Generalized Complementarity Problems Based on Generalized Fisher-Burmeister Functions as Unconstrained Optimization\textsuperscript{1}

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Abstract

In this article, we consider an unconstrained minimization reformulation of the generalized complementarity problem \text{GCP}(f, g) based on the generalized Fisher-Burmeister function. Starting with $C^1$ functions $f$ and $g$, we show under certain conditions any stationary point of the unconstrained minimization problem is a solution to \text{GCP}(f, g).

Key words: Generalized complementarity problem, \text{GCP} function, generalized FB function, merit function, unconstrained minimization, stationary point.

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1 Introduction

We consider a generalized complementarity problem corresponding to $C^1$ functions $f$ and $g$, denoted by $GCP(f, g)$, which is to find a vector $x^* \in \mathbb{R}^n$ such that

$$f(x^*) \geq 0, \quad g(x^*) \geq 0 \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle = 0$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

For the formulation, numerical methods, and applications of $GCP(f, g)$, see [12], [14], [18] and the references cited therein. Also $GCP(f, g)$ covers some well known problems studied in the literature in the last decade; for example, if $g(x) = x$, then $GCP(f, g)$ reduces to the nonlinear complementarity problem $NCP(f)$. By taking in $NCP(f)$ $f(x) = Mx + q$ with $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, then $NCP(f)$ is called a linear complementarity problem $LCP(M, q)$. Also, if $g(x) = x - W(x)$ with some $W : \mathbb{R}^n \rightarrow \mathbb{R}^n$, then $GCP(f, g)$ is known as the quasi/implicit complementarity problem, see e.g., [14], [17], [19].

These problems have numerous applications in diverse fields such as optimization, engineering, economics and other areas, see e.g., [4], [5], [8], [9], [11], [20], and the references therein.

A function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called a GCP function if

$$\phi(a, b) = 0 \iff ab = 0, \quad a \geq 0, \quad b \geq 0.$$ 

For the problem $GCP(f, g)$, we define

$$\Phi(x) = \begin{bmatrix} \phi(f_1(x), g_1(x)) \\
\vdots \\
\phi(f_i(x), g_i(x)) \\
\vdots \\
\phi(f_n(x), g_n(x)) \end{bmatrix}$$

and, call $\Phi(x)$ a GCP function for $GCP(f, g)$.

Our goal from this paper is to study a generalized complementarity problem $GCP(f, g)$ based on the generalized Fisher-Burmeister function when the underlying functions $f$ and $g$ are $C^1$. By considering a GCP function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ associated with $GCP(f, g)$ and its merit function

$$\Psi(x) := \frac{1}{2} \|\Phi(x)\|^2,$$
so that
\[ \bar{x} \text{ solves } \text{GCP}(f, g) \iff \Phi(\bar{x}) = 0 \iff \Psi(\bar{x}) = 0. \]

If we assume GCP\((f, g)\) has at least one solution, then a vector \(\bar{x} \in \mathbb{R}^n\) solves GCP\((f, g)\) if and only if it is a global/local minimizer (a stationary point) of the unconstrained minimization problem
\[
\min_{x \in \mathbb{R}^n} \Psi(x).
\]

In this paper, we show how, under appropriate \(P_0(P)\), positive definite (semidefinite)-conditions on \(H\)-differentials of \(f\) and \(g\), finding local/global minimum of \(\Psi\) (or a ‘stationary point’ of \(\Psi\)) leads to a solution of the given generalized complementarity problem. Further, we show that how our results unify/extend various similar results proved in the literature for nonlinear complementarity problem when the underlying functions are \(C^1\).

## 2 Preliminaries

Throughout this paper, we regard vectors in \(\mathbb{R}^n\) as column vectors. We denote the inner-product between two vectors \(x\) and \(y\) in \(\mathbb{R}^n\) by either \(x^Ty\) or \(\langle x, y \rangle\). Vector inequalities are interpreted componentwise. For a matrix \(A\), \(A_i\) denotes the \(i\)th row of \(A\). For a differentiable function \(f : \mathbb{R}^n \to \mathbb{R}^m\), \(\nabla f(\bar{x})\) denotes the Jacobian matrix of \(f\) at \(\bar{x}\).

We need the following definition from [4].

**Definition 2.1** [(i)] A matrix \(A \in \mathbb{R}^{n \times n}\) is called semimonotone (\(E_0\)) (strictly semimonotone (\(E\)))-matrix if
\[
\forall x \in \mathbb{R}^n, x \neq 0, \text{ there exists } i \text{ such that } x_i \neq 0 \text{ and } x_i(Ax)_i \geq 0 (> 0).
\]

[(ii)] A matrix \(A \in \mathbb{R}^{n \times n}\) is called \(P_0\) (\(P\))-matrix if
\[
\forall x \in \mathbb{R}^n, x \neq 0, \text{ there exists } i \text{ such that } x_i \neq 0 \text{ and } x_i(Ax)_i \geq 0 (> 0).
\]

In [21], the author generalized the concepts of monotonicity, \(P_0\)-property and their variants for functions and use them to establish some conditions to get a solution for generalized complementarity problem when the underlying functions \(f\) and \(g\) are \(H\)-differentiable. .

Let us recall the following definitions from [21].
Definition 2.2 For functions \( f, g : \mathbb{R}^n \to \mathbb{R}^n \), we say that \( f \) and \( g \) are:

(a) Relatively monotone if
\[
\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0 \text{ for all } x, y \in \mathbb{R}^n.
\]

(b) Relatively strictly monotone if
\[
\langle f(x) - f(y), g(x) - g(y) \rangle > 0 \text{ for all } x, y \in \mathbb{R}^n.
\]

(c) Relatively strongly monotone if there exists a constant \( \mu > 0 \) such that
\[
\langle f(x) - f(y), g(x) - g(y) \rangle \geq \mu \|x - y\|^2 \text{ for all } x, y \in \mathbb{R}^n.
\]

(d) Relatively \( P_0 \) (\( P \))-functions if for any \( x \neq y \) in \( \mathbb{R}^n \),
\[
\max_{1 \leq i \leq n} |f(x) - f(y)|,|g(x) - g(y)| \geq (>)0.
\]

(e) Relatively uniform (\( P \))-functions if there exists a constant \( \eta > 0 \) such that for any \( x, y \in \mathbb{R}^n \),
\[
\max_{1 \leq i \leq n} |f(x) - f(y)|,|g(x) - g(y)| \geq \eta \|x - y\|^2.
\]

Note that relatively strongly monotone functions are relatively strictly monotone, and relatively strictly monotone functions are relatively monotone. Also we note that every relatively monotone (strictly monotone) function is a relatively \( P_0(P) \)-function.

There are some relations between \( f, g \) and \( f \circ g^{-1} \) when \( g \) is one-to-one and onto, which are given in [21].

Lemma 2.1 Suppose that \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) and \( g \) is one-to-one and onto. Define \( h : \mathbb{R}^n \to \mathbb{R}^n \) where \( h := f \circ g^{-1} \). The following hold:

(a) \( f \) and \( g \) are relatively (strictly) monotone if and only if \( h \) is (strictly) monotone.

(b) If \( g \) is Lipschitz-continuous, and \( f \) and \( g \) are relatively strongly monotone then \( h \) is strongly monotone.

(c) \( f \) and \( g \) are relatively \( P_0 \) (\( P \))-functions if and only if \( h \) is \( P_0 \) (\( P \))-function.

(d) If \( g \) is Lipschitz-continuous, and \( f \) and \( g \) are relatively uniform (\( P \))-functions, then \( h \) is uniform (\( P \))-function.

The following result is from [16].
Theorem 2.1 Under the following conditions, \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a \( P_0(\mathbb{P}) \)-function. \( f \) is Fréchet differentiable on \( \mathbb{R}^n \) and for every \( x \in \mathbb{R}^n \), the Jacobian matrix \( \nabla f(x) \) is a \( P_0(\mathbb{P}) \)-matrix.

Remark 2.1 Based on some results in [16], we note the following. For \( \mathbb{P} \)-conditions, the converse statements in the above theorem are usually false.

3 Minimizing the merit function

Over the past two decades, a variety of NCP-functions have been studied, see [10] and references therein. Among which, some families of NCP functions [2, 1, 13] based on the Fisher-Burmeister function with \( p \)-norm are proposed. The family NCP functions are proposed in [2]:

\[
\phi_p(a, b) := a + b - \|(a, b)\|_p
\]

where \( p \) is any fixed real number in the interval \((1, +\infty)\) and \( \|(a, b)\|_p \) denotes the \( p \)-norm of \((a, b)\), i.e., \( \|(a, b)\|_p = \sqrt[p]{|a|^p + |b|^p} \). Based on the functions (4), some more NCP functions are introduced in [1]:

\[
\phi_1(a, b) := \phi_p(a, b) + \alpha a_+ b_+, \alpha > 0.
\]

\[
\phi_2(a, b) := \phi_p(a, b) + \alpha (ab)_+, \alpha > 0.
\]

\[
\phi_3(a, b) := \sqrt{[\phi_p(a, b)]^2 + \alpha (a_+ b_+)^2}, \alpha > 0.
\]

\[
\phi_4(a, b) := \sqrt{[\phi_p(a, b)]^2 + \alpha [(ab)_+]^2}, \alpha > 0.
\]

Our objective in this article is to study GCP functions based on these NCP functions. For given \( C^1 \)-functions \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \), we consider the associated GCP function \( \Phi \) and the corresponding merit function

\[
\Psi^*_*(\bar{x}) := \frac{1}{2} \|\Phi^*_*(\bar{x})\|^2 = \sum_{i=1}^n \psi_i(f_i(\bar{x}), g_i(\bar{x})),
\]

where \( \psi_i \) are the \( \psi \)-functions associated with the \( i \)-th constraint.
where
\[ \Phi^*(\bar{x}) := \begin{pmatrix} \phi^*(f_1(\bar{x}), g_1(\bar{x})) \\ \vdots \\ \phi^*(f_n(\bar{x}), g_n(\bar{x})) \end{pmatrix}, \]
(10)
and
\[ \psi^*(a, b) := \frac{1}{2} \phi^*(a, b)^2, \]
(11)
with \( * \in \{1, p, 1, 2, 3, 4\} \).

It should be recalled that
\[ \Psi^*(\bar{x}) = 0 \iff \Phi^*(\bar{x}) = 0 \iff \bar{x} \text{ solves GCP}(f, g). \]

In the following proposition, we give favorable properties for \( \psi \).

**Proposition 3.1** Let \( \psi \in \{\psi_{1,p}, \psi_1, \psi_2, \psi_3, \psi_4\} \) be defined in (9). Then \( \psi \) has the following favorable properties:

(a) \( \psi \) is a nonnegative, i.e., \( \psi(a, b) \geq 0 \) for all \( (a, b) \in \mathbb{R}^2 \).
(b) \( \psi \) is continuously differentiable everywhere.
(c) \( \nabla_a \psi(a, b) \cdot \nabla_b \psi(a, b) \geq 0 \) for all \( (a, b) \in \mathbb{R}^2 \).
(d) \( \psi(a, b) = 0 \iff \nabla \psi(a, b) = 0 \iff \nabla_a \psi(a, b) = 0 \iff \nabla_b \psi(a, b) = 0 \).

**Proof.** When \( \psi = \psi_{1,p} \), the results (a)-(d) can be obtained from [2, Proposition 3.2 (a)-(e)] respectively. When \( \psi \in \{\psi_1, \psi_2, \psi_3, \psi_4\} \), the results (a)-(d) can be obtained from [1, Proposition 3.3 (a)-(d)] respectively. \( \square \)

Now we minimize the merit function under \( P_0 \)-conditions.

**Theorem 3.1** Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable. Assume \( g \) is one-to-one, onto, and \( \nabla g(x) \) is nonsingular for all \( x \in \mathbb{R}^n \). Suppose \( \Phi \) is a GCP function of \( f \) and \( g \) and \( \Psi := \frac{1}{2} \| \Phi \|^2 \) :

(i) \( \Psi \) is continuously differentiable,
(ii) \( (\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0 \), for any \( x \in \mathbb{R}^n \),
(iii) \( (\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \neq 0 \) whenever \( \Phi_i(x) \neq 0 \),
(iv) \((\nabla_a \Psi(f(x), g(x)))_i = 0 \iff (\nabla_b \Psi(f(x), g(x)))_i = 0 \iff \Phi_i(x) = 0.\)

Suppose that \(\nabla g(x)^{-1} \nabla f(x)\) is a \(P_0\)-matrix for any \(x \in \mathbb{R}^n\), then \(x^*\) is a stationary point of \(\Psi\) if and only if \(x^*\) is a solution of \(GCP(f, g)\).

**Proof.** “\(\Rightarrow\)” Suppose that \(x^*\) is a solution of \(GCP(f, g)\), then \(\Phi(x^*) = 0\), and from the property (iv), we have
\[
\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,
\]
that is, \(x^*\) is a stationary point of \(\Psi\).

“\(\Leftarrow\)”

Suppose that \(x^*\) is a stationary point of \(\Psi\), i.e.,
\[
\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,
\]
then
\[
\nabla g(x^*)^{-1} \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla_b \Psi(f(x^*), g(x^*)) = 0. \tag{12}
\]

We want to prove that \(x^*\) is a solution of \(GCP(f, g)\), i.e., \(\Phi(x^*) = 0\). Suppose not, i.e., \(\Phi(x^*) \neq 0\), then \(\exists U \neq \emptyset\) and \(U \subseteq I := \{1, 2, \ldots, n\}\) such that \(\Phi_i(x^*) \neq 0, \forall i \in U\), and \(\Phi_i(x^*) = 0, \forall i \in I \setminus U\). We have
\[
(\nabla_a \Psi(f(x^*), g(x^*)))_i \neq 0, (\nabla_b \Psi(f(x^*), g(x^*)))_i \neq 0, \forall i \in U \tag{13}
\]
and \((\nabla_a \Psi(f(x^*), g(x^*)))_i = 0, (\nabla_b \Psi(f(x^*), g(x^*)))_i = 0, \forall i \in I \setminus U\) from the property (iv). Since \(\nabla g(x^*)^{-1} \nabla f(x^*)\) is a \(P_0\)-matrix, then for \(\nabla_a \Psi(f(x^*), g(x^*)) \neq 0, \exists i_0 \in U\) such that
\[
(\nabla_a \Psi(f(x^*), g(x^*))_{i_0} \nabla g(x^*)^{-1} \nabla f(x^*) (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \geq 0. \tag{14}
\]

From \([12]\) and \([14]\),
\[
(\nabla_a \Psi(f(x^*), g(x^*))_{i_0} (\nabla_b \Psi(f(x^*), g(x^*))_{i_0}
= - (\nabla_a \Psi(f(x^*), g(x^*))_{i_0} \nabla g(x^*)^{-1} \nabla f(x^*) (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0}
\leq 0.
\]

By the property (ii), then we have
\[
(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} (\nabla_b \Psi(f(x^*), g(x^*)))_{i_0} = 0.
\]

which contradicts \([13]\). Hence, the proof is complete. \qed

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Remark 3.1: From Proposition 3.1, we note that Theorem 3.1 is applicable to GCP functions in (4)-(8).

If we state the above results for GCP function based on the generalized Fisher-Burmeister function (4) and replace the p-norm to 2-norm, then Theorem 3.1 reduces to Theorem 3.2 in [15]. And, when \( g(x) = x \), GCP\((f, g)\) reduces to NCP\((f)\) and Theorem 3.1 reduces to Prop. 3.4 in [7]. Also, When \( g(x) = x \), our result extends/generalizes a result obtained by Geiger and Kanzow [6] for NCP\((f)\) under monotonicity of a function.

In Theorem 3.1, if we consider GCP functions in (5)-(8) and \( g(x) = x \), GCP\((f, g)\) reduces to NCP\((f)\) and Theorem 3.1 reduces to Prop. 3.4 in [1].

Since every positive semidefinite matrix is also a \( P_0 \)-matrix, the proof of the following theorem will follow from Theorem 3.1.

Theorem 3.2 Suppose \( f: \mathbb{R}^n \to \mathbb{R}^n \) and \( g: \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable. Assume \( g \) is one-to-one, onto, and \( \nabla g(x) \) is nonsingular for all \( x \in \mathbb{R}^n \). Suppose \( \Phi \) is a GCP function of \( f \) and \( g \) and \( \Psi := \frac{1}{2}\|\Phi\|^2 \) satisfies:

(i) \( \Psi \) is continuously differentiable,
(ii) \((\nabla_a \Psi(f(x), g(x)))(\nabla_b \Psi(f(x), g(x)))\) \(\geq 0\), for any \( x \in \mathbb{R}^n \),
(iii) \((\nabla_a \Psi(f(x), g(x)))(\nabla_b \Psi(f(x), g(x))) \neq 0\) whenever \( \Phi_i(x) \neq 0 \),
(iv) \((\nabla_a \Psi(f(x), g(x))) = 0 \iff (\nabla_b \Psi(f(x), g(x))) = 0 \iff \Phi_i(x) = 0 \).

Suppose that \( \nabla g(x)^{-1}\nabla f(x) \) is a positive semidefinite matrix for any \( x \in \mathbb{R}^n \), then \( x^* \) is a stationary point of \( \Psi \) if and only if \( x^* \) is a solution of GCP\((f, g)\).

From Lemma 2.1 and view of Theorem 2.1 we now state two consequences of the above theorems.

Corollary 3.1 Suppose \( f: \mathbb{R}^n \to \mathbb{R}^n \) and \( g: \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable. Assume \( g \) is one-to-one, onto, and \( \nabla g(x) \) is nonsingular for all \( x \in \mathbb{R}^n \). Suppose \( \Phi \) is a GCP function of \( f \) and \( g \) and \( \Psi := \frac{1}{2}\|\Phi\|^2 \) satisfies:

(i) \( \Psi \) is continuously differentiable,
(ii) \((\nabla_a \Psi(f(x), g(x)))(\nabla_b \Psi(f(x), g(x)))\) \(\geq 0\), for any \( x \in \mathbb{R}^n \),
(iii) \((\nabla_a \Psi(f(x), g(x)))(\nabla_b \Psi(f(x), g(x))) \neq 0\) whenever \( \Phi_i(x) \neq 0 \),
(iv) \((\nabla_a \Psi(f(x), g(x))) = 0 \iff (\nabla_b \Psi(f(x), g(x))) = 0 \iff \Phi_i(x) = 0 \).
Suppose that \( f \) and \( g \) are relatively \( P_0 \)-functions. Then \( x^* \) is a stationary point of \( \Psi \) if and only if \( x^* \) is a solution of GCP\((f,g)\).

**Proof.** Since \( g \) is a one-to-one and onto, and \( f \) and \( g \) are relatively \( P_0 \)-functions, by Lemma 2.1 the mapping \( f \circ g^{-1} \) is \( P_0 \)-function which implies \( \nabla f(x^*)\nabla g(x^*)^{-1} \) is \( P_0 \)-matrix, see Theorem 2.1. The proof follows from Theorem 3.1. \(\square\)

**Corollary 3.2** Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable. Assume \( g \) is strongly monotone. Suppose \( \Phi \) is a GCP function of \( f \) and \( g \) and \( \Psi := \frac{1}{2} \| \Phi \|^2 \) satisfies:

(i) \( \Psi \) is continuously differentiable,

(ii) \( (\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0 \), for any \( x \in \mathbb{R}^n \),

(iii) \( (\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \neq 0 \) whenever \( \Phi_i(x) \neq 0 \),

(iv) \( (\nabla_a \Psi(f(x), g(x)))_i = 0 \Leftrightarrow (\nabla_b \Psi(f(x), g(x)))_i = 0 \Leftrightarrow \Phi_i(x) = 0 \).

Suppose that \( f \) and \( g \) are relatively \( P_0 \)-functions. Then \( x^* \) is a stationary point of \( \Psi \) if and only if \( x^* \) is a solution of GCP\((f,g)\).

**Proof.** Since \( g \) is a strongly monotone and \( C^1 \), then it is a homeomorphism from \( \mathbb{R}^n \) onto itself and the \( \nabla g(x^*) \) is positive definite matrix (see [10]). Thus \( \nabla g(x^*) \) is nonsingular and the proof follows from Corollary 3.1. \(\square\)

**Remark 3.2** If we state the above results for GCP function based on the generalized Fischer-Burmeister function (4), and \( g(x) = x \), GCP\((f,g)\) reduces to NCP\((f)\) and Corollary 3.1 reduces to Prop. 3.4 in [2].

**Theorem 3.3** Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable. Assume \( g \) is one-to-one, onto, and \( \nabla g(x) \) is nonsingular for all \( x \in \mathbb{R}^n \). Suppose \( \Phi \) is a GCP function of \( f \) and \( g \) and \( \Psi := \frac{1}{2} \| \Phi \|^2 \) satisfies:

(i) \( \Psi \) is continuously differentiable,

(ii) \( (\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0 \), for any \( x \in \mathbb{R}^n \),

(iii) \( (\nabla_a \Psi(f(x), g(x)))_i \neq 0 \) whenever \( \Phi_i(x) \neq 0 \),

(iv) \( (\nabla_a \Psi(f(x), g(x)))_i = (\nabla_b \Psi(f(x), g(x)))_i = 0 \) whenever \( \Phi_i(x) = 0 \).
Suppose that $\nabla g(x)^{-1}\nabla f(x)$ is a $\mathbf{P}$-matrix for any $x \in \mathbb{R}^n$, then $x^*$ is a stationary point of $\Psi$ if and only if $x^*$ is a solution of $GCP(f, g)$.

**Proof.** “$\Leftarrow$” Suppose that $x^*$ is a solution of $GCP(f, g)$, then $\Phi(x^*) = 0$, and from the property (iv), we have

$$
\nabla \Psi(x^*) = \nabla f(x^*)\nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b \Psi(f(x^*), g(x^*)) = 0,
$$
i.e., $x^*$ is a stationary point of $\Psi$.

“$\Rightarrow$” Suppose that $x^*$ is a stationary point of $\Psi$, i.e.,

$$
\nabla \Psi(x^*) = \nabla f(x^*)\nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b \Psi(f(x^*), g(x^*)) = 0,
$$
then

$$
\nabla g(x^*)^{-1}\nabla f(x^*)\nabla_a \Psi(f(x^*), g(x^*)) + \nabla_b \Psi(f(x^*), g(x^*)) = 0. \tag{15}
$$

We want to prove that $x^*$ is a solution of $GCP(f, g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $\subseteq I := \{1, 2, \ldots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I\setminus U$. From the property (iii), we get

$$
(\nabla_a \Psi(f(x^*), g(x^*)))_i \neq 0, \forall i \in U. \tag{16}
$$

Since $\nabla g(x^*)^{-1}\nabla f(x^*)$ is a $\mathbf{P}$-matrix, then for $\nabla_a \Psi(f(x^*), g(x^*)) \neq 0, \exists i_0 \in U$ such that

$$
(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)/(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} > 0. \tag{17}
$$

From (15) and (17),

$$
(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0}(\nabla_b \Psi(f(x^*), g(x^*)))_{i_0}
= - (\nabla_a \Psi(f(x^*), g(x^*))_{i_0}\nabla g(x^*)^{-1}\nabla f(x^*)/(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0}
< 0,
$$

which contradicts the property (ii). Thus, the proof is complete.

**Remark 3.3** Note that Theorem 3.1 is applicable to $GCP$ functions in (4)-(8) in view of Proposition 3.1.

**Theorem 3.4** Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable. Assume $g$ is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in \mathbb{R}^n$. Suppose $\Phi$ is a $GCP$ function of $f$ and $g$ and $\Psi := \frac{1}{2}\|\Phi\|^2$ satisfies:

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(i) $\Psi$ is continuously differentiable,
(ii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, for any $x \in \mathbb{R}^n$,
(iii) $(\nabla_a \Psi(f(x), g(x)))_i \neq 0$ whenever $\Phi_i(x) \neq 0$,
(iv) $(\nabla_a \Psi(f(x), g(x)))_i = (\nabla_b \Psi(f(x), g(x)))_i = 0$ whenever $\Phi_i(x) = 0$.

Suppose that $\nabla g(x)^{-1} \nabla f(x)$ is a positive definite-matrix for any $x \in \mathbb{R}^n$, then $x^*$ is a stationary point of $\Psi$ if and only if $x^*$ is a solution of GCP($f, g$).

**Proof.** Since every positive definite matrix is also a $P$-matrix, the proof of Theorem 3.4 follows from Theorem 3.3.

Before stating the results of the subsequent theorems, we need the following definition.

**Definition 3.1** A vector $\bar{x}$ is said to be feasible (strictly feasible) for GCP($f, g$) if $f(\bar{x}) \geq 0 (> 0)$, and $g(\bar{x}) \geq 0 (> 0)$.

In the following theorems, we minimize the merit function under semi-monotone ($E_0$)-conditions and strictly semi-monotone ($E$)-conditions, respectively.

**Theorem 3.5** Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuously differentiable. Assume $g$ is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in \mathbb{R}^n$. Suppose $\Phi$ is a GCP function of $f$ and $g$ and $\Psi := \frac{1}{2} \|\Phi\|^2$ satisfies:

(i) $\Psi$ is continuously differentiable,
(ii) $(\nabla_a \Psi(f(x), g(x)))_i (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, for any $x \in \mathbb{R}^n$,
(iii) $(\nabla_a \Psi(f(x), g(x)))_i > 0, (\nabla_b \Psi(f(x), g(x)))_i > 0$, whenever $f_i(x) > 0, g_i(x) > 0$,
(iv) $(\nabla_a \Psi(f(x), g(x)))_i = (\nabla_b \Psi(f(x), g(x)))_i = 0$ whenever $\Phi_i(x) = 0$.

Suppose that $\nabla g(x)^{-1} \nabla f(x)$ is a $E_0$-matrix for any $x \in \mathbb{R}^n$ and $x^*$ is a feasible point of GCP($f, g$). Then $x^*$ is a stationary point of $\Psi$ if and only if $x^*$ is a solution of GCP($f, g$).

**Proof.** “$\Rightarrow$” Suppose that $x^*$ is a solution of GCP($f, g$), then $\Phi(x^*) = 0$, and from the property (iv), we have

$$\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,$$

that is, $x^*$ is a stationary point of $\Psi$.
"⇒" Suppose that $x^*$ is a stationary point of $\Psi$, i.e.,

$$
\nabla \Psi(x^*) = \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*) \nabla_b \Psi(f(x^*), g(x^*)) = 0,
$$

then

$$
\nabla g(x^*)^{-1} \nabla f(x^*) \nabla_a \Psi(f(x^*), g(x^*)) + \nabla_b \Psi(f(x^*), g(x^*)) = 0. \tag{18}
$$

We want to prove that $x^*$ is a solution of $GCP(f, g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $U \subseteq I := \{1, 2, \ldots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I \setminus U$. We have $f_i(x) > 0, g_i(x) > 0, \forall i \in U$, by $x^*$ is a feasible point of $GCP(f, g)$ and the definition of GCP function. From the properties (iii) and (iv), we get

$$
(\nabla_a \Psi(f(x^*), g(x^*)))_i > 0, \forall i \in U \text{ and } (\nabla_a \Psi(f(x^*), g(x^*)))_i = 0, \forall i \in I \setminus U. \tag{19}
$$

Since $\nabla g(x^*)^{-1} \nabla f(x^*)$ is an $E_0$-matrix, then for $\nabla_a \Psi(f(x^*), g(x^*)) \geq 0$ and $\nabla_a \Psi(f(x^*), g(x^*)) \neq 0$, $\exists i_0 \in U$ such that

$$
(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \nabla g(x^*)^{-1} \nabla f(x^*) (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \geq 0. \tag{20}
$$

From (18) and (20),

$$
(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} (\nabla_b \Psi(f(x^*), g(x^*)))_{i_0}
= -(\nabla_a \Psi(f(x^*), g(x^*)))_{i_0} \nabla g(x^*)^{-1} \nabla f(x^*) (\nabla_a \Psi(f(x^*), g(x^*)))_{i_0}
\leq 0,
$$

which contradicts the property (iii). Hence, the proof is complete. \qed

**Remark 3.4** In view Proposition 3.1, Theorem 3.5 is applicable to GCP functions in [4]-[8].

**Theorem 3.6** Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuously differentiable. Assume $g$ is one-to-one, onto, and $\nabla g(x)$ is nonsingular for all $x \in \mathbb{R}^n$. Suppose $\Phi$ is a GCP function of $f$ and $g$ and $\Psi := \frac{1}{2} \|\Phi\|^2$ satisfies:

(i) $\Psi$ is continuously differentiable,

(ii) $(\nabla_a \Psi(f(x), g(x)))_i > 0, (\nabla_b \Psi(f(x), g(x)))_i \geq 0$, when $f_i(x) > 0, g_i(x) > 0,$

(iii) $(\nabla_a \Psi(f(x), g(x)))_i = (\nabla_b \Psi(f(x), g(x)))_i = 0$ whenever $\Phi_i(x) = 0$. 

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Suppose that $\nabla g(x)^{-1}\nabla f(x)$ is a $E$-matrix for any $x \in \mathbb{R}^n$ and $x^*$ is a strictly feasible point of $GCP(f, g)$. Then $x^*$ is a stationary point of $\Psi$ if and only if $x^*$ is a solution of $GCP(f, g)$.

**Proof.** $\Leftarrow$ Suppose that $x^*$ is a solution of $GCP(f, g)$, then $\Phi(x^*) = 0$, and from the property (iii), we have

$$\nabla \Psi(x^*) = \nabla f(x^*)\nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b \Psi(f(x^*), g(x^*)) = 0,$$

that is, $x^*$ is a stationary point of $\Psi$.

$\Rightarrow$ Suppose that $x^*$ is a stationary point of $\Psi$, i.e.,

$$\nabla \Psi(x^*) = \nabla f(x^*)\nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b \Psi(f(x^*), g(x^*)) = 0,$$

then

$$\nabla g(x^*)^{-1}\nabla f(x^*)\nabla_a \Psi(f(x^*), g(x^*)) + \nabla g(x^*)\nabla_b \Psi(f(x^*), g(x^*)) = 0. \tag{21}$$

We want to prove that $x^*$ is a solution of $GCP(f, g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\exists U \neq \emptyset$ and $U \subseteq I := \{1, 2, \ldots, n\}$ such that $\Phi_i(x^*) \neq 0, \forall i \in U$, and $\Phi_i(x^*) = 0, \forall i \in I \setminus U$. Since $x^*$ is a feasible point of $GCP(f, g)$ and using the definition of GCP function, we have $f_i(x) > 0, g_i(x) > 0, \forall i \in U$. From the properties (ii) and (iii), it is implied that

$$(\nabla \Psi(f(x^*), g(x^*)))_i > 0, \forall i \in U \text{ and } (\nabla \Psi(f(x^*), g(x^*)))_i = 0, \forall i \in I \setminus U. \tag{22}$$

Since $\nabla g(x^*)^{-1}\nabla f(x^*)$ is an $E$-matrix, then for $\nabla \Psi(f(x^*), g(x^*)) \geq 0$ and $\nabla \Psi(f(x^*), g(x^*)) \neq 0$, $\exists i_0 \in U$ such that

$$(\nabla \Psi(f(x^*), g(x^*)))_{i_0} \nabla g(x^*)^{-1}\nabla f(x^*)(\nabla \Psi(f(x^*), g(x^*)))_{i_0} > 0. \tag{23}$$

From \[21\] and \[23\],

$$-(\nabla \Psi(f(x^*), g(x^*)))_{i_0} \nabla g(x^*)^{-1}\nabla f(x^*)(\nabla \Psi(f(x^*), g(x^*)))_{i_0} < 0,$$

which contradicts the property (ii). The proof is complete. \[\square\]

**Remark 3.5** We note that Theorem 3.6 is applicable to the GCP functions \[4\]-\[8\].

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Concluding Remarks

In this paper, we considered a generalized complementarity problem corresponding to $C^1$-differentiable functions, with an associated GCP function $\Phi$ and a merit function $\Psi = \frac{1}{2}||\Phi||^2$, we showed under certain conditions the global/local minimum or a stationary point of $\Psi$ is a solution of $\text{GCP}(f, g)$.

Our results recover/extend various well known results stated for nonlinear complementarity problem based on the generalized on generalized Fisher-Burmeister functions.

We note here that similar methodologies can be carried out for the following GCP functions:

1. $\phi_{\theta,p}(a, b) := a + b - \sqrt{\theta(|a|^p + |b|^p)} + (1 - \theta)|a - b|^p, \theta \in (0, 1]$, based on NCP proposed in [13].

2. It is clear that when $\theta = 1$, $\phi_{\theta,p}(a, b)$ will reduce to (4) and denote it by $\phi_{1,p}(a, b) = \phi_p(a, b) = a + b - \|(a, b)\|_p$.

3. $\phi_{\alpha,\theta,p}(a, b) := \frac{\alpha}{4}((ab)_+)^2 + \frac{1}{2} \phi_{\theta,p}(a, b)^2, \alpha \geq 0$

where $\phi_{\alpha,\theta,p}(a, b) : R^2 \rightarrow R_+$. This GCP function based on NCP function suggested in [3].

References


