define $f : V(G) \rightarrow \{0, 1\}$ as follows:

$f(u) = 0$; $f(v) = 1$;

<u>For $1 \leq i \leq n$:</u>
$f(u_i) = 0$;
$f(u'_i) = 0$;

$f(u_i'') = 0;$
$f(v_i) = 1;$
$f(v_i') = 1;$
$f(v_i'') = 1;$

<u>For $1 \leq j \leq 2$:</u>

$$f(w_j') = \begin{cases} 0, & j \equiv 1(\text{mod } 2); \\ 1, & otherwise. \end{cases}$$
$$f(w_j'') = \begin{cases} 0, & j \equiv 1(\text{mod } 2); \\ 1, & otherwise. \end{cases}$$

In view of above defined labeling patterns we have

$v_f(0) = v_f(1) = 3n + 3$, $e_f(0) = e_f(1) + 1 = 4n + 4$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence, G is a product cordial graph.

Illustration 2.6: The graph obtained by duplicating each vertex of $B_{3,3}$ by an edge and its product cordial labeling is shown in *Figure 3*.

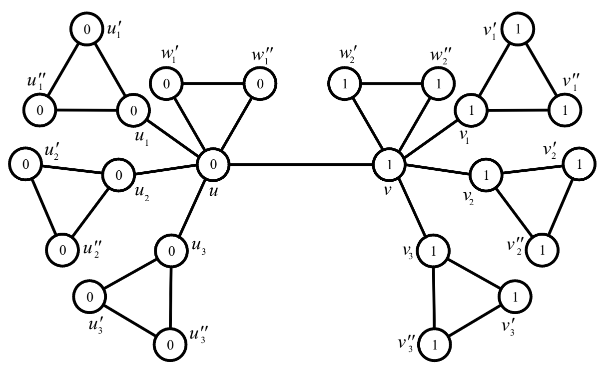

Figure 3

Theorem 2.7: $B^2_{n,n}$ is not a product cordial graph.

Proof: For the graph $B^2_{n,n}$, $|V(B^2_{n,n})| = 2n + 2$ and $|E(B^2_{n,n})| = 4n + 1$. Here degree of every vertex is at least 2. In order to make $B^2_{n,n}$ a product cordial graph, we have to assign label 0 to $n + 1$ vertices out of $2n + 2$ vertices which will generate atleast $2n + 2$ edges having label 0. Therefore $|e_f(0) - e_f(1)| = (2n + 2) - (2n - 1) = 3 > 1$. Hence $B^2_{n,n}$ is not a product cordial graph.

Theorem 2.8: $D_2(B_{n,n})$ is not a product cordial graph.

Proof: For the graph $D_2(B_{n,n})$, $|V(D_2(B_{n,n}))| = 4n+4$ and $|E(D_2(B_{n,n}))| = 8n+4$. Here degree of every vertex is at least 2. In order to make $D_2(B_{n,n})$ a product cordial graph, we have to assign label 0 to exactly $2n+2$ vertices out of $4n+4$. Hence these vertex label generate atleast $4n+4$ edges with label 0. Therefore $|e_f(0)-e_f(1)| = (4n+4)-(4n) = 4 > 1$. Hence $D_2(B_{n,n})$ is not a product cordial graph.

3. Concluding Remarks

The bistar is a tree and trees are product cordial graphs as reported in Sundaram *et al.* [6] while we have investigated product cordial labeling for the larger graphs obtained from bistar(tree) by means of some graph operations.

References

[1] I. Cahit, Cordial Graphs: A weaker version of graceful and harmonious Graphs, Ars Combinatoria, 23(1987), 201-207.
[2] G. Chartrand and L. Lesniak, Graphs and Digraphs, 4/e, CRC Press, 2004.
[3] J. A. Gallian, A dynamic survey of graph labeling, The Electronics Journal of Combinatorics, 16(# DS6), 2013.
[4] R. L. Graham and N. J. A. Sloane, On additive basis and harmonious graphs, SIAM Journal on Algebraic and Discrete Methods, 1(4) (1980), 382-404.
[5] G. Ringel, Problem 25 in: Theory of graphs and its applications, Proceedings of Symposium, Smolenice 1963, Prague (1964), 162.
[6] M. Sundaram, R. Ponraj and S. Somsundaram, Product cordial labeling of graphs, Bull. Pure and Applied Sciences(Mathematics and Statistics), 23E (2004), 155-163.
[7] S. K. Vaidya and C. M. Barasara, Further results on product cordial labeling, International Journal of Math. Combin., 3 (2012), 64-71.
[8] S. K. Vaidya and N. A. Dani, Some new product cordial graphs, Journal of App. Comp. Sci. Math., 8(4) (2010), 63-66.
[9] S. K. Vaidya and K. K. Kanani, Some cycle related product cordial graph, International Journal of Algorithms, Comp. and Math., 3(1) (2010), 109-116.
[10] S. K. Vaidya and K. K. Kanani, Some new product cordial graphs, Mathematics Today, 27 (2011), 64-70.
[11] S. K. Vaidya and N. B. Vyas, Product Cordial Labeling in the Context of Tensor Product of Graphs, Journal of Mathematics Research, 3(3) (2011), 83-88.