Generalized Complementarity Problems Based on Generalized Fisher-Burmeister Functions as Unconstrained Optimization[1]

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Abstract

In this article, we consider an unconstrained minimization reformulation of the generalized complementarity problem GCP(f, g) based on the generalized Fisher-Burmeister function. Starting with C^1 functions f and g, we show under certain conditions any stationary point of the unconstrained minimization problem is a solution to GCP(f, g).

Key words: Generalized complementarity problem, GCP function, generalized FB function, merit function, unconstrained minimization, stationary point.
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1 Introduction

We consider a generalized complementarity problem corresponding to C^1 functions f and g, denoted by GCP(f,g), which is to find a vector $x^* \in \Re^n$ such that

$$f(x^*) \geq 0, \quad g(x^*) \geq 0 \quad \text{and} \quad \langle f(x^*), g(x^*)\rangle = 0 \tag{1}$$

where $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$.

For the formulation, numerical methods, and applications of GCP(f,g), see [12], [14], [18] and the references cited therein. Also GCP(f,g) covers some well known problems studied in the literature in the last decade; for example, if $g(x) = x$, then GCP(f,g) reduces to the nonlinear complementarity problem NCP(f). By taking in NCP(f) $f(x) = Mx + q$ with $M \in R^{n\times n}$ and a vector $q \in R^n$, then NCP(f) is called a linear complementarity problem LCP(M,q). Also, if $g(x) = x - W(x)$ with some $W : R^n \to R^n$, then GCP(f,g) is known as the quasi/implicit complementarity problem, see e.g., [14], [17], [19].

These problems have numerous applications in diverse fields such as optimization, engineering, economics and other areas, see e.g., [4], [5], [8], [9], [11], [20], and the references therein.

A function $\phi : R^2 \to R$ is called a GCP function if

$$\phi(a,b) = 0 \Leftrightarrow ab = 0, a \geq 0,\ b \geq 0.$$

For the problem GCP(f,g), we define

$$\Phi(x) = \begin{bmatrix} \phi(f_1(x), g_1(x)) \\ \vdots \\ \phi(f_i(x), g_i(x)) \\ \vdots \\ \phi(f_n(x), g_n(x)) \end{bmatrix} \tag{2}$$

and, call $\Phi(x)$ a GCP function for GCP(f,g).

Our goal from this paper is to study a generalized complementarity problem GCP(f,g) based on the generalized Fisher-Burmeister function when the underlying functions f and g are C^1. By considering a GCP function $\Phi : R^n \to R^n$ associated with GCP(f,g) and its merit function

$$\Psi(x) := \frac{1}{2}\|\Phi(x)\|^2. \tag{3}$$

so that

$$\bar{x} \text{ solves GCP}(f,g) \ \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \Psi(\bar{x}) = 0.$$

If we assume GCP(f,g) has at least one solution, then a vector $\bar{x} \in R^n$ solves GCP(f,g) if and only if it is a global/local minimizer (a stationary point) of the unconstrained minimization problem

$$\min_{x \in R^n} \Psi(x).$$

In this paper, we show how, under appropriate $\mathbf{P}_0(\mathbf{P})$, positive definite (semidefinite)-conditions on H-differentials of f and g, finding local/global minimum of Ψ (or a 'stationary point' of Ψ) leads to a solution of the given generalized complementarity problem. Further, we show that how our results unify/extend various similar results proved in the literature for nonlinear complementarity problem when the underlying functions are C^1.

2 Preliminaries

Throughout this paper, we regard vectors in R^n as column vectors. We denote the inner-product between two vectors x and y in R^n by either $x^T y$ or $\langle x, y \rangle$. Vector inequalities are interpreted componentwise. For a matrix A, A_i denotes the ith row of A. For a differentiable function $f : R^n \to R^m$, $\nabla f(\bar{x})$ denotes the Jacobian matrix of f at $\bar{x}$.

We need the following definition from [4].

Definition 2.1 [(i)] *A matrix $A \in \Re^{n\times n}$ is called semimonotone ($\mathbf{E_0}$) (strictly semimonotone ($\mathbf{E}$))-matrix if*

$$\forall x \in \Re^n_+, x \neq 0, \textit{there exists } i \textit{ such that} \quad x_i \neq 0 \textit{and} \quad x_i(Ax)_i \geq 0 (> 0).$$

[(ii)] *A matrix $A \in \Re^{n\times n}$ is called $\mathbf{P_0}$ ($\mathbf{P}$))-matrix if*

$$\forall x \in \Re^n, x \neq 0, \textit{there exists } i \textit{ such that} \quad x_i \neq 0 \textit{and} \quad x_i(Ax)_i \geq 0 (> 0).$$

In [21], the author generalized the concepts of monotonicity, $\mathbf{P_0}$-property and their variants for functions and use them to establish some conditions to get a solution for generalized complementarity problem when the underlying functions f and g are H-differentiable. .

Let us recall the following definitions from [21].

Definition 2.2 *For functions $f, g : \Re^n \to \Re^n$, we say that f and g are:*
(a) Relatively monotone if

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0 \text{ for all } x, y \in \Re^n.$$

(b) Relatively strictly monotone if

$$\langle f(x) - f(y), g(x) - g(y) \rangle > 0 \text{ for all } x, y \in \Re^n.$$

(c) Relatively strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq \mu \|x - y\|^2 \text{ for all } x, y \in \Re^n.$$

(d) Relatively $\mathbf{P_0}$ ($\mathbf{P}$)-functions if for any $x \neq y$ in $\Re^n$,

$$\max_{i: x_i \neq y_i} [f(x) - f(y)]_i [g(x) - g(y)]_i \geq (>) 0.$$

(e) Relatively uniform $(\mathbf{P})$-functions if there exists a constant $\eta > 0$ such that for any $x, y \in \Re^n$,

$$\max_{1 \leq i \leq n} [f(x) - f(y)]_i [g(x) - g(y)]_i \geq \eta \|x - y\|^2.$$

Note that relatively strongly monotone functions are relatively strictly monotone, and relatively strictly monotone functions are relatively monotone. Also we note that every relatively monotone (strictly monotone) function is a relatively $\mathbf{P}_0(\mathbf{P})$-function.

There are some relations between f, g and $f \circ g^{-1}$ when g is one-to-one and onto, which are given in [21].

Lemma 2.1 *Suppose that $f, g : \Re^n \to \Re^n$ and g is one-to-one and onto. Define $h : \Re^n \to \Re^n$ where $h := f \circ g^{-1}$. The following hold:*
(a) f and g are relatively (strictly) monotone if and only if h is (strictly) monotone.
(b) If g is Lipschitz-continuous, and f and g are relatively strongly monotone then h is strongly monotone.
(c) f and g are relatively $\mathbf{P_0}$ ($\mathbf{P}$))-functions if and only if h is $\mathbf{P_0}$ ($\mathbf{P}$))-function.
(d) If g is Lipschitz-continuous, and f and g are relatively uniform ($\mathbf{P}$))-functions, then h is uniform ($\mathbf{P}$))-function.

The following result is from [16].

Theorem 2.1 *Under the following conditions, $f : R^n \to R^n$ is a $\mathbf{P}_0(\mathbf{P})$-function. f is Fréchet differentiable on R^n and for every $x \in R^n$, the Jacobian matrix $\nabla f(x)$ is a $\mathbf{P}_0(\mathbf{P})$-matrix.*

Remark 2.1 *Based on some results in [16], we note the following. For $\mathbf{P}$-conditions, the the converse statements in the above theorem are usually false.*

3 Minimizing the merit function

Over the past two decades, a variety of NCP-functions have been studied, see [10] and references therein. Among which, some families of NCP functions [2, 1, 13] based on the Fisher-Burmeister function with p-norm are proposed. The family NCP functions are proposed in [2]:

$$\phi_p(a, b) := a + b - \|(a, b)\|_p \tag{4}$$

where p is any fixed real number in the interval $(1, +\infty)$ and $\|(a, b)\|_p$ denotes the p-norm of (a, b), i.e., $\|(a, b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. Based on the functions (4), some more NCP functions are introduced in [1]:

$$\phi_1(a, b) := \phi_p(a, b) + \alpha a_+ b_+, \alpha > 0. \tag{5}$$

$$\phi_2(a, b) := \phi_p(a, b) + \alpha(ab)_+, \alpha > 0. \tag{6}$$

$$\phi_3(a, b) := \sqrt{[\phi_p(a, b)]^2 + \alpha(a_+ b_+)^2}, \alpha > 0. \tag{7}$$

$$\phi_4(a, b) := \sqrt{[\phi_p(a, b)]^2 + \alpha[(ab)_+]^2}, \alpha > 0. \tag{8}$$

Our objective in this article is to study GCP functions based on these NCP functions. For given C^1- functions $f : R^n \to R^n$ and $g : R^n \to R^n$, we consider the associated GCP function Φ and the corresponding merit function

$$\Psi_*(\bar{x}) := \frac{1}{2}\|\Phi_*(\bar{x})\|^2 = \sum_{i=1}^{n} \psi_*(f_i(\bar{x}), g_i(\bar{x})), \tag{9}$$

where

$$\Phi_*(\bar{x}) := \begin{pmatrix} \phi_*(f_1(\bar{x}), g_1(\bar{x})) \\ \vdots \\ \phi_*(f_n(\bar{x}), g_n(\bar{x})) \end{pmatrix}, \tag{10}$$

and

$$\psi_*(a,b) := \frac{1}{2}\phi_*(a,b)^2, \tag{11}$$

with $* \in \{\{1,p\},1,2,3,4\}$.

It should be recalled that

$$\Psi_*(\bar{x}) = 0 \Leftrightarrow \Phi_*(\bar{x}) = 0 \Leftrightarrow \bar{x} \text{ solves GCP}(f,g).$$

In the following proposition, we give favorable properties for ψ.

Proposition 3.1 *Let $\psi \in \{\psi_{1,p}, \psi_1, \psi_2, \psi_3, \psi_4\}$ be defined in (9). Then ψ has the following favorable properties:*

(a) *ψ is a nonnegative, i.e., $\psi(a,b) \geq 0$ for all $(a,b) \in \Re^2$.*

(b) *ψ is continuously differentiable everywhere.*

(c) *$\nabla_a\psi(a,b) \cdot \nabla_b\psi(a,b) \geq 0$ for all $(a,b) \in \Re^2$.*

(d) $\psi(a,b) = 0 \Leftrightarrow \nabla\psi(a,b) = 0 \Leftrightarrow \nabla_a\psi(a,b) = 0 \Leftrightarrow \nabla_b\psi(a,b) = 0.$

Proof. When $\psi = \psi_{1,p}$, the results (a)-(d) can be obtained from [2, Proposition 3.2 (a)-(e)] respectively. When $\psi \in \{\psi_1, \psi_2, \psi_3, \psi_4\}$, the results (a)-(d) can be obtained from [1, Proposition 3.3 (a)-(d)] respectively. □

Now we minimize the merit function under $\mathbf{P_0}$-conditions.

Theorem 3.1 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$:*

(i) *Ψ is continuously differentiable,*

(ii) *$(\nabla_a\Psi(f(x), g(x)))_i(\nabla_b\Psi(f(x), g(x)))_i \geq 0$, for any $x \in \Re^n$,*

(iii) *$(\nabla_a\Psi(f(x), g(x)))_i(\nabla_b\Psi(f(x), g(x)))_i \neq 0$ whenever $\Phi_i(x) \neq 0$,*

(iv) $(\nabla_a \Psi(f(x), g(x)))_i = 0 \Leftrightarrow (\nabla_b \Psi(f(x), g(x)))_i = 0 \Leftrightarrow \Phi_i(x) = 0.$

Suppose that $\nabla g(x)^{-1} \nabla f(x)$ *is a* $\mathbf{P_0}$*-matrix for any* $x \in \Re^n$*, then* x^* *is a stationary point of* Ψ *if and only if* x^* *is a solution of* $GCP(f, g)$.

Proof. "$\Leftarrow$" Suppose that x^* is a solution of $GCP(f, g)$, then $\Phi(x^*) = 0$, and from the property (iv), we have

$$\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,$$

that is, x^* is a stationary point of Ψ.

"$\Rightarrow$"

Suppose that x^* is a stationary point of Ψ, i.e.,

$$\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,$$

then

$$\nabla g(x^*)^{-1} \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla_b \Psi(f(x^*), g(x^*)) = 0. \tag{12}$$

We want to prove that x^* is a solution of $GCP(f, g)$, i.e., $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $U \subseteq I := \{1, 2, \ldots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I \backslash U$. We have

$$(\nabla_a \Psi(f(x^*), g(x^*)))_i \neq 0, (\nabla_b \Psi(f(x^*), g(x^*)))_i \neq 0, \forall i \in U \tag{13}$$

and $(\nabla_a \Psi(f(x^*), g(x^*)))_i = 0, (\nabla_b \Psi(f(x^*), g(x^*)))_i = 0, \forall i \in I \backslash U$ from the property (iv). Since $\nabla g(x^*)^{-1} \nabla f(x^*)$ is a P_0-matrix, then for $\nabla_a \Psi(f(x^*), g(x^*)) \neq 0$, $\exists i_0 \in U$ such that

$$(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \nabla g(x^*)^{-1} \nabla f(x^*) (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \geq 0. \tag{14}$$

From (12) and (14),

$$\begin{aligned} & (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} (\nabla_b \Psi(f(x^*), g(x^*)))_{i_0} \\ = \; & -(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \nabla g(x^*)^{-1} \nabla f(x^*) (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \\ \leq \; & 0. \end{aligned}$$

By the property (ii), then we have

$$(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} (\nabla_b \Psi(f(x^*), g(x^*)))_{i_0} = 0.$$

which contradicts (13). Hence, the proof is complete. □

Remark 3.1 • *From Proposition 3.1, we note that Theorem 3.1 is applicable to GCP functions in (4)-(8).*
• *If we state the above results for GCP function based on the generalized Fischer-Burmeister function (4) and replace the p-norm to 2-norm, then Theorem 3.1 reduces to Theorem 3.2 in [15]. And, when $g(x) = x$, GCP(f, g) reduces to NCP(f) and Theorem 3.1 reduces to Prop. 3.4 in [7]. Also, When $g(x) = x$, our result extends/generalizes a result obtained by Geiger and Kanzow [6] for NCP(f) under monotonicity of a function.*
• *In Theorem 3.1, if we consider GCP functions in (5)-(8) and $g(x) = x$, GCP(f, g) reduces to NCP(f) and Theorem 3.1 reduces to Prop. 3.4 in [1].*

Since every positive semidefinite matrix is also a $\mathbf{P_0}$-matrix, the proof of the following theorem will follow from Theorem 3.1.

Theorem 3.2 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, *for any* $x \in \Re^n$,

(iii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \neq 0$ *whenever* $\Phi_i(x) \neq 0$,

(iv) $(\nabla_a \Psi(f(x), g(x)))_i = 0 \Leftrightarrow (\nabla_b \Psi(f(x), g(x)))_i = 0 \Leftrightarrow \Phi_i(x) = 0$.

Suppose that $\nabla g(x)^{-1} \nabla f(x)$ is a positive semidefinite-matrix for any $x \in \Re^n$, then x^* is a stationary point of Ψ if and only if x^* is a solution of $GCP(f, g)$.

From Lemma 2.1 and view of Theorem 2.1, we now state two consequences of the above theorems

Corollary 3.1 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, *for any* $x \in \Re^n$,

(iii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \neq 0$ *whenever* $\Phi_i(x) \neq 0$,

(iv) $(\nabla_a \Psi(f(x), g(x)))_i = 0 \Leftrightarrow (\nabla_b \Psi(f(x), g(x)))_i = 0 \Leftrightarrow \Phi_i(x) = 0$.

Suppose that f and g are relatively $\mathbf{P_0}$-functions. Then x^* is a stationary point of Ψ if and only if x^* is a solution of $GCP(f,g)$.

Proof. Since g is a one-to-one and onto, and f and g are relatively $\mathbf{P_0}$-functions, by Lemma 2.1, the mapping $f \circ g^{-1}$ is $\mathbf{P_0}$-function which implies $\nabla f(x^*)\nabla g(x^*)^{-1}$ is $\mathbf{P}_0$-matrix, see Theorem 2.1. The proof follows from Theorem 3.1. □

Corollary 3.2 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is strongly monotone. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a\Psi(f(x),g(x)))_i(\nabla_b\Psi(f(x),g(x)))_i \geq 0$*, for any* $x \in \Re^n$*,*

(iii) $(\nabla_a\Psi(f(x),g(x)))_i(\nabla_b\Psi(f(x),g(x)))_i \neq 0$ *whenever* $\Phi_i(x) \neq 0$*,*

(iv) $(\nabla_a\Psi(f(x),g(x)))_i = 0 \Leftrightarrow (\nabla_b\Psi(f(x),g(x)))_i = 0 \Leftrightarrow \Phi_i(x) = 0$.

Suppose that f and g are relatively $\mathbf{P_0}$-functions. Then x^* is a stationary point of Ψ if and only if x^* is a solution of $GCP(f,g)$.

Proof. Since g is a strongly monotone and C^1, then it is a homeomorphism from R^n onto itself and the $\nabla g(x^*)$ is positive definite matrix (see [16]). Thus $\nabla g(x^*)$ is nonsingular and the proof follows from Corollary 3.1. □

Remark 3.2 *If we state the above results for GCP function based on the generalized Fischer-Burmeister function (4), and $g(x) = x$, GCP(f,g) reduces to NCP(f) and Corollary 3.1 reduces to Prop. 3.4 in [2].*

Theorem 3.3 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a\Psi(f(x),g(x)))_i(\nabla_b\Psi(f(x),g(x)))_i \geq 0$*, for any* $x \in \Re^n$*,*

(iii) $(\nabla_a\Psi(f(x),g(x)))_i \neq 0$ *whenever* $\Phi_i(x) \neq 0$*,*

(iv) $(\nabla_a\Psi(f(x),g(x)))_i = (\nabla_b\Psi(f(x),g(x)))_i = 0$ *whenever* $\Phi_i(x) = 0$.

Suppose that $\nabla g(x)^{-1}\nabla f(x)$ is a $\mathbf{P}$-matrix for any $x \in \Re^n$, then x^* is a stationary point of Ψ if and only if x^* is a solution of $GCP(f, g)$.

Proof. "$\Leftarrow$" Suppose that x^* is a solution of $GCP(f, g)$, then $\Phi(x^*) = 0$, and from the property (iv), we have

$$\nabla\Psi(x^*) = \nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b\Psi(f(x^*), g(x^*)) = 0,$$

i.e., x^* is a stationary point of Ψ.

"$\Rightarrow$" Suppose that x^* is a stationary point of Ψ, i.e.,

$$\nabla\Psi(x^*) = \nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b\Psi(f(x^*), g(x^*)) = 0,$$

then

$$\nabla g(x^*)^{-1}\nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla_b\Psi(f(x^*), g(x^*)) = 0. \tag{15}$$

We want to prove that x^* is a solution of $GCP(f, g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $\subseteq I := \{1, 2, \dots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I \backslash U$. From the property (iii), we get

$$(\nabla_a\Psi(f(x^*), g(x^*)))_i \neq 0, \forall i \in U. \tag{16}$$

Since $\nabla g(x^*)^{-1}\nabla f(x^*)$ is a $\mathbf{P}$-matrix, then for $\nabla_a\Psi(f(x^*), g(x^*)) \neq 0$, $\exists i_0 \in U$ such that

$$(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0} > 0. \tag{17}$$

From (15) and (17),

$$\begin{aligned} & (\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}(\nabla_b\Psi(f(x^*), g(x^*)))_{i_0} \\ = \ & -(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0} \\ < \ & 0, \end{aligned}$$

which contradicts the property (ii). Thus, the proof is complete. □

Remark 3.3 *Note that Theorem 3.1 is applicable to GCP functions in (4)-(8) in view of Proposition 3.1.*

Theorem 3.4 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, *for any* $x \in \Re^n$,

(iii) $(\nabla_a \Psi(f(x), g(x)))_i \neq 0$ *whenever* $\Phi_i(x) \neq 0$,

(iv) $(\nabla_a \Psi(f(x), g(x)))_i = (\nabla_b \Psi(f(x), g(x)))_i = 0$ *whenever* $\Phi_i(x) = 0$.

Suppose that $\nabla g(x)^{-1} \nabla f(x)$ *is a positive definite-matrix for any* $x \in \Re^n$, *then* x^* *is a stationary point of* Ψ *if and only if* x^* *is a solution of* $GCP(f, g)$.

Proof. Since every positive definite matrix is also a **P**-matrix, the proof of Theorem 3.4 follows from Theorem 3.3. □

Before stating the results of the subsequent theorems, we need the following definition.

Definition 3.1 *A vector* $\bar{x}$ *is said to be feasible (strictly feasible) for GCP(*f, g*) if* $f(\bar{x}) \geq 0 (> 0)$, *and* $g(\bar{x}) \geq 0 (> 0)$.

In the following theorems, we minimize the merit function under semi-monotone ($\mathbf{E_0}$)-conditions and strictly semi-monotone (**E**)-conditions, respectively.

Theorem 3.5 *Suppose* $f : \Re^n \to \Re^n$ *and* $g : \Re^n \to \Re^n$ *are continuously differentiable. Assume* g *is one-to-one, onto, and* $\nabla g(x)$ *is nonsingular for all* $x \in R^n$. *Suppose* Φ *is a GCP function of* f *and* g *and* $\Psi := \frac{1}{2}\|\Phi\|^2$ *satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, *for any* $x \in \Re^n$,

(iii) $(\nabla_a \Psi(f(x), g(x)))_i > 0, (\nabla_b \Psi(f(x), g(x)))_i > 0$, *whenever* $f_i(x) > 0, g_i(x) > 0$,

(iv) $(\nabla_a \Psi(f(x), g(x)))_i = (\nabla_b \Psi(f(x), g(x)))_i = 0$ *whenever* $\Phi_i(x) = 0$.

Suppose that $\nabla g(x)^{-1} \nabla f(x)$ *is a* $\mathbf{E_0}$*-matrix for any* $x \in \Re^n$ *and* x^* *is a feasible point of GCP(*f, g*). Then* x^* *is a stationary point of* Ψ *if and only if* x^* *is a solution of* $GCP(f, g)$.

Proof. "⇐" Suppose that x^* is a solution of $GCP(f, g)$, then $\Phi(x^*) = 0$, and from the property (iv), we have

$$\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,$$

that is, x^* is a stationary point of Ψ.

"⇒" Suppose that x^* is a stationary point of Ψ, i.e.,

$$\nabla\Psi(x^*) = \nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b\Psi(f(x^*), g(x^*)) = 0,$$

then

$$\nabla g(x^*)^{-1}\nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla_b\Psi(f(x^*), g(x^*)) = 0. \tag{18}$$

We want to prove that x^* is a solution of $GCP(f, g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $U \subseteq I := \{1, 2, \ldots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I\backslash U$. We have $f_i(x) > 0, g_i(x) > 0, \forall i \in U$, by x^* is a feasible point of GCP(f, g) and the definition of GCP function. From the properties (iii) and (iv), we get

$$(\nabla_a\Psi(f(x^*), g(x^*)))_i > 0, \forall i \in U \text{ and } (\nabla_a\Psi(f(x^*), g(x^*)))_i = 0, \forall i \in I\backslash U. \tag{19}$$

Since $\nabla g(x^*)^{-1}\nabla f(x^*)$ is an $\mathbf{E_0}$-matrix, then for $\nabla_a\Psi(f(x^*), g(x^*)) \geq 0$ and $\nabla_a\Psi(f(x^*), g(x^*)) \neq 0$, $\exists i_0 \in U$ such that

$$(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0} \geq 0. \tag{20}$$

From (18) and (20),

$$\begin{aligned} & (\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}(\nabla_b\Psi(f(x^*), g(x^*)))_{i_0} \\ = \; & -(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0} \\ \leq \; & 0, \end{aligned}$$

which contradicts the property (iii). Hence, the proof is complete. □

Remark 3.4 *In view Proposition 3.1, Theorem 3.5 is applicable to GCP functions in (4)-(8).*

Theorem 3.6 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume g is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ is a GCP function of f and g and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:*

(i) Ψ *is continuously differentiable,*

(ii) $(\nabla_a\Psi(f(x), g(x)))_i > 0, (\nabla_b\Psi(f(x), g(x)))_i \geq 0,$ *when* $f_i(x) > 0, g_i(x) > 0,$

(iii) $(\nabla_a\Psi(f(x), g(x)))_i = (\nabla_b\Psi(f(x), g(x)))_i = 0$ *whenever* $\Phi_i(x) = 0.$

Suppose that $\nabla g(x)^{-1}\nabla f(x)$ is a **E***-matrix for any $x \in \Re^n$ and x^* is a strictly feasible point of GCP(f,g). Then x^* is a stationary point of Ψ if and only if x^* is a solution of $GCP(f,g)$.*

Proof. "$\Leftarrow$" Suppose that x^* is a solution of $GCP(f,g)$, then $\Phi(x^*) = 0$, and from the property (iii), we have

$$\nabla\Psi(x^*) = \nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b\Psi(f(x^*), g(x^*)) = 0,$$

that is, x^* is a stationary point of Ψ.

"$\Rightarrow$" Suppose that x^* is a stationary point of Ψ, i.e.,

$$\nabla\Psi(x^*) = \nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b\Psi(f(x^*), g(x^*)) = 0,$$

then

$$\nabla g(x^*)^{-1}\nabla f(x^*)\nabla_a\Psi(f(x^*), g(x^*)) + \nabla_b\Psi(f(x^*), g(x^*)) = 0. \tag{21}$$

We want to prove that x^* is a solution of $GCP(f,g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $U \subseteq I := \{1, 2, \ldots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I\backslash U$. Since x^* is a feasible point of GCP(f,g) and using the definition of GCP function, we have $f_i(x) > 0, g_i(x) > 0, \forall i \in U$. From the properties (ii) and (iii), it is implied that

$$(\nabla_a\Psi(f(x^*), g(x^*)))_i > 0, \forall i \in U \text{ and } (\nabla_a\Psi(f(x^*), g(x^*)))_i = 0, \forall i \in I\backslash U. \tag{22}$$

Since $\nabla g(x^*)^{-1}\nabla f(x^*)$ is an **E**-matrix, then for $\nabla_a\Psi(f(x^*), g(x^*)) \geq 0$ and $\nabla_a\Psi(f(x^*), g(x^*)) \neq 0$, $\exists i_0 \in U$ such that

$$(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0} > 0. \tag{23}$$

From (21) and (23),

$$\begin{aligned} & (\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}(\nabla_b\Psi(f(x^*), g(x^*)))_{i_0} \\ = \;& -(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)(\nabla_a\Psi(f(x^*), g(x^*)))_{i_0} \\ < \;& 0, \end{aligned}$$

which contradicts the property (ii). The proof is complete. □

Remark 3.5 *We note that Theorem 3.6 is applicable to the GCP functions (4)-(8).*

Concluding Remarks

In this paper, we considered a generalized complementarity problem corresponding to C^1-differentiable functions, with an associated GCP function Φ and a merit function $\Psi = \frac{1}{2}||\Phi||^2$, we showed under certain conditions the global/local minimum or a stationary point of Ψ is a solution of GCP(f, g).

Our results recover/extend various well known results stated for nonlinear complementarity problem based on the generalized on generalized Fisher-Burmeister functions.

We note here that similar methodologies can be carried out for the following GCP functions:

(1) $\phi_{\theta,p}(a,b) := a + b - \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a-b|^p},\ \theta \in (0,1]$,
based on NCP proposed in [13].

(2) It is clear that when $\theta = 1$, $\phi_{\theta,p}(a,b)$ will reduce to (4) and denote it by

$$\phi_{1,p}(a,b) = \phi_p(a,b) = a + b - \|(a,b)\|_p.$$

(3) $\phi_{\alpha,\theta,p}(a,b) := \frac{\alpha}{2}[(ab)_+]^2 + \frac{1}{2}\,\phi_{\theta,p}(a,b)^2, \alpha \geq 0$
where $\phi_{\alpha,\theta,p}(a,b) : R^2 \to R_+$. This GCP function based on NCP function suggested in [3].

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