A finite replenishment model with trade credit and variable deterioration for fixed lifetime products

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Abstract
This article explores a finite replenishment model with variable deterioration for fixed lifetime products. In this model, suppliers offer trade credit period to the retailers in order to increase the demand of their products. During the credit period, the retailers can earn more by selling their products. The interest on purchasing cost is charged for the delay of payment by the retailers. Some of the items may deteriorate in the course of time. The purpose of this model is to minimize the total cost of the system. Some numerical examples along with graphical representations are pointed out to illustrate the model. Finally, through numerical examples, a sensitivity analysis shows the influence of the key model parameters.

Keywords: Inventory; trade-credit policy; variable deterioration; fixed lifetime.

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1 Introduction

Along with the globalization of the market and increasing competition, the enterprises always take trade credit financing policy to promote sales, increase the market share, and reduce the current inventory levels. As a result, the trade credit financing play an important role in business as a source of funds after Banks or other financial institutions. In the traditional inventory economic order quantity (EOQ) model, it is assumed that the retailer must pay for the products when receiving them. In practice the suppliers often provide the delayed payment time for the payment of the amount owed. Usually, there is no interest charged for the retailer if the outstanding amount is paid in the allowable delay. However, if the payment is unpaid in full by the end of the permissible delay period, interest is charged on the outstanding amount.

Many researchers have studied inventory models in which the seller permits its buyers to delay payment without charging interest (i.e., trade credit). Goyal (1985) was the first proponent for developing an EOQ model under the conditions of permissible delay in payments (i.e., credit period). Shah (1993) considered a stochastic inventory model when items in inventory deteriorate and delays in payments are permissible. Aggarwal and Jaggi (1995) extended Goyal’s model to consider deteriorating items. Jamal et al. (1997) further generalized the EOQ model to allow for shortages. Teng (2002) established an easy analytical closed-form solution to the problem. Huang (2003) extended the trade credit problem to the case in which a supplier offers its retailer a credit period, and the retailer in turn provides another credit period to its customers. Liao (2008) extended Huang’s model to an EPQ model for deteriorating items. Teng (2009) provided the optimal ordering policies for a retailer who offers distinct trade credits to its good and bad customers. Lately, Teng et al. (2012) generalized traditional constant demand to non-decreasing demand. Several relevant articles related to this subject are Huang (2010), Kreng and Tan (2010, 2011), Zhou et al. (2012), and others.

In supply chain management, it is too difficult to preserve deteriorating items for all business sectors. Many products such as fruits, vegetables, high-tech products, pharmaceuticals, and volatile liquids not only deteriorate continuously due to evaporation, obsolescence and spoilage but also have their expiration dates, i.e., the product will have a maximum lifetime which is time bound. However, only a few researchers take the expiration date of a deteriorating item into consideration. In the present article, we consider the replenishment policies for inventory which are subject to deteriorate continuously and also have their expiration dates.

Many researchers have studied inventory models for deteriorating items such as volatile liquids, blood banks, medicines, electronic components and fashion goods. Ghare and Schrader (1963) were the first proponents for developing an inventory model with an exponentially decaying product. They categorized decaying inventory into three types: direct spoilage, physical depletion and deterioration. Misra (1975) developed an economic order quantity (EOQ) model with a Weibull deterioration rate for the perishable product. Dave and Patel (1981) considered an EOQ model for deteriorating items with time-proportional demand. Sachan (1984) then generalized the EOQ model by considering shortages. Teng et al. (2002) further generalized the EOQ model for deteriorating items to any time varying continuous demand. Yang and Wee (2003) established an economic production quantity (EPQ) model for deteriorating items. Goyal and Giri (2001) provided a review on trends in modeling of deteriorating inventory. Min et al. (2010) studied an EOQ model for deteriorating items under stock-dependent demand and two-level trade credit. Recently, Teng et al. (2011) generalized Soni and Shah (2008) to allow for deteriorating items with stock-dependent demand and progressive payment scheme. Many related articles to deteriorating items can be found in Atici et al. (2013), Bakker et al. (2012), Goyal and Giri (2003), Dye and Hsieh (2012), Dye et al. (2007), Liao (2008), Chang et al. (2010), Yang et al. (2010), Skouri et al. (2012), and their references. However, none of the above mentioned papers take the maximum lifetime into consideration. In reality, every product has its maximum lifetime or expiration date.

In this paper, we derive the retailer's optimal lot size policies in an EPQ model in which (1) the retailer offers a trade credit of M years to his/her customers, (2) a deteriorating product not only deteriorates continuously but also has its maximum lifetime, and (3) the replenishment rate is finite. Under these conditions, we model the retailer's inventory system as a cost minimization problem. Some theorems are developed to determine retailer's optimal ordering policies and numerical examples along with graphical representations are pointed out to illustrate the model. Finally, we study sensitivity analysis of the effects on the optimal solution with respect to each parameter, and then provide some managerial insights.

2 Notation and Assumptions

The following notations and assumptions are used throughout.

2.1 Notations

D = demand rate per year

P = replenishment rate per year, $P > D$

$\rho = 1 - \frac{D}{P} > 0$

h = unit stock holding cost per year excluding interest charges

A = ordering cost per order

c = unit purchasing cost

s = unit selling price ($s > c$)

M = the trade credit period in years

I_e = interest earned per \$ per year

I_c = interest charged per \$ in stocks per year

$\theta(t)$ = deterioration rate

t_1 = the time at which the production stops in a cycle

T = the cycle time in years

$TRC(T)$ = the annual total relevant cost, which is a function of T

T^* = the optimal cycle time of $TRC(T)$

2.2 Assumptions

1) Replenishment rate, P, is known and constant.
2) Demand rate, D, is known and constant and always $P > D$.
3) Shortages are not allowed and lead time is negligible.
4) Time horizon is infinite.
5) The deterioration rate is time dependent as $\theta(t) = \frac{1}{1+L-t}$, where $L > t$ and L is the maximum lifetime of products at which the total on-hand inventory deteriorates. When t increases, $\theta(t)$ increases and $\lim_{t\to L} \theta(t) = 1$.
6) During the time the account is not settled, generated sales revenue is deposited in an interest bearing account. When $T \geq M$, the account is settled at $T = M$ and we start paying for the interest charges on the items in stock. When $T \leq M$, the account is settled at $T = M$ and we do not need to pay any interest charges.

Given the above notation and assumptions, it is possible to formulate the retailer's annual total cost as a function of the replenishment cycle time T for deteriorating items with maximum lifetime into a mathematical model.

3 Model Formulation

A constant production rate starts at $t = 0$, and continues up to $t = t_1$ where the inventory level reaches the maximum level. Production then stops at $t = t_1$, and the inventory gradually depletes to zero at the end of the production cycle $t = T$ due to deterioration and consumption. Thereafter, during the time interval $(0, t_1)$, the system is subject to the effect of production, demand and deterioration. The graphical representation of this inventory system is clearly depicted in Fig. 1.

Then, the change in the inventory level can be described by the following differential equation:

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = P - D; \quad 0 \leq t \leq t_1 \tag{1}$$

with initial condition $I_1(0) = 0$.

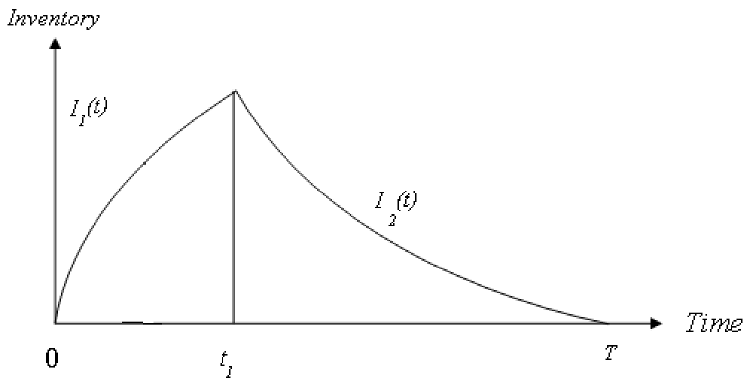


Figure 1: Graphical representation of inventory system.

On the other hand, in the time interval (t_1, T), the system is affected by the combined effect of demand and deterioration. Hence, the change in the inventory level is governed by the following differential equation:

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -D; \quad t_1 \le t \le T \tag{2}$$

The solution of the differential equations (1) and (2) are respectively represented by

$$I_1(t) = (P - D)(1 + L - t)\ln\left(\frac{1+L}{1+L-t}\right), \qquad 0 \le t \le t_1, \tag{3}$$

and

$$I_2(t) = D(1 + L - t)\ln\left(\frac{1+L-t}{1+L-T}\right), \qquad t_1 \le t \le T. \tag{4}$$

In addition, using the boundary condition at $t = t_1$, $I_1(t_1) = I_2(t_2)$, we obtain the following equations:

$$t_1 = (1 + L) - (1 + L - T)^{\frac{D}{P}}(1 + L)^{\frac{P-D}{P}}. \tag{5}$$

3.1 Determination of the annual total cost function

In this section, we shall derive the annual total relevant cost which consists of the following elements: annual ordering cost, annual stock-holding cost (excluding interest charges), annual cost due to deteriorated units, annual interest payable and annual interest earned. These components are evaluated as in the following:

1. Annual ordering cost $= \frac{A}{T}$.
2. Annual stock-holding cost (excluding interest charges)

$$HC = \frac{h}{T}\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right]$$
$$= \frac{h(P-D)}{T}\left[\left(\frac{\ln(1+L)}{2}+\frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] + \frac{hP(1+L-t_1)^2}{2T} ln(1+L-t_1)$$
$$+\frac{hD}{T}\left[-\ln(1+L-T)\frac{(1+L-T)^2}{2} + \left(\frac{\ln(1+L-T)}{2}+\frac{1}{4}\right)(2+2L-t_1-T)(t_1-T)\right]. \qquad (6)$$

Since $I_1(t_1) = I_2(t_1)$, which implies equation (7) can be rearranged as follows:

Annual stock-holding cost (excluding interest charges) $= \frac{h}{\theta T}(Pt_1 - DT)$.

3. Annual cost due to deteriorated units $= \frac{c}{T}(Pt_1 - DT)$.
4. There are three cases to occur in costs of interest charges for the items kept in stock per year.

Case 1. $M \le t_1$ i.e. $(1+L) - (1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}} \le T$ (as shown Fig. 2)

Annual interest payable is

$$= \frac{cI_c}{T}\left[\int_M^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right]$$
$$= \frac{cI_c(P-D)}{T}\left[\left(\frac{\ln(1+L)}{2}+\frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] + \frac{hP(1+L-t_1)^2}{2T}\ln(1+L-t_1)$$
$$+\frac{hD}{T}\left[-\ln(1+L-T)\frac{(1+L-T)^2}{2} + \left(\frac{\ln(1+L-T)}{2}+\frac{1}{4}\right)(2+2L-t_1-T)(t_1-T)\right].$$

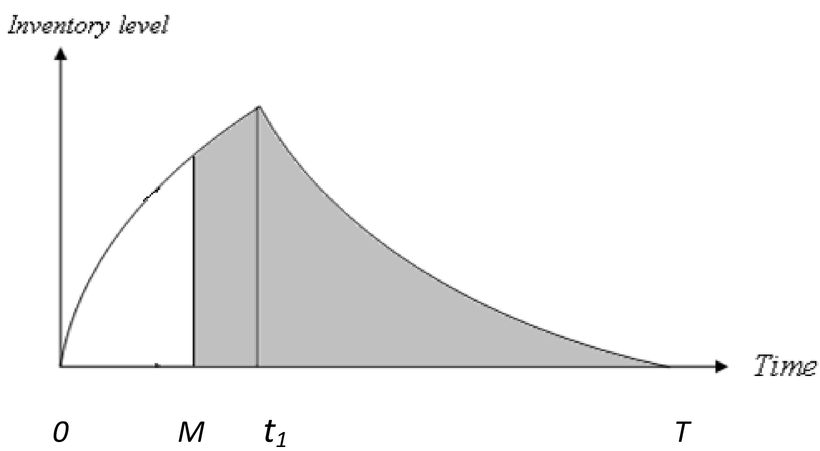


Figure 2: Total accumulation of interest payable when $T \ge T_M$

Case 2. $t_1 \le M \le T$ i.e. $M \le T \le (1+L) - (1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}$ (as shown Fig. 3)

In this case, annual interest payable is

$$= \frac{cI_c}{T}\int_{t_1}^{T} I_2(t)dt = \frac{cI_cD}{T}\left[\begin{array}{c}\ln(1+L-M)(1+L-M)^2 - \ln(1+L-T)(1+L-T)^2 \\ -(T-M)(2+2L-M-T)\left(\frac{1}{2}+\ln(1+L-T)\right)\end{array}\right].$$

Case 3. $0 \le T \le M$ (as shown in Fig. 4).

In this case, no interest charges are paid for the items.

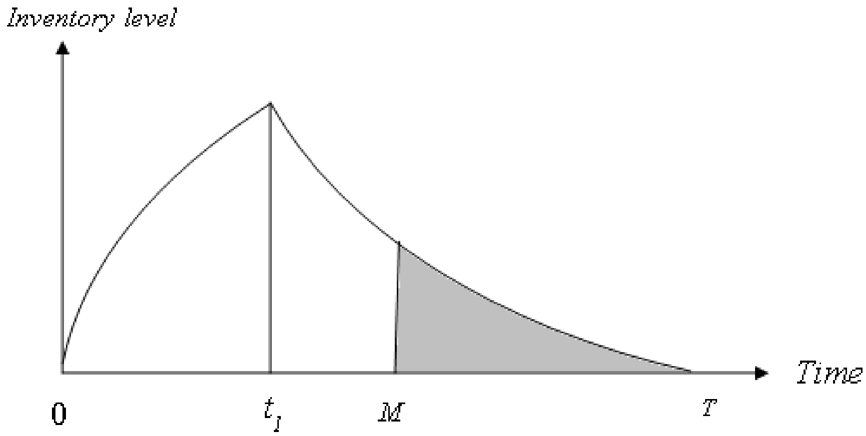


Figure 3: Total accumulation of interest payable when $M \leq T \leq T_M$

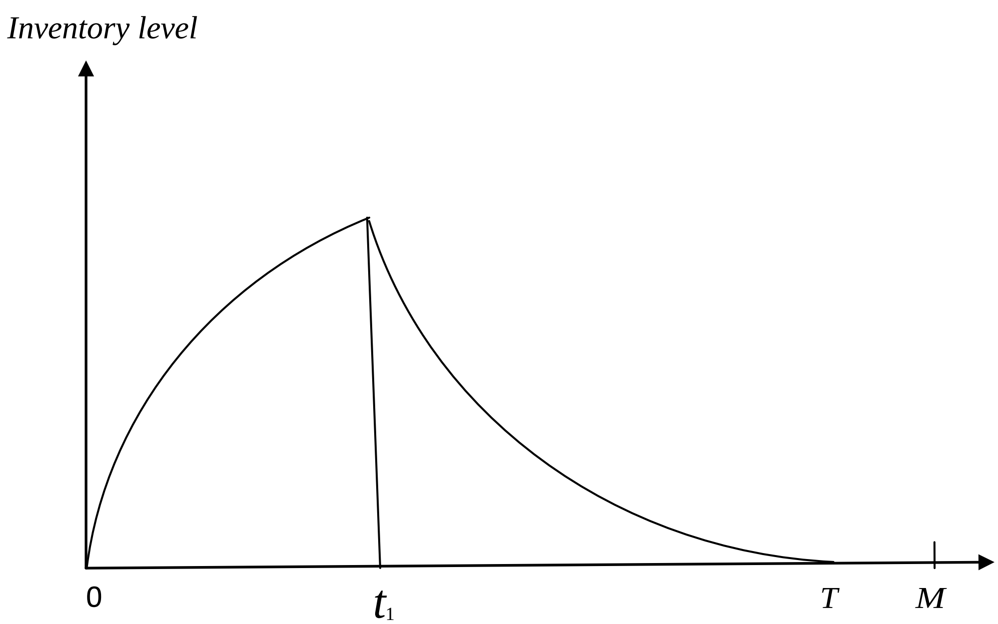


Figure 4: Total accumulation of interest payable when $T \leq M$

5. There are three cases to occur in interest earned per year.

Case 1. $M \leq t_1$ i.e. $(1+L)-(1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}} \leq T$ (as shown Fig. 5)

Annual interest earned $= \frac{sI_eDM^2}{2T}$.

Case 2. $t_1 \leq M \leq T$ i.e. $M \leq T \leq (1+L)-(1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}$

Similar as Case 1, annual interest earned $= \frac{sI_eDM^2}{2T}$.

Case 3. $0 \leq T \leq M$

As shown in Fig. 6, annual interest earned $= \frac{sI_e}{T}\left[\frac{DT^2}{2} + DT(M-T)\right] = sI_eD\left(M - \frac{T}{2}\right)$.

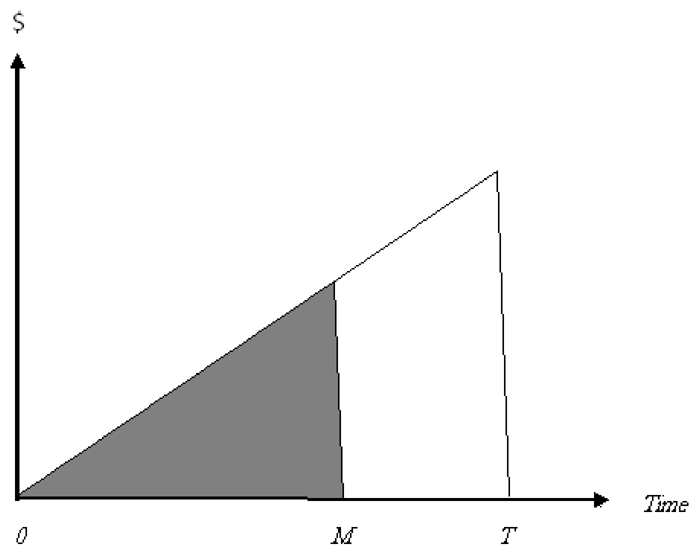


Figure 5: Total amount of interest earned when $M \leq T$

Figure 6: Total amount of interest earned when $T \leq M$

From the above arguments, the annual total relevant cost incurred at the retailer, $TRC(T)$, is $TRC(T)$ = ordering cost + stock-holding cost + deterioration cost + interest payable – interest earned.

$$TRC(T) = \begin{cases} TRC_1(T); & \text{if } T \geq (1+L) - (1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}, \\ TRC_2(T); & \text{if } M \leq T \leq (1+L) - (1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}, \\ TRC_3(T); & \text{if } 0 \leq T \leq M, \end{cases} \quad (7)$$

$$\begin{aligned} TRC_1(T) = {} & \frac{A}{T} + \frac{P(h+cI_c)(1+L-t_1)^2}{2T}\ln(1+L-t_1) - \frac{sI_eDM^2}{2T} + \frac{cPt_1}{T} - cD \\ & + \frac{h(P-D)}{T}\left[\left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] \\ & + \frac{(h+cI_c)D}{T}\left[-\ln(1+L-T)\frac{(1+L-T)^2}{2} + \left(\frac{\ln(1+L-T)}{2} + \frac{1}{4}\right)(2+2L-t_1-T)(t_1-T)\right] \\ & + \frac{cI_c(P-D)}{T}\left[\left(\frac{\ln(1+L)}{2} - \frac{1}{4}\right)(2+2L-t_1-M)(t_1-M) - \ln(1+L-M)\frac{(1+L-M)^2}{2}\right], \quad (8) \end{aligned}$$

$$\begin{aligned} TRC_2(T) = {} & \frac{A}{T} + \frac{h(P-D)}{T}\left[\left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] + \frac{hP(1+L-t_1)^2}{2T}\ln(1+ \\ & L-t_1) + \frac{hD}{T}\left[-\ln(1+L-T)\frac{(1+L-T)^2}{2} + \left(\frac{\ln(1+L-T)}{2} + \frac{1}{4}\right)(2+2L-t_1-T)(t_1-T)\right] + \\ & \frac{cDI_c}{2T}\left[-(T-M)(2+2L-M-T)\left(\frac{1}{2} + \ln(1+L-T)\right) + \ln(1+L-M)(1+L-M)^2 - \right. \\ & \left. \ln(1+L-T)(1+L-T)^2\right] - \frac{sI_eDM^2}{2T} + \frac{c}{T}(Pt_1 - DT) \quad (9) \end{aligned}$$

$$\begin{aligned} TRC_3(T) = {} & \frac{A}{T} + \frac{h(P-D)}{T}\left[\left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right] + \frac{hP(1+L-t_1)^2}{2T}\ln(1+ \\ & L-t_1) + \frac{hD}{T}\left[-\ln(1+L-T)\frac{(1+L-T)^2}{2} + \left(\frac{\ln(1+L-T)}{2} + \frac{1}{4}\right)(2+2L-t_1-T)(t_1-T)\right] - \\ & sI_eD\left(M - \frac{T}{2}\right) + \frac{c}{T}(Pt_1 - DT). \quad (10) \end{aligned}$$

Let $T_M = (1+L) - (1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}$. Since $TRC_1(T_M) = TRC_2(T_M)$ and $TRC_2(M) = TRC_3(M)$, $TRC(T)$ is continuous and well-defined on $T > 0$. All $TRC_1(T), TRC_2(T), TRC_3(T)$ and $TRC(T)$ are defined on $T > 0$.

4 SOLUTION PROCEDURE

To find the optimal solution, say $\left(T^*, TRC(T^*)\right)$, the following procedures are considered.

Definition 1: A function $f(x)$ defined on an open interval (a,b) is said to be convex if for $x, y \in (a,b)$ and each λ, $0 \leq \lambda \leq 1$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

Intermediate Value Theorem (In Real Analysis):

Let g be a continuous function on the closed interval $[a,b]$ and let $g(a).g(b) < 0$. Then there exists a number $c \in (a,b)$ such that $g(c) = 0$.

Lemma 1. If $f(t)$ is a continuous function on (a,b) and if $\frac{df}{dt}$ is non-decreasing, then $f(t)$ is convex.

Proof: Given x, y with $a<x<y<b$, define a function g on $[0,1]$ by $g(t)=tf(y)+(1-t)f(x)-f(ty+(1-t)x)$.

Our goal is to show that g is non-negative on $[0,1]$. Now g is continuous and $g(0)=g(1)=0$. Moreover, $\frac{dg(t)}{dt}=f(y)-f(x)-(y-x)\frac{df}{dt}$.

For $t+h>t$,

$$\frac{dg(t+h)}{dt}-\frac{dg(t)}{dt}=-(y-x)\left[\frac{df(t+h)}{dt}-\frac{df(t)}{dt}\right].$$

Since $\frac{df}{dt}$ is non decreasing, $\frac{df(t+h)}{dt}-\frac{df(t)}{dt}>0$. It implies that $\frac{dg(t)}{dt}$ is non-increasing on $[0,1]$. Let c be a point where g assumes its minimum on $[0,1]$ If $c=1$, then $g(t)\geq g(1)=0$ on $[0,1]$. In this case, g is non-negative. Suppose that $c\in(a,b)$. Since g has a local minimum at c, we have $\frac{dg(c)}{dt}\geq 0$. But $\frac{dg(t)}{dt}$ is non-increasing and so $\frac{dg(t)}{dt}\geq 0$ on $[0,c]$. Consequently, g is non-decreasing on $[0,c]$ and hence $g(c)\leq g(0)=0$, then the minimum of g on $[0,1]$ is non-negative and so $g>0$ on $[0,1]$. That is $f(ty+(1-t)x)\leq tf(y)+(1-t)f(x)$ on $[0,1]$. This implies $f(t)$ is convex.

4.1 Determination of the optimal replenishment cycle length *T*

The objective here is to find the optimal cycle time to minimize the annual total relevant cost.

Case 1. $T \geq (1+L)-(1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}$

In order to find the optimal solution T^* for the case of $T \geq (1+L)-(1+L-T)^{\frac{D}{P}}(1+L)^{\frac{P-D}{P}}$, we derive the first-order necessary condition for $TRC_1(T)$ in equation (8) to be minimized is

$\frac{dTRC_1(T)}{dT}=\frac{f_1(T)}{T^2}=0$,

where

$$f_1(T) = (h + cI_c)DT\left[(2 + 2L - t_1 - T)\left(\frac{t_1-T}{2(1+L-T)} - \frac{\ln(1+L-T)}{2}\right) - (1 + L - T) - \ln(1 + L - T)(t_1 + 1 + L - 2T)\right] - A - h(P - D)\left[\left(\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2 + 2L - t_1)t_1 - \ln(1 + L)\frac{(1+L)^2}{2}\right] - \left(\frac{hP}{2} + cPI_c\right)(1 + L - t_1)^2\frac{\ln(1+L-t_1)}{2} - (h + cI_c)D\left[-(1 + L - T)^2\frac{\ln(1+L-T)}{2} + \left(\frac{\ln(1+L-T)}{2} + \frac{1}{4}\right)(2 + 2L - t_1 - T)(t_1 - T)\right] - cPt_1 + \frac{sI_eDM^2}{2} - cI_c(P - D)\left[\left(-\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2 + 2L - t_1 - M)(t_1 - M) - \ln(1 + L - M)\frac{(1+L-M)^2}{2}\right] - cI_cD\left[\left(\frac{1}{4} + \frac{\ln(1+L-T)}{2}\right)(2 + 2L - t_1 - T)(T - t_1) + \ln(1 + L - T)\frac{(1+L-M)^2}{2}\right]. \quad (11)$$

Then both $f_1(T)$ and $\frac{dTRC_1(T)}{dT}$ have the same sign and domain. The optimal value of T, say T_1^*, can be obtained by solving the equation $f_1(T) = 0$. From equation (11) we obtain $\frac{df_1(T)}{dT} > 0$, if $T > 0$. Hence $f_1(T)$ is increasing on $(0, \infty)$ and so $\frac{dTRC_1(T)}{dT}$ is increasing on $(0, \infty)$. From lemma 1, $TRC_1(T)$ is convex function on $(0, \infty)$. On the other hand, $\lim_{T\to\infty} f_1(T) = \infty > 0$ and from equation (11), we obtain

$$f_1(0) = -\left[A + h(P - D)\left[\left(\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2 + 2L - t_1)t_1 - \ln(1 + L)\frac{(1+L)^2}{2}\right] + \left(\frac{hP}{2} + cPI_c\right)(1 + L - t_1)^2\frac{\ln(1+L-t_1)}{2} + hD\left[-(1 + L)^2\frac{\ln(1+L)}{2} + \left(\frac{\ln(1+L)}{2} + \frac{1}{4}\right)(2 + 2L - t_1)t_1\right] + cPt_1 - \frac{sI_eDM^2}{2} + cI_c(P - D)\left[\left(-\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2 + 2L - t_1 - M)(t_1 - M) - \ln(1 + L - M)\frac{(1+L-M)^2}{2}\right]\right].$$

Hence we see that

$$\frac{dTRC_1(T)}{dT}\begin{cases} < 0; & \text{if } T \in (0, T_1^*) \quad (a) \\ = 0; & \text{if } T = T_1^* \quad (b) \\ > 0; & \text{if } T \in (T_1^*, \infty) \quad (c) \end{cases} \quad (12)$$

Based upon the above arguments, we sure that the optimal solution, T_1^*, not only exists but also is unique.

The similar procedure as described in case 1 can be applied to the remaining two cases.

Case 2. $M \leq T \leq (1 + L) - (1 + L - T)^{\frac{D}{P}}(1 + L)^{\frac{P-D}{P}}$

Similarly, for the case 1, the first-order necessary condition for $TRC_2(T)$ in equation (9) to be minimized is

$\frac{dTRC_2(T)}{dT} = \frac{f_2(T)}{T^2} = 0,$

where

$$f_2(T) = -\Big[A + h(P-D)\Big\{\Big(\frac{1}{4} + \frac{\ln(1+L)}{2}\Big)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big\} + hP(1+L-t_1)^2\frac{\ln(1+L-t_1)}{2} - hD\Big\{(1+L-T)^2\frac{\ln(1+L-T)}{2} - \Big(\frac{\ln(1+L-T)}{2} + \frac{1}{4}\Big)(2+2L-t_1-T)(t_1-T)\Big\} + \frac{cDI_c}{2}\{-(T-M)(2+2L-M-T)\}\Big(\frac{1}{2} + \ln(1+L-T)\Big) + \ln(1+L-M)(1+L-M)^2 - (1+L-T)^2\ln(1+L-T) - \frac{sI_eDM^2}{2} + cPt_1 - \frac{cI_cDT}{2}\Big\{(2T+M-2L-2) - (3T-M-2L-2)\ln(1+L-T) - (2L+2-T-M)\Big((T-M)\frac{1}{1+L-T} + \ln(1+L-T)\Big)\Big\}\Big] \quad (13)$$

Then both $f_2(T)$ and $\frac{dTRC_2(T)}{dT}$ have the same sign and domain. The optimal value of T, say T_2^*, can be obtained by solving the equation $f_2(T) = 0$. From equation (13) we obtain $\frac{df_2(T)}{dT} > 0$, if $T > 0$. Hence $f_2(T)$ is increasing on $(0, \infty)$ and so $\frac{dTRC_2(T)}{dT}$ is increasing on $(0, \infty)$. From lemma 1, $TRC_2(T)$ is convex function on $(0, \infty)$. On the other hand, $\lim_{T\to\infty} f_2(T) = \infty > 0$ and from equation (11), we obtain

$$f_2(0) = -\Big[A + h(P-D)\Big\{\Big(\frac{1}{4} + \frac{\ln(1+L)}{2}\Big)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\Big\} + hP(1+L-t_1)^2\frac{\ln(1+L-t_1)}{2} - hD\Big\{(1+L)^2\frac{\ln(1+L)}{2} + \Big(\frac{1}{4} - \frac{\ln(1+L)}{2}\Big)(2+2L-t_1)t_1\Big\} + \frac{cDI_c}{2}\{M(2+2L-M)\}\Big(\frac{1}{2} + \ln(1+L)\Big) + \ln(1+L-M)(1+L-M)^2 - (1+L)^2\ln(1+L) - \frac{sI_eDM^2}{2} + cPt_1\Big]$$

Hence we see that

$$\frac{dTRC_2(T)}{dT}\begin{cases} < 0; & \text{if } T \in (0, T_2^*) \quad (a) \\ = 0; & \text{if } T = T_2^* \quad (b) \\ > 0; & \text{if } T \in (T_2^*, \infty) \quad (c) \end{cases} \quad (14)$$

Based upon the above arguments, we sure that the optimal solution, T_2^*, not only exists but also is unique.

Case 3. $0 < M \leq T$

Likewise, for the case of $0 < M \leq T$, the first-order necessary condition for $TRC_3(T)$ in equation (10) to be minimized is

$\frac{dTRC_3(T)}{dT} = \frac{f_3(T)}{T^2} = 0,$

where

$$f_3(T) = hDT\left[(2+2L-t_1-T)\left(\frac{t_1-T}{2(1+L-T)} - \frac{\ln(1+L-T)}{2}\right) - (1+L-T) - \ln(1+L-T)(t_1 + 1 + L - 2T)\right] - A - h(P-D)\left\{\left(\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right\} - \frac{hP(1+L-t_1)^2}{2}\ln(1+L-t_1) - hD\left\{-(1+L-T)^2\frac{\ln(1+L-T)}{2} + \left(\frac{\ln(1+L-T)}{2} + \frac{1}{4}\right)(2+2L-t_1-T)(t_1-T)\right\} + \frac{sI_eDT}{2} - cPt_1. \quad (13)$$

Then both $f_3(T)$ and $\frac{dTRC_3(T)}{dT}$ have the same sign and domain. The optimal value of T, say T_3^*, can be found by solving the equation $f_3(T) = 0$. From equation (15) we obtain $\frac{df_3(T)}{dT} > 0$, if $T > 0$. Hence $f_3(T)$ is increasing on $(0,\infty)$ and so $\frac{dTRC_3(T)}{dT}$ is increasing on $(0,\infty)$. From lemma 1, $TRC_3(T)$ is convex function on $(0,\infty)$. On the other hand, $\lim_{T\to\infty} f_3(T) = \infty > 0$ and from equation (13), we obtain

$$f_3(0) = -\left[A + h(P-D)\left\{\left(\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2+2L-t_1)t_1 - \ln(1+L)\frac{(1+L)^2}{2}\right\} + hP(1+L-t_1)^2\frac{\ln(1+L-t_1)}{2} + hD\left\{-(1+L)^2\frac{\ln(1+L)}{2} + \left(\frac{1}{4} + \frac{\ln(1+L)}{2}\right)(2+2L-t_1)t_1\right\} + cPt_1\right].$$

Hence we see that

$$\frac{dTRC_3(T)}{dT}\begin{cases} < 0; \quad \text{if} \quad T \in (0, T_3^*) & (a) \\ = 0; \quad \text{if } T = T_3^* & (b) \\ > 0; \quad \text{if } T \in (T_3^*, \infty) & (c) \end{cases} \quad (16)$$

Based upon the above arguments, the intermediate value theorem yields that the optimal solution, T_3^*, not only exists but also is unique.

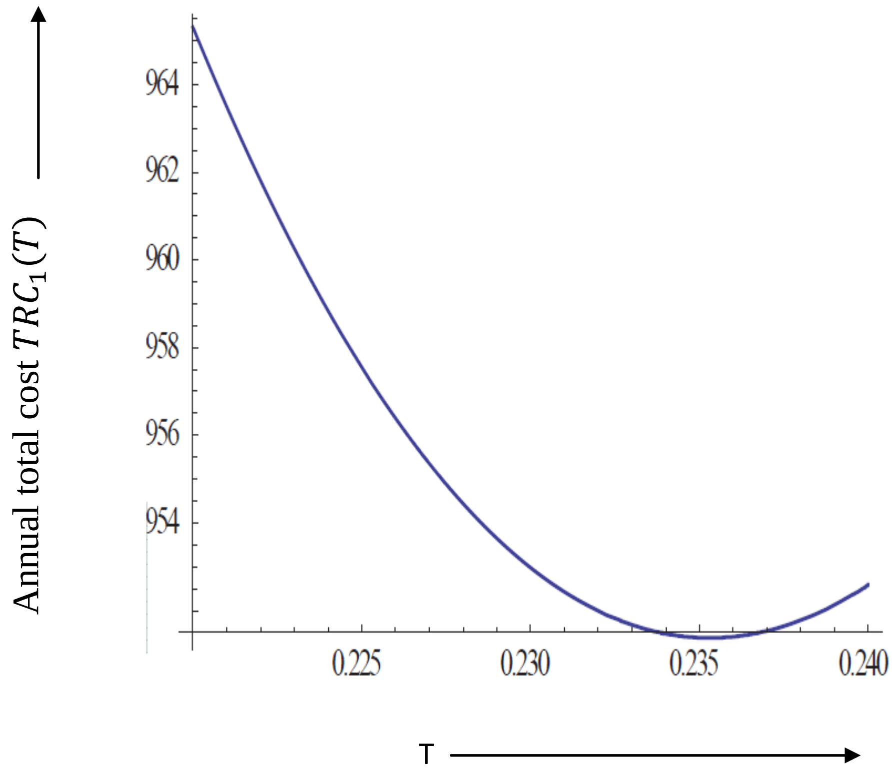
In the next section, we provide several numerical examples in order to illustrate theoretical results.

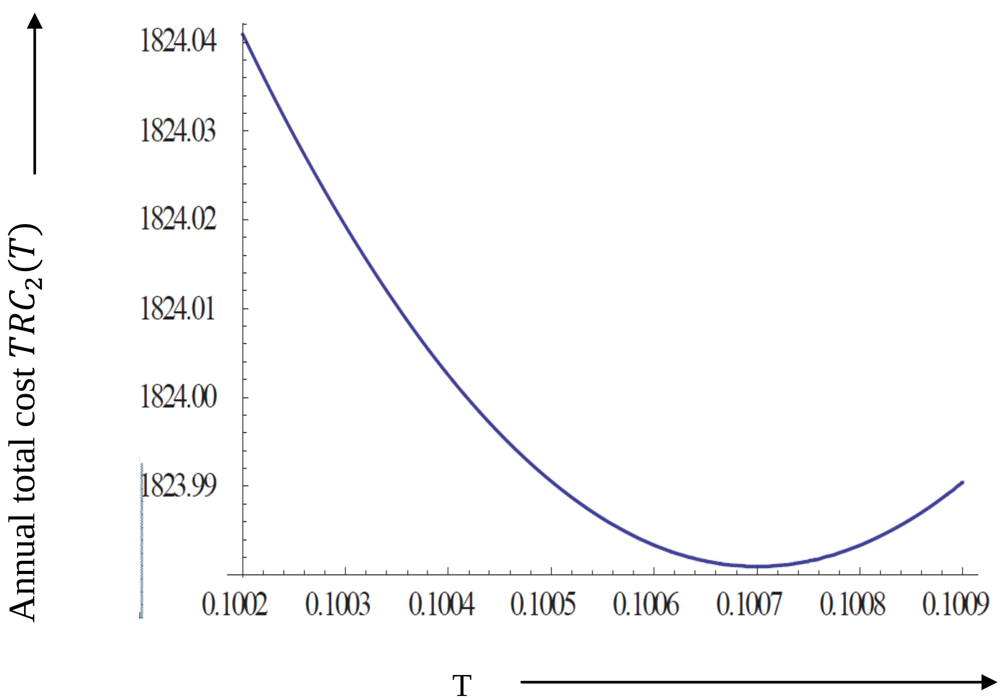
5 Numerical Examples

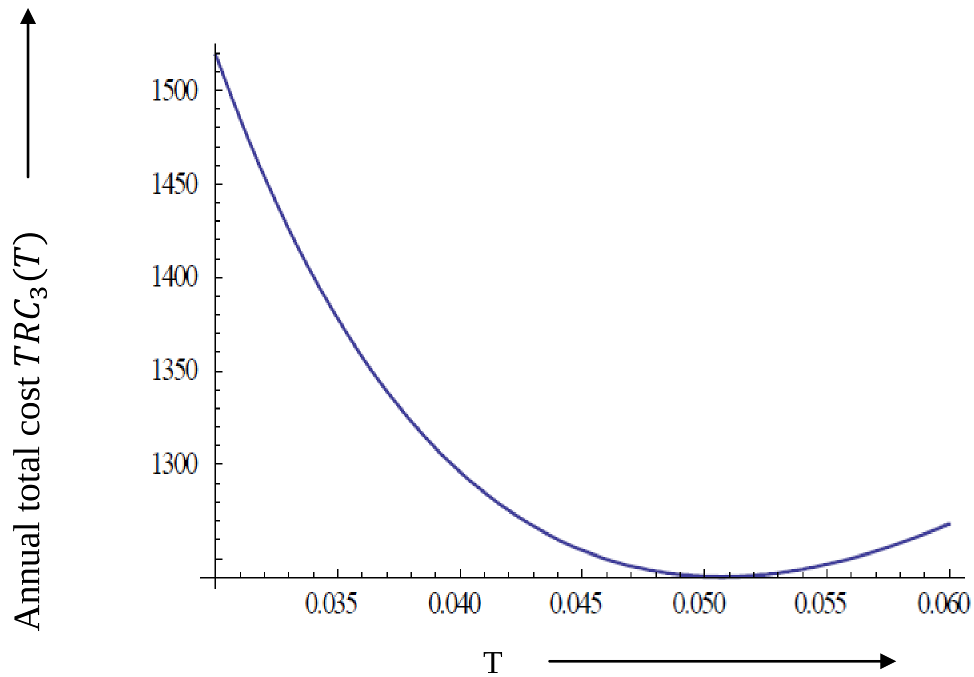
The following numerical examples are given to illustrate the proposed model.

Example 1: Let $A = \$150/\text{order}$, $D = 2500$ units/year, $P = 3000$ units/year, $s = \$75/\text{unit}$, $c = \$50/\text{unit}$, $h = \$15/\text{unit/year}$, $I_c = \$0.15$/\$/year, $I_e = \$0.1$/\$/year, and $M = 0.1$ year, $L = 6$ year, then the optimal solution is $T^* = T_1^* = 0{,}235297$ year and $TRC(T^*) = TRC(T_1^*) = \$951.3795$. Figure 7 indicates the minimum of the annual total cost $TRC_1(T)$ at the optimal cycle time (T).

Example 2: Let $A = \$100/\text{order}$, $D = 2500$ units/year, $P = 3500$ units/year, $s = \$75/\text{unit}$, $c = \$50/\text{unit}$, $h = \$15$/unit/year, $I_c = \$0.24$/\$/year, $I_e = \$0.15$/\$/year, and $M = 0.1$ year, $L = 4$ year, then the optimal solution is $T^* = T_2^* = 0.100713$ year and $TRC(T^*) = TRC(T_2^*) = \$1823.9783$. Figure 8 indicates the minimum of the annual total cost $TRC_2(T)$ at the optimal cycle time (T).







Example 3: Let $A = \$50/\text{order}$, $D = 2500$ units/year, $P = 4000$ units/year, $s = \$100/\text{unit}$, $c = \$50/\text{unit}$, $h = \$15$/unit/year, $I_c = \$0.24$/\$/year, $I_e = \$0.15$/\$/year, and $M = 0.8$ year, $L = 1$ year, then the optimal solution is $T^* = T_3^* = 0.051152$ year and $TRC(T^*) = TRC(T_3^*) = \$1240.0683$. Figure 9 indicates the minimum of the annual total cost $TRC_3(T)$ at the optimal cycle time (T).

5.1 Effect of changing the inventory model parameters

Here, we consider the following example. Let $A = \$150/\text{order}$, $D = 2500$ units/year, $P = 3000$ units/year, $s = \$75/\text{unit}$, $c = \$50/\text{unit}$, $h = \$15$/unit/year, $I_c = \$0.15$/\$/year, $I_e = \$0.1$/\$/year, and $M = 0.1$ year, $L = 6$ year. The sensitivity analysis is performed by varying the different parameters and is given in Table 1. It is important to discuss the influence of the key model parameters on the optimal solutions. The effect of changing the parameters is shown in Table 1. Based on Table 1, we have the following comments.

1) As ordering cost, A, increases, the replenishment cycle time, T^*, and the optimal annual total cost, $TRC(T^*)$, all significantly increase. The economic interpretation is as follows:

the retailer needs to order more to reduce the number of orders if the ordering cost is more expensive.

Table 1: Sensitivity analysis for various inventory parameters

Parameters		T^*	$TRC(T^*)$
	150	0.1073512	919.6189
	175	0.1159494	1143.498
A	200	0.1297641	1349.422
	225	0.1471730	1529.892
	250	0.1627314	1691.170
	55	0.1471703	1529.892
	65	0.1387404	1442.488
s	75	0.1297641	1349.422
	85	0.1201235	1249.426
	95	0.1159514	1143.498
	3000	0.1297715	1349.422
	3500	0.1131585	1659.965
P	4000	0.1066763	1874.611
	4500	0.1023335	2033.490
	5000	0.1003526	2110.577
	4	0.1239379	1414.609
	5	0.1261742	1388.906
L	6	0.1285384	1362.712
	7	0.1310294	1335.997
	8	0.1316883	1329.233
	10	0.1451091	1206.857
	12	0.1383513	1265.809
h	15	0.1297715	1349.422
	18	0.1226054	1428.150
	21	0.1184113	1503.102
	0.05	0.1608769	1671.863
	0.07	0.1492055	1550.971
Ie	0.09	0.1365436	1419.795
	0.11	0.1226095	1275.162
	0.13	0.1149108	1116.430

2) The larger the value of the unit selling price, s, the smaller the value of the optimal cycle time, T^*, and the smaller the value of the annual total relevant cost, $TRC(T^*)$. That is, when the unit selling price is increasing, the retailer should order less to gain more frequently the benefits of trade credit.

3) As production rate, P, increases, $TRC(T^*)$ increases; so it is not advisable to increase the production rate without prior knowledge of the demands.
4) A higher value of the maximum lifetime of products, L, results in higher value for the optimal cycle time, T^*, and a lower value for the annual total relevant cost, $TRC(T^*)$. This shows that if the maximum lifetime L is higher, then it is worth to increase the cycle time in order to increase the sales and decease the annual total cost.
5) When holding cost (h) increases, it is seen that cycle length, T^*, decreases whereas the optimal annual total cost, $TRC(T^*)$, increases. Thus, when the holding cost increases, the retailer shortens the cycle time and reduces order quantity to maintain the profit gained by maintaining the threshold credit period.
6) Our computational results show that a higher the rate of interest earned I_e, the lower the optimal cycle time, T^*, and the annual total relevant cost, $TRC(T^*)$. A simple economic interpretation is as follows: a higher value of I_e implies a higher value of the benefit from the permissible delay. Consequently, the retailer should order less and more frequently reap the benefits of the permissible delay.

6 Conclusions

In this paper, we formulated an inventory model for time varying deterioration for the fixed lifetime products under permissible delay in payment. Some of the total items, which are collected from the supplier, may have deteriorated with the increase of time since allowing a trade-credit period does not imply the selling of all products within time-boundary. Therefore, there may be a deteriorating factor which will be time-dependent. In the inventory literature, most of the deterioration is considered as constant or exponential along with the infinite replenishment rate. In this model, the author develops an EOQ model with time varying deterioration rate as $\theta(t) = \frac{1}{1+L-t}$, where L is the maximum lifetime of the product, i.e. when $t = L$, $\theta(t) = 1$ along with finite replenishment rate. When t increases, $\theta(t)$ will also increase. After formulating the model, we find the associated cost function which we have to minimize. The model is derived analytically. Some numerical examples and graphical representations are considered to illustrate the proposed model. The effect of the model parameters on the optimal replenishment time and on the optimal total variable cost per unit time are investigated using numerical examples. To the author's best knowledge, such a type of model has not yet been discussed in the existing literature.

The results of this paper not only provide a valuable reference for decision-makers in planning production and controlling inventory but also provide a useful model for organizations that use the decision rule to improve their total operating costs in the real world. Finally, there are several extensions of this work that could constitute future research related in this field. One immediate possible extension could be to discuss the effect of inflation. The model may be an extended inventory to the multi-item EOQ model. These are, among others, some models of ongoing future research.

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