AMO - Advanced Modeling and Optimization, Volume 16, Number 2, 2014

SEQUENCE OF FUZZY DIVERGENCE MEASURES AND INEQUALITIES

Dr. Vijay Prakash Tomar, Assistant Professor,
Department of Mathematics, Deenbandhu Chhotu Ram University
of Science & Technology, Murthal 131039, Haryana, India
vijay1976pratap@rediffmail.com

Mrs. Anshu Ohlan[1], Doctoral Research Scholar,
Department of Mathematics, Deenbandhu Chhotu Ram University
of Science & Technology, Murthal 131039,Haryana, India
anshu.gahlawat@yahoo.com

Abstract

Fuzzy divergence measures and inequalities have recently been widely applied in the fuzzy comprehensive evaluation and information theory. In view of the importance of fuzzy information measures and fuzzy inequalities, the paper presents a sequence of fuzzy mean difference divergence measures along with the proof of their validity. Further, it introduces a sequence of inequalities among some of these fuzzy divergence measures. Finally, a numerical example is given to verify the inequalities of proposed fuzzy mean difference divergence measures.

Keywords Fuzzy set, Fuzzy entropy, Fuzzy divergence measure, Fuzzy mean divergence measures, Inequalities.

1. Introduction

Entropy is one of the key measures of information. The word "entropy" to measure an uncertain degree of the randomness in a probability distribution was first use by [Shannon, 1948]. Entropy as a measure of fuzziness was first introduced by [Zadeh, 1968]. There is an intrinsic similarity between two equations but Shannon entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment, but the entropy of fuzzy set describes the degree of fuzziness in a fuzzy set. The concept of fuzzy sets proposed by [Zadeh, 1968] has proven useful in the context of pattern recognition, image processing, speech recognition, bioinformatics, fuzzy aircraft control, feature selection, decision making, etc.

During the last six decades, Entropy, as a very important notion of measuring fuzziness degree or uncertain information in fuzzy set theory, has received a great attention. [De Luca and Termini, 1972] introduced the measure of fuzzy entropy corresponding to [Shannon, 1948]. Corresponding to [Boekee and Lubbe, 1980] $R-$norm information measure, [Hooda, 2004] proposed the $R-$norm fuzzy measure of information. Further, [Tomar and Ohlan, 2014] provide two new more flexible generalizations of $R-$ norm fuzzy measure of information. A measure of directed divergence between two probability distributions is provided by [Kullback and Leibler, 1951]. Some of the axioms to describe the measure of directed divergence between fuzzy sets is presented by [Bhandari and Pal, 1993], which was corresponding to [Kullback and Leibler, 1951] measure of directed divergence. Thereafter, many other researchers have also studied the fuzzy divergence measures in

[1] Corresponding author. +91-8053477499.

*** AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

different ways and provide their application in different areas. The divergence measure based on exponential operation is introduced by [Fan and Xie, 1999] and studied its relation with divergence measure introduced in [Bhandari and Pal, 1993]. Its application in the area of automated leukocyte recognition is presented by [Ghosh et al., 2010]. The special classes of divergence measures and the link between fuzzy and probabilistic uncertainty is studied by [Montes et al., 2002]. Two fuzzy divergence measures corresponding to [Ferreri, 1980] probabilistic measure of directed divergence were proposed by [Parkash et al., 2006]. Three families of fuzzy divergence measures corresponding to [Taneja, 2005] Arithmetic-Geometric divergence measure were proposed by [Bhatia and Singh, 2012]. In the recent years, many authors have introduced various divergence measures between fuzzy sets.

Motivated by the above-mentioned work we introduce a sequence of fuzzy mean difference divergence measures and established the inequalities among them to explore the fuzzy inequalities. The advantage of establishing the inequalities is to make the computational work much simpler. We think that the technique of inequalities provides a better comparison among fuzzy mean divergence measures.

The remainder of the paper is organized as follows. In Section 2, we present preliminaries on fuzzy divergence measures and fuzzy mean divergence measures. A sequence of fuzzy mean difference divergence measures and the essential properties for their validity are provided in Section 3. Section 4 provides some of inequalities among some of the proposed fuzzy divergence measures. In the same Section, the proof of these inequalities is also presented. In Section 5, a numerical example is given to verify the inequalities. Finally, concluding remarks are drawn in Section 6.

2. PRELIMINARIES ON FUZZY DIVERGENCE MEASURES

Fuzziness, a feature of uncertainty, results from the lack of sharp difference of being or not being a member of the set, i.e., the boundaries of the set under consideration are not sharply defined. A fuzzy set A defined on a universe of discourse *X* is given as [Zadeh, 1965]:

$$A=\left\{\left\langle x,\mu_A(x)\right\rangle / x\in X\right\}$$

where $\mu_A : X \to [0,1]$ is the membership function of *A*. The membership value $\mu_A(x)$ describes the degree of the belongingness of $x \in X$ in *A*. When $\mu_A(x)$ is valued in {0, 1}, it is the characteristic function of a crisp (non-fuzzy) set. The measure of information defined by [Shannon, 1948] is given by

$$H(P)=-\sum_{i=1}^{n} p_i \log p_i \tag{1}$$

Taking into consideration the concept of fuzzy sets, [De Luca and Termini, 1972] introduced the measure of fuzzy entropy corresponding to Shannon's entropy given in (1) as

$$H(A)=-\sum_{i=1}^{n}[\mu_A(\mathrm{x_i})\log\mu_A(\mathrm{x_i})+(1-\mu_A(\mathrm{x_i}))\log(1-\mu_A(\mathrm{x_i}))] \tag{2}$$

The measure of directed divergence of probability distribution $P=(p_1,p_2,...p_n)$ from probability distribution $Q=(q_1,q_2,...q_n)$ is obtained by [Kullback and Leibler, 1951] is as

$$D(P:Q)=\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} \tag{3}$$

Measure of fuzzy divergence between two fuzzy sets gives the difference between two fuzzy sets and this measure of distance/difference between two fuzzy sets is called the fuzzy divergence measure.

Sequence of fuzzy divergence measures and inequalities

The measure of fuzzy directed divergence corresponding to (3) is introduced by [Bhandari and Pal, 1993] is as

$$I(A:B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right] \qquad (4)$$

Table 1 presents the fuzzy mean divergence measures corresponding to seven geometrical mean measures given in [Taneja, 2012].

Table 1
Fuzzy Mean Divergence Measures

Sr. No.	**Fuzzy Mean Divergence Measure**	**Definition**
1.	Fuzzy Arithmetic Mean Measure	$A(A,B)=\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}+\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}\right)$
2.	Fuzzy Geometric Mean Measure	$G(A,B)=\sum_{i=1}^{n}\left(\sqrt{\mu_A(x_i)\mu_B(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right)$
3.	Fuzzy Harmonic Mean Measure	$H(A,B)=\sum_{i=1}^{n}\left(\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right)$
4.	Fuzzy Heronian Mean Measure	$N(A,B)=\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}+\mu_B(x_i)}{3}+\frac{(1-\mu_A(x_i))+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}+(1-\mu_B(x_i))}{3}\right)$
5.	Fuzzy Contra-harmonic Mean Measure	$C(A,B)=\sum_{i=1}^{n}\left(\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right)$
6.	Fuzzy Root-mean-square Mean Measure	$S(A,B)=\sum_{i=1}^{n}\left(\sqrt{\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{2}}+\sqrt{\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2}}\right)$
7.	Fuzzy Centroidal Mean Measure	$R(A,B)=\sum_{i=1}^{n}\left(\frac{2(\mu_A^2(x_i)+\mu_A(x_i)\mu_B(x_i)+\mu_B^2(x_i))}{3(\mu_A(x_i)+\mu_B(x_i))}+\frac{2\left((1-\mu_A(x_i))^2+(1-\mu_A(x_i))(1-\mu_B(x_i))+(1-\mu_B(x_i))^2\right)}{3(2-\mu_A(x_i)-\mu_B(x_i))}\right)$

We have the following Lemma in fuzzy context corresponding to the Lemma of [Taneja, 2005]:

Schwarz's Lemma 1: Let $f_1, f_2 : I \subset R_+ \to R$ be two convex functions satisfying the assumptions:

i) $f_1\left(\frac{1}{2}\right) = f_1'\left(\frac{1}{2}\right) = 0,\ \ f_2\left(\frac{1}{2}\right) = f_2'\left(\frac{1}{2}\right) = 0$;

ii) f_1 and f_2 are twice differentiable in R_+;

iii) there exist the real constants α, β such that $0 \le \alpha < \beta$ and $\alpha \le \frac{f_1''(z)}{f_2''(z)} \le \beta$, $f_2''(z) > 0$, for all $z > 0$

then we have the inequalities:

$$\alpha \varphi_{f_2}(a,b) \le \varphi_{f_1}(a,b) \le \beta \varphi_{f_2}(a,b)$$

for all $a, b \in (0,1)$, where the function $\varphi_{(.)}(a,b)$ is defined as

$$\varphi_f(a,b) = af\left(\frac{b}{a}\right),\ \ a, b > 0.$$

3. A SEQUENCE OF FUZZY MEAN DIFFERENCE DIVERGENCE MEASURES

Corresponding to fuzzy mean divergence measures defined in Section 2, we propose a sequence of fuzzy mean difference divergence measures as follows:

(1) $D_{CS}(A,B) =$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right] - \left[\sqrt{\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{2}} + \sqrt{\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2}}\right]\right\}$$

(2) $D_{CN}(A,B) =$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right] - \left[\frac{\mu_A(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}+\mu_B(x_i)}{3} + \frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}\right]\right\}$$

(3) $D_{CG}(A,B) =$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right] - \left[\sqrt{\mu_A(x_i)\mu_B(x_i)} + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]\right\}$$

(4) $D_{CR}(A,B) =$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right] - \left[\frac{2(\mu_A^2(x_i)+\mu_A(x_i)\mu_B(x_i)+\mu_B^2(x_i))}{3(\mu_A(x_i)+\mu_B(x_i))} + \frac{2((1-\mu_A(x_i))^2+(1-\mu_A(x_i))(1-\mu_B(x_i))+(1-\mu_B(x_i))^2)}{3(2-\mu_A(x_i)-\mu_B(x_i))}\right]\right\}$$

(5) $D_{CA}(A,B) =$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right] - \left[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2} + \frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}\right]\right\}$$

<u>Sequence of fuzzy divergence measures and inequalities</u>

(6) $D_{CH}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)}\right]-\left[\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]\right\}$$

(7) $D_{SA}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\sqrt{\frac{(\mu_A^2(x_i)+\mu_B^2(x_i))}{2}}+\sqrt{\frac{((1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2)}{2}}\right]-\left[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\right]\right\}$$

(8) $D_{SN}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\sqrt{\frac{(\mu_A^2(x_i)+\mu_B^2(x_i))}{2}}+\sqrt{\frac{((1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2)}{2}}\right]-\left[\frac{\mu_A(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}+\mu_B(x_i)}{3}+\frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}\right]\right\}$$

(9) $D_{SG}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\sqrt{\frac{(\mu_A^2(x_i)+\mu_B^2(x_i))}{2}}+\sqrt{\frac{((1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2)}{2}}\right]-\left[\sqrt{\mu_A(x_i)\mu_B(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]\right\}$$

(10) $D_{SH}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\sqrt{\frac{(\mu_A^2(x_i)+\mu_B^2(x_i))}{2}}+\sqrt{\frac{((1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2)}{2}}\right]-\left[\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]\right\}$$

(11) $D_{RA}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{2(\mu_A^2(x_i)+\mu_A(x_i)\mu_B(x_i)+\mu_B^2(x_i))}{3(\mu_A(x_i)+\mu_B(x_i))}+\frac{2((1-\mu_A(x_i))^2+(1-\mu_A(x_i))(1-\mu_B(x_i))+(1-\mu_B(x_i))^2)}{3(2-\mu_A(x_i)-\mu_B(x_i))}\right]-\left[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\right]\right\}$$

(12) $D_{RN}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{2(\mu_A^2(x_i)+\mu_A(x_i)\mu_B(x_i)+\mu_B^2(x_i))}{3(\mu_A(x_i)+\mu_B(x_i))}+\frac{2((1-\mu_A(x_i))^2+(1-\mu_A(x_i))(1-\mu_B(x_i))+(1-\mu_B(x_i))^2)}{3(2-\mu_A(x_i)-\mu_B(x_i))}\right]-\left[\frac{\mu_A(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}+\mu_B(x_i)}{3}+\frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}\right]\right\}$$

(13) $D_{RG}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{2(\mu_A^2(x_i)+\mu_A(x_i)\mu_B(x_i)+\mu_B^2(x_i))}{3(\mu_A(x_i)+\mu_B(x_i))}+\frac{2((1-\mu_A(x_i))^2+(1-\mu_A(x_i))(1-\mu_B(x_i))+(1-\mu_B(x_i))^2)}{3(2-\mu_A(x_i)-\mu_B(x_i))}\right]-\left[\sqrt{\mu_A(x_i)\mu_B(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]\right\}$$

(14) $D_{RH}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{2(\mu_A^2(x_i)+\mu_A(x_i)\mu_B(x_i)+\mu_B^2(x_i))}{3(\mu_A(x_i)+\mu_B(x_i))}+\frac{2((1-\mu_A(x_i))^2+(1-\mu_A(x_i))(1-\mu_B(x_i))+(1-\mu_B(x_i))^2)}{3(2-\mu_A(x_i)-\mu_B(x_i))}\right]-\left[\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]\right\}$$

(15) $D_{AN}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\right]-\left[\frac{\mu_A(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}+\mu_B(x_i)}{3}+\frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}\right]\right\}$$

(16) $D_{AG}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\right]-\left[\sqrt{\mu_A(x_i)\mu_B(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]\right\}$$

(17) $D_{AH}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\right]-\left[\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}+\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]\right\}$$

(18) $D_{NG}(A,B)=$

$$\sum_{i=1}^{n}\left\{\left[\frac{\mu_A(x_i)+\sqrt{\mu_A(x_i)\mu_B(x_i)}+\mu_B(x_i)}{3}+\frac{2-\mu_A(x_i)-\mu_B(x_i)+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}{3}\right]-\left[\sqrt{\mu_A(x_i)\mu_B(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]\right\}$$

Theorem 1: All the measures defined above are valid measure of fuzzy directed divergence.

Proof: (i) **Non-negativity:** The condition of non-negativity clearly holds.

(ii) **Symmetry:** All the purposed measures are symmetric in nature.

(iii) **Convexity:**
We now prove the condition of convexity

1) $$\frac{\partial^2 D_{CS}}{\partial\mu_A^2}=$$

$$4\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)-\frac{1}{2}\left(\sqrt{\frac{2}{\mu_A^2+\mu_B^2}}+\sqrt{\frac{2}{(1-\mu_A)^2+(1-\mu_B)^2}}\right)-\frac{1}{2}\left(\mu_A^2\sqrt{\frac{\mu_A^2+\mu_B^2}{2}}+(1-\mu_A)^2\sqrt{\frac{(1-\mu_A)^2+(1-\mu_B)^2}{2}}\right)>0$$

2) $$\frac{\partial^2 D_{CN}}{\partial\mu_A^2}=4\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)+\frac{1}{12}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

3) $$\frac{\partial^2 D_{CG}}{\partial\mu_A^2}=4\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)+\frac{1}{2}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

4) $$\frac{\partial^2 D_{CR}}{\partial\mu_A^2}=\frac{8}{3}\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

5) $$\frac{\partial^2 D_{CA}}{\partial\mu_A^2}=4\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

6) $$\frac{\partial^2 D_{CH}}{\partial\mu_A^2}=8\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

7) $$\frac{\partial^2 D_{AG}}{\partial\mu_A^2}=\frac{1}{4}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

8) $$\frac{\partial^2 D_{AH}}{\partial\mu_A^2}=4\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

9) $$\frac{\partial^2 D_{AN}}{\partial\mu_A^2}=\frac{1}{12}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

10) $$\frac{\partial^2 D_{SA}}{\partial \mu_A^2}=\left(\mu_A^2\sqrt{\frac{\mu_A^2+\mu_B^2}{2}}+\sqrt{\frac{2}{\mu_A^2+\mu_B^2}}+\sqrt{\frac{2}{(1-\mu_A)^2+(1-\mu_B)^2}}+(1-\mu_A)^2\sqrt{\frac{(1-\mu_A)^2+(1-\mu_B)^2}{2}}\right)>0$$

11) $$\frac{\partial^2 D_{NG}}{\partial \mu_A^2}=\frac{1}{6}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

12) $$\frac{\partial^2 D_{RA}}{\partial \mu_A^2}=\frac{4}{3}\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

13) $$\frac{\partial^2 D_{RG}}{\partial \mu_A^2}=\frac{4}{3}\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)+\frac{1}{2}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

14) $$\frac{\partial^2 D_{RH}}{\partial \mu_A^2}=\frac{16}{3}\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

15) $$\frac{\partial^2 D_{SN}}{\partial \mu_A^2}=$$
$$\left(\mu_A^2\sqrt{\frac{\mu_A^2+\mu_B^2}{2}}+\sqrt{\frac{2}{\mu_A^2+\mu_B^2}}+\sqrt{\frac{2}{(1-\mu_A)^2+(1-\mu_B)^2}}+(1-\mu_A)^2\sqrt{\frac{(1-\mu_A)^2+(1-\mu_B)^2}{2}}\right)+\frac{1}{12}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

16) $$\frac{\partial^2 D_{RN}}{\partial \mu_A^2}=\frac{4}{3}\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)+\frac{1}{12}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

17) $$\frac{\partial^2 D_{SG}}{\partial \mu_A^2}=$$
$$\left(\mu_A^2\sqrt{\frac{\mu_A^2+\mu_B^2}{2}}+\sqrt{\frac{2}{\mu_A^2+\mu_B^2}}+\sqrt{\frac{2}{(1-\mu_A)^2+(1-\mu_B)^2}}+(1-\mu_A)^2\sqrt{\frac{(1-\mu_A)^2+(1-\mu_B)^2}{2}}\right)+\frac{1}{4}\left(\frac{\sqrt{\mu_B}}{\mu_A^{3/2}}+\frac{\sqrt{1-\mu_B}}{(1-\mu_A)^{3/2}}\right)>0$$

18) $$\frac{\partial^2 D_{SH}}{\partial \mu_A^2}=\left(\mu_A^2\sqrt{\frac{\mu_A^2+\mu_B^2}{2}}+\sqrt{\frac{2}{\mu_A^2+\mu_B^2}}+\sqrt{\frac{2}{(1-\mu_A)^2+(1-\mu_B)^2}}+(1-\mu_A)^2\sqrt{\frac{(1-\mu_A)^2+(1-\mu_B)^2}{2}}\right)$$
$$+4\left(\frac{\mu_B^2}{(\mu_A+\mu_B)^3}+\frac{(1-\mu_B)^2}{(2-\mu_A-\mu_B)^3}\right)>0$$

Thus, all the measures defined above are valid measures of fuzzy directed divergence.

4. INEQUALITIES AMONG FUZZY MEAN DIFFERENCE DIVERGENCE MEASURES

Theorem 2: The above defined fuzzy divergence measures admit the following inequalities:

$$D_{SA} \le \left\{ \begin{matrix} \frac{3}{4} D_{SN} \\ \frac{1}{3} D_{SH} \le \frac{3}{4} D_{CR} \end{matrix} \right\} \le \left\{ \begin{matrix} \frac{3}{7} D_{CN} \le \left\{ \begin{matrix} D_{CS} \\ \frac{1}{3} D_{CG} \le \frac{3}{5} D_{RG} \end{matrix} \right. \\ \frac{1}{2} D_{SG} \le \frac{3}{5} D_{RG} \end{matrix} \right\} \le 3 D_{AN}$$

Proof: The proof of the above theorem is based on Lemma 1 and is given in parts in the following propositions.

Proposition 1: We have $D_{SA} \le \frac{3}{4} D_{SN}$

Proof: Let us consider the function $g_{SA\_SN}(z) = \frac{f''_{SA}(z)}{f''_{SN}(z)}$, where

$$f''_{SA}(z) = \frac{\sqrt{2}}{\left(z^2 + (1-z)^2\right)^{3/2}} \text{ and } f''_{SN}(z) = \frac{\sqrt{2}}{\left(z^2 + (1-z)^2\right)^{3/2}} + \frac{1}{6(z-z^2)^{3/2}}.$$

This gives

$$g'_{SA\_SN}(z) = \frac{3\sqrt{2}(2z-1)(z-z^2)^{1/2}(2z^2-2z+1)^{1/2}(4z^2-4z-1)}{\left[6\sqrt{2}(z-z^2)^{3/2} + (2z^2-2z+1)^{3/2}\right]^2} \begin{cases} >0 & for\ z < 1/2 \\ <0 & for\ z > 1/2 \end{cases}$$

And we have $\beta_{SA\_SN} = \sup_{z\in[0,1]} g_{SA\_SN}(z) = g_{SA\_SN}\left(\frac{1}{2}\right) = \frac{3}{4}$ (5)

By the application of Lemma 1 with (5), the proof holds.

Proposition 2: We have $D_{SA} \le \frac{1}{3} D_{SH}$

Proof: Let us consider the function $g_{SA\_SH}(z) = \frac{f''_{SA}(z)}{f''_{SH}(z)}$, where

$$f''_{SA}(z) = \frac{\sqrt{2}}{\left(z^2 + (1-z)^2\right)^{3/2}} \text{ and } f''_{SH}(z) = 8 + \frac{\sqrt{2}}{\left(2z^2 - 2z + 1\right)^{3/2}}.$$

This gives

$$g'_{SA\_SH}(z) = -\frac{12\sqrt{2}(2z^2-2z+1)^{1/2}(4z-2)}{\left[8(2z^2-2z+1)^{3/2} + \sqrt{2}\right]^2} \begin{cases} >0 & for\ z < 1/2 \\ <0 & for\ z > 1/2 \end{cases}$$

And we have $\beta_{SA\_SH} = \sup_{z\in[0,1]} g_{SA\_SH}(z) = g_{SA\_SH}\left(\frac{1}{2}\right) = \frac{1}{3}$ (6)

By the application of Lemma 1 with (6), the proof holds.

Proposition 3: We have $D_{SH} \le \frac{9}{4} D_{CR}$

Proof: Let us consider the function $g_{SH\_CR}(z) = \frac{f''_{SH}(z)}{f''_{CR}(z)}$, where

$$f''_{SH}(z) = 8 + \frac{\sqrt{2}}{\left(2z^2 - 2z + 1\right)^{3/2}} \text{ and } f''_{CR}(z) = \frac{16}{3}.$$

This gives

$$g'_{SH\_CR}(z) = -\frac{46\sqrt{2}(2z-1)}{49(2z^2 - 2z + 1)^{5/2}} \begin{cases} > 0 & for\, z < 1/2 \\ < 0 & for\, z > 1/2 \end{cases}$$

And we have $\beta_{SH\_CR} = \sup_{z \in [0,1]} g_{SH\_CR}(z) = g_{SH\_CR}\left(\frac{1}{2}\right) = \frac{9}{4}$ (7)

By the application of Lemma 1 with (7), the proof holds.

Proposition 4: We have $D_{CR} \le \frac{4}{7} D_{CN}$

Proof: Let us consider the function $g_{CR\_CN}(z) = \frac{f''_{CR}(z)}{f''_{CN}(z)}$, where

$$f''_{CR}(z) = \frac{16}{3} \text{ and } f''_{CN}(z) = 8 + \frac{1}{6\left(z - z^2\right)^{3/2}}.$$

This gives

$$g'_{CR\_CN}(z) = \frac{48(z - z^2)^{1/2}(1 - 2z)}{\left[48(z - z^2)^{3/2} + 1\right]^2} \begin{cases} > 0 & for\, z < 1/2 \\ < 0 & for\, z > 1/2 \end{cases}$$

And we have $\beta_{CR\_CN} = \sup_{z \in [0,1]} g_{CR\_CN}(z) = g_{CR\_CN}\left(\frac{1}{2}\right) = \frac{4}{7}$ (8)

By the application of Lemma 1 with (8), the proof holds.

Proposition 5: We have $D_{CR} \le \frac{2}{3} D_{SG}$

Proof: Let us consider the function $g_{CR\_CN}(z) = \frac{f''_{CR}(z)}{f''_{SG}(z)}$, where

$$f''_{CR}(z) = \frac{16}{3} \text{ and } f''_{SG}(z) = \frac{4}{(4z^2 - 4z + 2)^{3/2}} + \frac{1}{2\left(z - z^2\right)^{3/2}}.$$

This gives

$$g'_{CR\_SG}(z)=\frac{8(2z-1)(z-z^2)^{1/2}(4z^2-4z+2)^{1/2}\left[32(z-z^2)^{5/2}-(4z^2-4z+2)^{5/2}\right]}{\left[8(z-z^2)^{3/2}+(4z^2-4z+2)^{3/2}\right]^2}\begin{cases}>0 & for\ z<1/2\\ <0 & for\ z>1/2\end{cases}$$

And we have $\beta_{CR\_SG}=\sup_{z\in[0,1]} g_{CR\_SG}(z)=g_{CR\_SG}\left(\frac{1}{2}\right)=\frac{2}{3}$ (9)

By the application of Lemma 1 with (9), the proof holds.

Proposition 6: We have $D_{SN}\le\frac{4}{7}D_{CN}$

Proof: Let us consider the function $g_{SN\_CN}(z)=\frac{f''_{SN}(z)}{f''_{CN}(z)}$, where

$f''_{SN}(z)=\frac{4}{\left(4z^2-4z+2\right)^{3/2}}+\frac{1}{6(z-z^2)^{3/2}}$ and $f''_{CN}(z)=8+\frac{1}{6\left(z-z^2\right)^{3/2}}$.

This gives $g'_{SN\_CN}(z)=\frac{72(z-z^2)^{1/2}(4z^2-4z+2)^{1/2}(1-2z)\left[1+96(z-z^2)^{5/2}-(4z^2-4z+2)^{5/2}\right]}{\left(48(z-z^2)^{3/2}+1\right)^2\left(4z^2-4z+2\right)^3}\begin{cases}>0 & for\ z<1/2\\ <0 & for\ z>1/2\end{cases}$

And we have $\beta_{SN\_CN}=\sup_{z\in[0,1]} g_{SN\_CN}(z)=g_{SN\_CN}\left(\frac{1}{2}\right)=\frac{4}{7}$ (10)

By the application of Lemma 1 with (10), the proof holds.

Proposition 7: We have $D_{SN}\le\frac{2}{3}D_{SG}$

Proof: Let us consider the function $g_{SN\_SG}(z)=\frac{f''_{SN}(z)}{f''_{SG}(z)}$, where

$f''_{SN}(z)=\frac{4}{\left(4z^2-4z+2\right)^{3/2}}+\frac{1}{6(z-z^2)^{3/2}}$ and $f''_{SG}(z)=\frac{4}{\left(4z^2-4z+2\right)^{3/2}}+\frac{1}{2(z-z^2)^{3/2}}$.

This gives $g'_{SN\_SG}(z)=\frac{144(z-z^2)^{1/2}(4z^2-4z+2)^{1/2}(1-2z)}{\left[24(z-z^2)^{3/2}+3(4z^2-4z+2)^{3/2}\right]^2}\begin{cases}>0 & for\ z<1/2\\ <0 & for\ z>1/2\end{cases}$

And we have $\beta_{SN\_SG}=\sup_{z\in[0,1]} g_{SN\_SG}(z)=g_{SN\_SG}\left(\frac{1}{2}\right)=\frac{2}{3}$ (11)

By the application of Lemma 1 with (11), the proof holds.

Proposition 8: We have $D_{CN}\le\frac{7}{3}D_{CS}$

Proof: Let us consider the function $g_{CN\_CS}(z) = \frac{f''_{CN}(z)}{f''_{CS}(z)}$, where

$$f''_{CN}(z) = 8 + \frac{1}{6\left(z - z^2\right)^{3/2}} \text{ and } f''_{CS}(z) = 8 - \frac{4}{(4z^2 - 4z + 2)^{3/2}}$$

And we have $\beta_{CN\_CS} = \sup_{z\in[0,1]} g_{CN\_CS}(z) = g_{CN\_CS}\left(\frac{1}{2}\right) = \frac{7}{3}$ (12)

By the application of Lemma 1 with (12), the proof holds.

Proposition 9: We have $D_{CS} \le 3D_{AN}$

Proof: Let us consider the function $g_{CS\_AN}(z) = \frac{f''_{CS}(z)}{f''_{AN}(z)}$, where

$$f''_{CS}(z) = 8 - \frac{4}{(4z^2 - 4z + 2)^{3/2}} \text{ and } f''_{AN}(z) = \frac{1}{6\left(z - z^2\right)^{3/2}}.$$

This gives $$g'_{CS\_AN}(z) = \frac{36(2z-1)(z-z^2)\left\{4(z-z^2)^{1/2} - (4z^2-4z+2)\left[2(4z^2-4z+2)^{3/2} - 1\right]\right\}}{(4z^2-4z+2)^{5/2}} \begin{cases} > 0 & for\ z < 1/2 \\ < 0 & for\ z > 1/2 \end{cases}$$

And we have $\beta_{CS\_AN} = \sup_{z\in[0,1]} g_{CS\_AN}(z) = g_{CS\_AN}\left(\frac{1}{2}\right) = 3$ (13)

By the application of Lemma 1 with (13), the proof holds.

Proposition 10: We have $D_{CN} \le \frac{7}{9} D_{CG}$

Proof: Let us consider the function $g_{CN\_CG}(z) = \frac{f''_{CN}(z)}{f''_{CG}(z)}$, where

$$f''_{CN}(z) = 8 + \frac{1}{6\left(z - z^2\right)^{3/2}} \text{ and } f''_{CG}(z) = 8 + \frac{1}{2\left(z - z^2\right)^{3/2}}$$

This gives $$g'_{CN\_CG}(z) = \frac{16(1-2z)(z-z^2)^{1/2}}{\left[16(z-z^2)^{3/2} + 1\right]^2} \begin{cases} > 0 & for\ z < 1/2 \\ < 0 & for\ z > 1/2 \end{cases}$$

And we have $\beta_{CN\_CG} = \sup_{z\in[0,1]} g_{CN\_CG}(z) = g_{CN\_CG}\left(\frac{1}{2}\right) = \frac{7}{9}$ (14)

By the application of Lemma 1 with (14), the proof holds.

Proposition 11: We have $D_{SG} \leq \frac{6}{5} D_{RG}$

Proof: Let us consider the function $g_{SG\_RG}(z) = \frac{f''_{SG}(z)}{f''_{RG}(z)}$, where

$$f''_{SG}(z) = \frac{4}{\left(4z^2 - 4z + 2\right)^{3/2}} + \frac{1}{2(z - z^2)^{3/2}} \text{ and } f''_{RG}(z) = \frac{4}{3} + \frac{1}{2(z - z^2)^{3/2}}$$

This gives

$$g'_{SG\_RG}(z) = \frac{9(2z-1)\left\{[(4z^2-4z+2)^{5/2} - 32(z-z^2)^{5/2}][8(z-z^2)^{3/2}+3] - 3[8(z-z^2)^{3/2} + (4z^2-4z+2)^{3/2}](4z^2-4z+2)\right\}}{2(4z^2-4z+2)^{5/2}(z-z^2)\left[8(z-z^2)^{3/2}+3\right]^2} \begin{cases} > 0 & \mathit{for}\ z < 1/2 \\ < 0 & \mathit{for}\ z > 1/2 \end{cases}$$

And we have $\beta_{SG\_RG} = \sup_{z\in[0,1]} g_{SG\_RG}(z) = g_{SG\_RG}\left(\frac{1}{2}\right) = \frac{6}{5}$ (15)

By the application of Lemma 1 with (15), the proof holds.

Proposition 12: We have $D_{CG} \leq \frac{9}{5} D_{RG}$

Proof: Let us consider the function $g_{CG\_RG}(z) = \frac{f''_{CG}(z)}{f''_{RG}(z)}$, where

$$f''_{CG}(z) = 8 + \frac{1}{2\left(z - z^2\right)^{3/2}} \text{ and } f''_{RG}(z) = \frac{4}{3} + \frac{1}{2(z - z^2)^{3/2}}.$$

This gives $g'_{CG\_RG}(z) = \frac{540(z-z^2)^{1/2}(1-2z)}{8(z-z^2)^{3/2}+3} \begin{cases} > 0 & \mathit{for}\ z < 1/2 \\ < 0 & \mathit{for}\ z > 1/2 \end{cases}$

And we have $\beta_{CG\_RG} = \sup_{z\in[0,1]} g_{CG\_RG}(z) = g_{CG\_RG}\left(\frac{1}{2}\right) = \frac{9}{5}$ (16)

By the application of Lemma 1 with (16), the proof holds.

Proposition 13: We have $D_{RG} \leq 5 D_{AN}$

Proof: Let us consider the function $g_{RG\_AN}(z) = \frac{f''_{RG}(z)}{f''_{AN}(z)}$, where

$$f''_{RG}(z) = \frac{4}{3} + \frac{1}{2(z - z^2)^{3/2}} \text{ and } f''_{AN}(z) = \frac{1}{6\left(z - z^2\right)^{3/2}}.$$

This gives $g'_{RG\_AN}(z) = 8(z - z^2)^{3/2} + 3 \begin{cases} > 0 & \mathit{for}\ z < 1/2 \\ < 0 & \mathit{for}\ z > 1/2 \end{cases}$

And we have $\beta_{RG\_AN} = \sup_{z\in[0,1]} g_{RG\_AN}(z) = g_{RG\_AN}\left(\frac{1}{2}\right) = 5$ (17)

By the application of Lemma 1 with (17), the proof holds.

5. NUMERICAL EXAMPLE

In this Section we numerically verify the inequalities established in Section 4. Let us consider two standard fuzzy sets:

$A = (0.5, 0.2, 0.3, 0.4, 0.1)$ and $B = (0.2, 0.1, 0.4, 0.4, 0.5)$.

From the calculated numerical values of fuzzy divergence measures given in Table 2, the inequalities proposed in theorem 2 are verified.

Table 2
Calculated Numerical Values of Fuzzy Divergence Measures in Inequalities for Fuzzy Sets A and B

D_{SA}	D_{SN}	D_{SH}	D_{SG}	D_{CR}	D_{CN}	D_{CS}	D_{CG}	D_{AN}	D_{RG}
0.1516	0.2091	0.4714	0.3241	0.2132	0.3773	0.1682	0.4923	0.0575	0.2791

6. CONCLUDING REMARKS

In this paper, we have proposed a sequence of fuzzy mean difference divergence measures. The validity of these fuzzy mean difference divergence measures is proved axiomatically. Furthermore, it establishes a sequence of inequalities among some of the proposed fuzzy divergence measures with their proof. The inequalities are also verified numerically with the help of a numerical example. The resulting fuzzy divergence measures are much simpler with the difference of the means involved. On establishing inequalities among them, they are therefore much more efficient computationally.

REFERENCES

1. Bhandari, D., Pal, N.R., (1993) Some new information measures for fuzzy sets. Information Sciences, vol. 67, nr. 3, pp.209– 228. http://www.sciencedirect.com/science/article/pii/002002559390073U
2. Bhatia, P. K., Singh, S. (2012) Three families of generalized fuzzy directed divergence. AMO-Advanced Modeling and Optimization, vol. 14, nr.3, pp.599– 614. *http://camo.ici.ro/journal/vol14/v14c14.pdf*
3. Boekee, D.E. and Van Der Lubbe, J.C.A., (1980) The *R*-norm information measure. Information and Control, vol. 45, pp.136– 155.
4. De Luca, A., Termini, S., (1972) A definition of non-probabilistic entropy in the setting of fuzzy set theory. Information and Control, vol. 20, nr. 4, pp.301– 312.
5. Fan, J., Xie, W., (1999) Distance measures and induced fuzzy entropy, Fuzzy Sets and Systems, vol. 104, pp. 305– 314.

6. Ferreri, C., (1980) Hyperentropy and related heterogeneity divergence and information measures. Statistica, vol. 40, nr. 2, pp.155– 168.

7. Ghosh, M., Das, D., Chakraborty, C. and Roy, A.K., (2010) Automated lecukocyte recognition using fuzzy divergence, Micron, vol. 41, pp.840– 846.

8. Hooda, D.S., (2004) On generalized measures of fuzzy entropy. Mathematica Slovaca, vol. 54, pp.315–325.

9. Kullback, S., Leibler, R.A., (1951) On information and sufficiency. Annals of Mathematical Statistics, vol. 22, nr. 1, pp.79– 86.

10. Montes, S., Couso, I., Gil, P. and Bertoluzza, C., (2002) Divergence measures between fuzzy sets, International Journal of Approximate Reasoning, vol. 30, pp.91– 105.

11. Parkash, O., Sharma, P. K., Kumar, S., (2006) Two new measures of fuzzy divergence and their properties. SQU Journal for Science, vol.11, pp.69– 77. *http://web.squ.edu.om/squjs/volum11/MATH041130corrected.pdf*

12. Shannon, C.E., (1948) A mathematical theory of communication. The Bell Syst. Tech. Journal, vol. 27, nr.3, pp. 379– 423. *http://cm.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf*

13. Taneja, I.J., (2005) On mean divergence measures. *http://arxiv.org/abs/math/0501298* , pp.1– 18.

14. Taneja, I.J., (2005) Refinement of Inequalities among means. *http://arxiv.org/abs/math/0505192*

15. Taneja, I.J., (2012) Inequalities having seven means and proportionality relations. *http://arxiv.org/abs/1203.2288*

16. Tomar, V.P., Ohlan A., (2014) Two new parametric generalized R– norm fuzzy information measures. International Journal of Computer Applications, vol. 93, nr. 13, pp. 22– 27.

17. Zadeh, L.A., (1965) Fuzzy sets. Information and Control, vol. 8, nr.3, pp. 338– 353. http://www.sciencedirect.com/science/article/pii/S001999586590241X

18. Zadeh, L.A., (1968) Probability measures of fuzzy events, Journal of Mathematical Analysis and Applications, vol. 23, pp. 421– 427.