AMO - Advanced Modeling and Optimization, Volume 16, Number 2, 2014

Representation formulae for Bertrand curves in Galilean and pseudo-Galilean 3-space

[1]Mahmut ERGÜT ,[1] Handan ÖZTEKİN and [2]Hülya GÜN BOZOK
[1]Department of Mathematics, Firat University, 23119 Elazığ/TURKEY
e-mail: mergut@firat.edu.tr , handanoztekin@gmail.com
[2]Department of Mathematics, Osmaniye Korkut Ata University, 80000 Osmaniye/TURKEY
e-mail: hulya-gun@hotmail.com

Abstract In this study, we give some characterization of Bertrand curves in Galilean and pseudo-Galilean space. We obtain representation formulae for Bertrand curves in Galilean and pseudo-Galilean space. Then we find that this Bertrand curves are also circular helices.

Keywords Bertrand curve, Galilean and pseudo-Galilean space, representation formulae.

1.Introduction

The notion of Bertrand curves was discovered by J. Bertrand in 1850, then they play an important role in classical differential geometry. Two curves which, at any point, have a common principal normal vector are called Bertrand curves. Bertrand curves are characterized as special curves whose curvature and torsion are in linear relation. Thus Bertrand curves may be regarded as 1-dimensional analogue of Weingarten surfaces. Throughout the years many mathematicians have studied Bertrand curves in different areas.

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. The fundamental concepts of Galilean geometry are expressed in [7], the pseudo-Galilean geometry like Galilean geometry which described in [3,14]. A necessary and sufficient condition that a curve to be Bertrand curve in Galilean 3-space G_3 is that the curve has constant torsion [2].

A curve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. Indeed, a helix is a special case of the general helix; if both the curvature $\kappa(s)$ and torsion $\tau(s)$ are non-zero constants, it is called a circular helix or simply a W-curve [9,11,13]. In fact, a circular helix is the simplest three-dimensional spiral.

Izumiya and Takeuchi have introduced the concept of Slant helix in Euclidean space by saying that the principal normal lines make a constant angle with a fixed direction [16]. A necessary and sufficient condition for a curve to be general helix in Galilean space is that ratio of curvature to torsion be constant [1].

In this paper we study representation formulae for Bertrand curves in Galilean and pseudo-Galilean space.

2.Preliminaries

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature (0,0,+,+) as in [5,17]. The absolute figure of Galilean geometry consist of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f.

Galilean scalar product can be written as

$$\langle v_1, v_2\rangle = \begin{cases} x_1 x_2 & ,if\ \ x_1 \neq 0 \vee x_2 \neq 0 \\ y_1 y_2 + z_1 z_2 & ,if\ \ x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.1}$$

where $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$. It leaves invariant the Galilean norm of the vector $v = (x, y, z)$ defined by

$$\|v\| = \begin{cases} x & ,if\ x \neq 0 \\ \sqrt{y^2 + z^2} & ,if\ x = 0 \end{cases} \tag{2.2}$$

[14].

If a curve C of the class C^r $(r \geq 3)$ is given by the parametrization

$$r = r\left(x, y(x), z(x)\right) \tag{2.3}$$

then x is a Galilean invariant the arc length on C.

The curvature is

$$\kappa(x)=\sqrt{y''^2+z''^2} \tag{2.4}$$

and torsion

$$\tau(x)=\frac{1}{\kappa^2(x)}\det\left(r'(x),r''(x),r'''(x)\right) \tag{2.5}$$

The orthonormal trihedron is defined

$$\begin{aligned} t(x)&=\left(1,y'(x),z'(x)\right)\\ n(x)&=\frac{1}{\kappa(x)}\left(0,y''(x),z''(x)\right)\\ b(x)&=\frac{1}{\kappa(x)}\left(0,-z''(x),y''(x)\right) \end{aligned} \tag{2.6}$$

The vectors t,n,b are called the vectors of tangent, principal normal and binormal line of, respectively. For their derivatives the following Frenet formulas hold

$$\begin{aligned} t'(x)&=\kappa(x)n(x)\\ n'(x)&=\tau(x)b(x)\\ b'(x)&=-\tau(x)n(x) \end{aligned} \tag{2.7}$$

[8].

Definition 2.1. Let α be a curve in Galilean 3-space and $\{t,n,b\}$ be the Frenet frame in Galilean 3-space along α. If κ and τ are positive constants along α, then α is called a circular helix with respect to Frenet frame [1].

Definition 2.2. Let α be a curve in Galilean 3-space and $\{t,n,b\}$ be the Frenet frame in Galilean 3-space along α. A curve α such that

$$\frac{\kappa}{\tau}=const$$

is called general helix with respect to Frenet frame [1].

Theorem 2.3. Let α be a curve in Galilean 3-space. Then α is a Bertrand curve if and only if α is a curve with constant torsion τ_α [2].

Remark 2.4. Similar definitions can be given in the pseudo-Galilean space.

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature (0,0,+,-), explained in [3]. The absolute of pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed hyperbolic involution of points of f.

As in [3], pseudo-Galilean inner product can be written as

$$\langle v_1, v_2\rangle = \begin{cases} x_1x_2 & ,if \;\; x_1 \neq 0 \vee x_2 \neq 0 \\ y_1y_2 - z_1z_2 & ,if \;\; x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.8}$$

where $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$. It leaves invariant the pseudo-Galilean norm of the vector $v = (x, y, z)$ defined by

$$\|v\| = \begin{cases} x & ,if \; x \neq 0 \\ \sqrt{|y^2 - z^2|} & ,if \; x = 0 \end{cases} \tag{2.9}$$

In pseudo-Galilean space a curve is given by $\alpha : I \to G_3^1$

$$\alpha(t) = (x(t), y(t), z(t)) \tag{2.10}$$

where $I \subseteq \mathbb{R}$ and $x(t), y(t), z(t) \in C^3$. A curve α given by (2.10) is admissible if $x'(t) \neq 0$ [3].

The curves in pseudo-Galilean space are characterized as follows [4]

An admissible curve in G_3^1 can be parametrized by arc length $t = s$, given in coordinate form

$$\alpha(s) = (s, y(s), z(s)) \tag{2.11}$$

For an admissible curve $\alpha : I \subseteq \mathbb{R} \to G_3^1$, the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(x) = \sqrt{|y''^2 - z''^2|} \tag{2.12}$$

$$\tau(s) = \frac{1}{\kappa^2(s)} \det(\alpha'(s), \alpha''(s), \alpha'''(s)) \tag{2.13}$$

The associated trihedron is given by

$$
\begin{aligned}
t(s) &= \alpha'(s) = (1, y'(s), z'(xs)) \\
n(s) &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\
b(s) &= \frac{1}{\kappa(s)}(0, z''(s), y''(s))
\end{aligned}
\tag{2.14}
$$

The vectors $t(s), n(s)$ and $b(s)$ are called the vectors of tangent, principal normal and binormal line of α, respectively. The curve α given by (2.11) is timelike if $n(s)$ is spacelike vector. For derivatives of tangent vector $t(s)$, principal normal vector $n(s)$ and binormal vector $b(s)$, respectively, the following Frenet formulas hold

$$
\begin{aligned}
t'(s) &= \kappa(s)n(s) \\
n'(s) &= \tau(s)b(s) \\
b'(s) &= \tau(s)n(s)
\end{aligned}
\tag{2.15}
$$

If the admissible curve β is given by $\beta(x) = (x, y(x), 0)$ and for this admissible curve the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(x) = y''(x) \tag{2.16}$$

$$\tau(x) = \frac{a_2'(x)}{a_3(x)} \tag{2.17}$$

where $a(x) = (0, a_2(x), a_3(x))$. The associated trihedron is given by

$$
\begin{aligned}
t(x) &= (1, y'(x), 0) \\
n(x) &= (0, a_2(x), a_3(x)) \\
b(x) &= (0, a_3(x), a_2(x))
\end{aligned}
\tag{2.18}
$$

For derivatives of tangent vector $t(s)$, principal normal vector $n(s)$ and binormal vector $b(s)$, respectively, the following Frenet formulas hold

$$
\begin{aligned}
t'(x) &= \kappa(x)(\cosh\phi(x)n(x) - \sinh\phi(x)b(x)) \\
n'(x) &= \tau(x)b(x) \\
b'(x) &= \tau(x)n(x)
\end{aligned}
\tag{2.19}
$$

where ϕ is the angle between $a(x)$ and the plane $z = 0$.

The unit pseudo-Galilean sphere is defined by

$S^2_{\mp} = \{u \in G^1_3 : \langle u,u \rangle = \mp 1\}$.

More information about pseudo-Galilean geometry can be found in [3].

3. Spherical curves and Bertrand curves in Galilean and pseudo-Galilean 3-space

In this section we give the method to construct Bertrand curves from spherical curves in Galilean and pseudo-Galilean 3-space.

Let $\gamma : I \to S^2$ be a unit speed spherical curve. In this section we denote σ as the arc-length parameter of γ. Let us denote $t(\sigma) = \dot{\gamma}(\sigma)$, and we call $t(\sigma)$ a unit tangent vector of γ at σ, where $\dot{\gamma} = \dfrac{d\gamma}{d\sigma}$. We now set a vector $s(\sigma) = \gamma(\sigma) \times t(\sigma)$. By definition we have an orthonormal frame $\{\gamma(\sigma), t(\sigma), s(\sigma)\}$ along γ. This frame is called Sabban frame of γ [10].

Theorem 3.1. Let $\gamma : I \to S^2_G$ be a unit speed spherical curve in Galilean 3-space. We denote σ as the arc-length parameter of γ. Then we have the following spherical Frenet-Serret formulae of γ:

$$\begin{aligned} \dot{\gamma}(\sigma) &= t(\sigma) \\ \dot{t}(\sigma) &= -\gamma(\sigma) + \kappa_g(\sigma) s(\sigma) \\ \dot{s}(\sigma) &= -\kappa_g(\sigma) t(\sigma) \end{aligned} \tag{3.1}$$

where $\kappa_g(\sigma)$ is the geodesic curvature of the curve γ in S^2_G which is given by $\kappa_g(\sigma) = \det(\gamma(\sigma), t(\sigma), \dot{t}(\sigma))$ or

$$\begin{aligned} \kappa_g(\sigma) &= \det(\gamma(\sigma), t(\sigma), \dot{t}(\sigma)) \\ &= \langle \gamma(\sigma) \times t(\sigma), \dot{t}(\sigma) \rangle \\ &= \langle s(\sigma), \dot{t}(\sigma) \rangle \end{aligned} \tag{3.2}$$

Also $\langle \dot{\gamma}(\sigma), t(\sigma) \rangle = 1$.

Theorem 3.2. Let $\gamma : I \to S^2_G$ be a unit speed spherical curve in pseudo-Galilean 3-space. We denote σ as the arc-length parameter of γ. Then we have the following spherical Frenet-Serret formulae of γ:

If $\langle \gamma(\sigma), \gamma(\sigma) \rangle = 1$, $\langle t(\sigma), t(\sigma) \rangle = 1$ and $\langle s(\sigma), s(\sigma) \rangle = -1$

$$\begin{aligned} \dot{\gamma}(\sigma) &= t(\sigma) \\ \dot{t}(\sigma) &= -\gamma(\sigma) - \kappa_g(\sigma) s(\sigma) \\ \dot{s}(\sigma) &= -\kappa_g(\sigma) t(\sigma) \end{aligned} \tag{3.3}$$

If $\langle \gamma(\sigma), \gamma(\sigma) \rangle = 1$, $\langle t(\sigma), t(\sigma) \rangle = -1$ and $\langle s(\sigma), s(\sigma) \rangle = 1$

$$\begin{aligned} \dot{\gamma}(\sigma) &= -t(\sigma) \\ \dot{t}(\sigma) &= \gamma(\sigma) + \kappa_g(\sigma) s(\sigma) \\ \dot{s}(\sigma) &= \kappa_g(\sigma) t(\sigma) \end{aligned} \tag{3.4}$$

If $\langle \gamma(\sigma), \gamma(\sigma) \rangle = -1$, $\langle t(\sigma), t(\sigma) \rangle = 1$ and $\langle s(\sigma), s(\sigma) \rangle = 1$

$$\begin{aligned} \dot{\gamma}(\sigma) &= t(\sigma) \\ \dot{t}(\sigma) &= \gamma(\sigma) + \kappa_g(\sigma) s(\sigma) \\ \dot{s}(\sigma) &= -\kappa_g(\sigma) t(\sigma) \end{aligned} \tag{3.5}$$

Theorem 3.3. Let $\gamma : I \to S_G^2$ be a unit speed spherical curve . Then

$$\tilde{\gamma}(\sigma) = a \int_{\sigma_0}^{\sigma} \gamma(u) du + a \tan\theta \int_{\sigma_0}^{\sigma} s(u) du + c \tag{3.6}$$

is a Bertrand curve, where a , θ are constant numbers and c is constant vector.

Proof. By using the method in [15] we calculate the curvature and torsion of $\tilde{\gamma}(\sigma)$.

In Galilean 3-space if we take the derivative of equation (3.6) with respect to σ , we have

$$\begin{aligned} \tilde{\gamma}'(\sigma) &= a\left(\gamma(\sigma) + \tan\theta s(\sigma)\right) \\ \tilde{\gamma}''(\sigma) &= \left(a - a\kappa_g \tan\theta\right) t(\sigma) \\ \tilde{\gamma}'''(\sigma) &= \left(a\kappa_g \tan\theta - a\right)\gamma(\sigma) + \kappa_g\left(a - a\kappa_g \tan\theta\right) s(\sigma) \end{aligned} \tag{3.7}$$

where $\kappa_g(\sigma)$ is the constant geodesic curvature of the curve γ in S_G^2. Then, by (2.4) and (2.5)

$$\kappa(\sigma) = \left|a - a\kappa_g \tan\theta\right| \text{ and } \tau(\sigma) = a\left(\kappa_g + \tan\theta\right)$$

so that $\tilde{\gamma}(\sigma)$ is a Bertrand curve and also it is a circular helix.

In pseudo-Galilean 3-space if we take the derivative of equation (3.6) with respect to σ, we have

Case I.

$$\begin{aligned}\tilde{\gamma}'(\sigma)&=a(\gamma(\sigma)+\tan\theta s(\sigma))\\ \tilde{\gamma}''(\sigma)&=(a-a\kappa_g\tan\theta)t(\sigma)\\ \tilde{\gamma}'''(\sigma)&=(a\kappa_g\tan\theta-a)\gamma(\sigma)+\kappa_g(a\kappa_g\tan\theta-a)s(\sigma)\end{aligned}\tag{3.8}$$

where $\kappa_g(\sigma)$ is the constant geodesic curvature of the curve γ in S_G^2. Then, by (2.12) and (2.13)

$$\kappa(\sigma)=|a-a\kappa_g\tan\theta| \text{ and } \tau(\sigma)=a(\tan\theta-\kappa_g)$$

so that $\tilde{\gamma}(\sigma)$ is a Bertrand curve and also it is a circular helix.

Case II.

$$\begin{aligned}\tilde{\gamma}'(\sigma)&=a(\gamma(\sigma)+\tan\theta s(\sigma))\\ \tilde{\gamma}''(\sigma)&=(a\kappa_g\tan\theta-a)t(\sigma)\\ \tilde{\gamma}'''(\sigma)&=(a\kappa_g\tan\theta-a)\gamma(\sigma)+\kappa_g(a\kappa_g\tan\theta-a)s(\sigma)\end{aligned}\tag{3.9}$$

where $\kappa_g(\sigma)$ is the constant geodesic curvature of the curve γ in S_G^2. Then, by (2.12) and (2.13)

$$\kappa(\sigma)=|a\kappa_g\tan\theta-a| \text{ and } \tau(\sigma)=a(\kappa_g-\tan\theta)$$

so that $\tilde{\gamma}(\sigma)$ is a Bertrand curve and also it is a circular helix.

Case III.

$$\begin{aligned}\tilde{\gamma}'(\sigma)&=a(\gamma(\sigma)+\tan\theta s(\sigma))\\ \tilde{\gamma}''(\sigma)&=(a-a\kappa_g\tan\theta)t(\sigma)\\ \tilde{\gamma}'''(\sigma)&=(a-a\kappa_g\tan\theta)\gamma(\sigma)+\kappa_g(a-a\kappa_g\tan\theta)s(\sigma)\end{aligned}\tag{3.10}$$

where $\kappa_g(\sigma)$ is the constant geodesic curvature of the curve γ in S_G^2. Then, by (2.12) and (2.13)

$$\kappa(\sigma)=|a-a\kappa_g\tan\theta| \text{ and } \tau(\sigma)=a(\kappa_g-\tan\theta)$$

so that $\tilde{\gamma}(\sigma)$ is a Bertrand curve and also it is a circular helix.

4. Representation formulae for Bertrand curves in Galilean and pseudo-Galilean 3-space

Lemma 4.1. If a spatial curve α has constant nonzero curvature κ_α and torsion τ_α, then the curve

$$\beta(s) = a\alpha(s) + b\left(-\frac{1}{\tau_\alpha^2}n(s) + \int b(s)ds\right) \tag{4.1}$$

is a Bertrand curve in Galilean and pseudo-Galilean 3-space.

Proof. In Galilean 3-space direct computations show that

$$\beta'(s) = at(s) + b\left(-\frac{1}{\tau_\alpha} + 1\right)b(s)$$
$$\beta''(s) = \left(a\kappa_\alpha + b(1-\tau_\alpha)\right)n(s)$$
$$\beta'''(s) = \left(a\kappa_\alpha + b(1-\tau_\alpha)\right)\tau_\alpha b(s)$$

From these

$$\kappa_\beta = \left|a\kappa_\alpha + b(1-\tau_\alpha)\right|$$

$$\tau_\beta = a\tau_\alpha$$

So $\beta(s)$ is a Bertrand curve in Galilean 3-space.

Now we show that $\beta(s)$ is also a Bertrand curve in pseudo-Galilean 3-space,

Case I.

$\langle n(s), n(s)\rangle = 1$ and $\langle b(s), b(s)\rangle = -1$

Direct computations show that

$$\beta'(s) = at(s) + b\left(-\frac{1}{\tau_\alpha} + 1\right)b(s)$$
$$\beta''(s) = \left(a\kappa_\alpha + b(\tau_\alpha - 1)\right)n(s)$$
$$\beta'''(s) = \left(a\kappa_\alpha + b(\tau_\alpha - 1)\right)\tau_\alpha b(s)$$

From these

$$\kappa_\beta = \left| a\kappa_\alpha + b(\tau_\alpha - 1) \right|$$

$$\tau_\beta = a\tau_\alpha$$

So $\beta(s)$ is a Bertrand curve in pseudo-Galilean 3-space.

Case II.

$\langle n(s), n(s) \rangle = -1$ and $\langle b(s), b(s) \rangle = 1$

Direct computations show that

$$\beta'(s) = at(s) + b\left(-\frac{1}{\tau_\alpha} + 1 \right) b(s)$$
$$\beta''(s) = \left(a\kappa_\alpha + b(\tau_\alpha - 1) \right) n(s)$$
$$\beta'''(s) = \left(a\kappa_\alpha + b(\tau_\alpha - 1) \right) \tau_\alpha b(s)$$

From these

$$\kappa_\beta = \left| -a\kappa_\alpha - b(\tau_\alpha - 1) \right|$$

$$\tau_\beta = a\tau_\alpha$$

So $\beta(s)$ is a Bertrand curve in pseudo-Galilean 3-space. Also this curve is a circular helix in Galilean and pseudo-Galilean 3-space.

Theorem 4.2. (Representation formulae) Let $u(\sigma)$ be a curve in the Galilean 3-space parametrised by arclength. Then define three spatial curves α, β and γ by $\alpha = a\int u(\sigma)d\sigma$, $\beta = a\cot\theta \int u(\sigma) \times du$ and $\gamma = \alpha + \beta$. Then α is constant curvature curve, β is constant torsion curve and γ is a Bertrand curve.

Proof. Let $u = u(\sigma)$ be a curve in G_3 parametrised by the arclength σ. Then $\{\xi = u', \eta = u \times u', u\}$ is a positive orthonormal frame field along u.

If $u = (1, y, z)$ then $y'^2 + z'^2 = 1$ and

$$u'' = -u + \lambda\eta \text{ and } u' \times u'' = -\lambda u' + \upsilon\eta$$

for some function λ and υ. From the definition of γ, we get

$$\gamma' = a(u + \cot\theta\eta)$$
$$\gamma'' = (a - a\lambda\cot\theta)\xi + a\upsilon\cot\theta\eta$$
$$\gamma''' = (a\lambda\cot\theta - a)u - a\lambda\upsilon\cot\theta\xi + \left(\lambda(a - a\lambda\cot\theta) + a\upsilon^2\cot\theta\right)\eta$$

Using these,

$$\kappa = \sqrt{(a - a\lambda\cot\theta)^2 + a^2\upsilon^2\cot^2\theta}$$

$$\tau = \frac{a\cot\theta(a - a\lambda\cot\theta)^2 + a\left[(a - a\lambda\cot\theta)\left(a\lambda - a\lambda^2 + a\upsilon^2\cot\theta\right) + a^3\lambda\upsilon^2\cot^2\theta\right]}{(a - a\lambda\cot\theta)^2 + a^2\upsilon^2\cot^2\theta}.$$

Then γ is a Bertrand curve.

Next, we compute the curvature of α and torsion of β.

Direct computations show that

$$\alpha' = au$$
$$\alpha'' = a\xi$$

and

$$\beta' = a\cot\theta\eta$$
$$\beta'' = -a\lambda\cot\theta u' + a\upsilon\cot\theta\eta$$
$$\beta''' = a\lambda\cot\theta u - a\lambda\upsilon\cot\theta u' + \left(a\upsilon^2\cot\theta - a\lambda^2\cot\theta\right)\eta$$

Hence

$$\kappa_\alpha = a$$

$$\tau_\beta = \frac{a\lambda^2\cot\theta}{\lambda^2 + \upsilon^2}$$

Theorem 4.3. (Representation formulae) Let $u(\sigma)$ be a curve in the pseudo-Galilean 3-space parametrised by arclength. Then define three spatial curves α, β and γ by $\alpha = a\int u(\sigma)d\sigma$, $\beta = a\cot\theta\int u(\sigma)\times du$ and $\gamma = \alpha + \beta$. Then α is constant curvature curve, β is constant torsion curve and γ is a Bertrand curve.

Proof. Let $u = u(\sigma)$ be a curve in G_3^1 parametrised by the arclength σ. Then $\{\xi = u', \eta = u \times u', u\}$ is a positive orthonormal frame field along u .

Case I.

If $u=(1,y,z)$, $\langle u\times u', u\times u'\rangle=1$ and $\langle u',u'\rangle=-1$ then $\left|y'^2-z'^2\right|=1$ and

$$u''=u+\lambda\eta \text{ and } u'\times u''=\lambda u'+\upsilon\eta$$

for some function λ and υ. From the definition of γ, we get

$$\begin{aligned}
\gamma' &= a\left(u+\cot\theta\eta\right)\\
\gamma'' &= \left(a+a\lambda\cot\theta\right)\xi+a\upsilon\cot\theta\eta\\
\gamma''' &= \left(a\lambda\cot\theta+a\right)u+a\lambda\upsilon\cot\theta\xi+\left(\lambda\left(a+a\lambda\cot\theta\right)+a\upsilon^2\cot\theta\right)\eta
\end{aligned}$$

Using these,

$$\kappa=\sqrt{-\left(a+a\lambda\cot\theta\right)^2+a^2\upsilon^2\cot^2\theta}$$

$$\tau=\frac{-a\cot\theta\left(a+a\lambda\cot\theta\right)^2+a\left[\left(a+a\lambda\cot\theta\right)\left(a\lambda+a\lambda^2+a\upsilon^2\cot\theta\right)-a^3\lambda\upsilon^2\cot^2\theta\right]}{a\upsilon^2\cot^2\theta-\left(a+a\lambda\cot\theta\right)^2}.$$

Then γ is a Bertrand curve.

Next, we compute the curvature of α and torsion of β.

Direct computations show that

$$\begin{aligned}
\alpha' &= au\\
\alpha'' &= a\xi
\end{aligned}$$

and

$$\begin{aligned}
\beta' &= a\cot\theta\eta\\
\beta'' &= a\lambda\cot\theta u'+a\upsilon\cot\theta\eta\\
\beta''' &= a\lambda\cot\theta u+a\lambda\upsilon\cot\theta u'+\left(a\upsilon^2\cot\theta+a\lambda^2\cot\theta\right)\eta
\end{aligned}$$

Hence

$$\kappa_\alpha=a$$

$$\tau_\beta=-\frac{a\lambda^2\cot\theta}{\lambda^2+\upsilon^2}$$

Case II.

If $u=(1,y,z)$, $\langle u\times u', u\times u'\rangle=-1$ and $\langle u',u'\rangle=1$ then $\left|y'^2-z'^2\right|=1$ and

$$u''=-u-\lambda\eta \text{ and } u'\times u''=-\lambda u'-\upsilon\eta$$

for some function λ and υ. From the definition of γ, we get

$$\begin{aligned}
\gamma' &= a\left(u + \cot\theta\eta\right) \\
\gamma'' &= \left(a - a\lambda\cot\theta\right)\xi - a\upsilon\cot\theta\eta \\
\gamma''' &= \left(a\lambda\cot\theta - a\right)u + a\lambda\upsilon\cot\theta\xi + \left(-\lambda\left(a - a\lambda\cot\theta\right) + a\upsilon^2\cot\theta\right)\eta
\end{aligned}$$

Using these,

$$\kappa = \sqrt{\left(a - a\lambda\cot\theta\right)^2 - a^2\upsilon^2\cot^2\theta}$$

$$\tau = \frac{a\cot\theta\left(a - a\lambda\cot\theta\right)^2 + a\left[\left(a - a\lambda\cot\theta\right)\left(-a\lambda + a\lambda^2 + a\upsilon^2\cot\theta\right) + a^3\lambda\upsilon^2\cot^2\theta\right]}{\left(a - a\lambda\cot\theta\right)^2 - a^2\upsilon^2\cot^2\theta}.$$

Then γ is a Bertrand curve.

Next, we compute the curvature of α and torsion of β.

Direct computations show that

$$\begin{aligned}
\alpha' &= au \\
\alpha'' &= a\xi
\end{aligned}$$

and

$$\begin{aligned}
\beta' &= a\cot\theta\eta \\
\beta'' &= -a\lambda\cot\theta u' - a\upsilon\cot\theta\eta \\
\beta''' &= a\lambda\cot\theta u + a\lambda\upsilon\cot\theta u' + \left(a\upsilon^2\cot\theta + a\lambda^2\cot\theta\right)\eta
\end{aligned}$$

Hence

$$\kappa_\alpha = a$$

$$\tau_\beta = \frac{a\lambda^2\cot\theta}{\lambda^2 - \upsilon^2}.$$

Corollary 4.4. The representation formulae for Bertrand curves in Galilean and pseudo-Galilean 3-space obtained in theorem 4.2. and theorem 4.3. show that this Bertrand curves are also circular helix.

References

[1] A. O. Öğrenmiş, M. Bektaş and M. Ergüt, *On the helices in the Galilean space* G_3, Iranian Journal of science & Technology, Transaction A, Vol.31, No:A2(2007), 177-181.

[2] A. O. Öğrenmiş, H. Öztekin and M. Ergüt, *Bertrand curves in Galilean space and their characterizations,* Kragujevac J. Math. 32(2009), 139-147.

[3] B. Divjak, *Curves in Pseudo-Galilean Geometry,* Annales Univ. Sci. Budapest. 41 (1998), 117-128.

[4] B. Divjak and Z. Milin Spus, *Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces,* Acta Math. Hungar, 98 (3) (2003), 203-215.

[5] E. Molnar, *The projective interpretation of the eight 3-dimensional homogeneous geometries*, Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry, 38 (2) (1997), 261-288.

[6] H. Balgetir Öztekin and M. Bektaş, *Representation formule for Bertrand curves in the Minkowski 3-space,* Scienta Magna. Vol. 6(2010), no.1, 89-96.

[7] I.M. Yaglom, A Simple Non-Euclidean Geometry and Its Physical Basis, Springer-Verlag, New York, 1979.

[8] I. Kamenarovic, *Existence theorems for ruled surfaces in the Galilean space* G_3 , Rad Hrvatskeakad. Znan. Umj. Mat. 10 (1991), 183-196.

[9] J. Monterde, *Salkowski curves revisted: A family of curves with constant curvature and non-constant torsion,* Comput. Aided Geomet. Design, 26(2009), 271-278.

[10] J. Koenderink, Solid shape, MIT Press, Cambridge, MA, (1990).

[11] K. Ilarslan and O. Boyacioglu: *Position vectors of a spacelike W-curve in Minkowski space* E_1^3, Bull. Korean Math. Soc. 44 (2007), 429–438.

[12] M. Babaarslan and Y. Yaylı, *The Characterizations of constant slope surfaces and Bertrand curves*, International Journal of the Physical Sciences. Vol. 6(8) (2011), 1868-1875.

[13] M. Barros, *General helices and a theorem of Lancret,* Proc. Amer. Math. Soc. 125(1997), 1503-1509.

[14] O. Röschel, Die Geometrie des Galileischen Raumes, Habilitationsschrift, Leoben, (1984).

[15] S. Izumiya and N. Takeuchi, *Generic properties of helices and Bertrand curves,* J. geom. 74(2002), 97-109.

[16] S. Izumiya , N. Takeuchi, New special curves and developable surfaces, Turk.J. Math. 28(5) (2004), 531—537 .

[17] Z. M. Sipus, *Ruled Weingarten surfaces in the Galilean space*, Periodica Mathematica Hungarica, 56 (2) (2008), 213-225.