Variational problems and l_1 exact exponential penalty function with $(p, r) - \rho - (\eta, \theta)$-invexity

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Abstract

In this paper, we formulate variational problems and establish weak, strong and converse duality theorems to relate solutions of the primal and dual problems with new generalized $(p, r) - \rho - (\eta, \theta)$-invexity assumptions on the functions involved. We also study the sufficient conditions for optimality and provide some examples to illustrate our works. We transform the constrained variational problem into an unconstrained one with the help of l_1 exact exponential penalty function method. The equivalence between sets of optimal solutions of the original variational problem and its associated exponential penalized variational problem is established with new type of generalized invexity assumptions.

Key words: $(p, r) - \rho - (\eta, \theta)$-invex function; variational problem; Mond-Weir type dual; sufficient optimal condition; l_1 exact exponential penalty function.

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1 Introduction

Calculus of variations is a field of mathematics that deals with extremizing functionals. Main interest in this problem is to find the curves that make the functional attain a minimum or maximum value and also satisfy the given constraints. The relationship between mathematical programming problems and variational problems was first explored by Hanson [14]. Thereafter variational programming problem has attracted the attention of many researchers. Most of the authors established the optimality conditions and duality results for variational problems. For example, Mond and Hanson [21] obtained the optimality conditions and duality results for scalar valued variational problems under convexity assumptions. Mond and Husain [22] studied duality results with pseudo-invexity and quasi-invexity assumptions. After that, some authors studied the mathematical programs involving several conflicting objectives, called multiobjective programming problems. Mishra and Mukherjee [23] studied same problems with generalized (F, ρ)-convexity. Nahak and Nanda [24] extended the optimality conditions and duality results for multiobjective variational problems under invexity assumptions. Bhatia and Kumar [9] established duality results to a wider class of functions, called B-vex function. More recently, Khazafi and Rueda [18] generalized the class of V-univex type I functions and studied multiobjective variational problems in the spirit of generalizations made by Aghezzaf and Khazafi [1].

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Most of the above contributions are based on sufficient optimality conditions and duality results for unconstrained variational problems. Sometimes, it is more easier to solve constrained variational problem rather than unconstrained one. Penalty function method is a technique, which is used to solve the constrained optimization problems. The usual strategy is to replace a constrained optimization problem by a series of unconstrained problems of penalty functions whose solutions ideally converge to the solutions of the original constrained problems (see, for example, [13]). Most of the authors studied exact penalty function method under the notions of convexity (see, for example, [11, 20]). However, there are many real world problems which are nonconvex in nature. In the recent years, some generalizations of convex functions have been derived. Antczak [5] established the equivalence between optimal solutions of original optimization problem and its exact penalized optimization problem under invexity assumptions. Further, under suitable r-invexity assumption, Antczak [4] established the same result. More recently, Antczak [6] introduced a new absolute value exact penalty function method, called the l_1 exact exponential penalty function method and established exact penalized optimization problem with suitable r-invexity assumptions.

In this paper, our objective is to establish the sufficient optimality conditions and duality results for variational problems with $(p, r) - \rho - (\eta, \theta)$-invex functions. We also solve much wider class of nonconvex constrained variational problem by reformulating its associated unconstrained penalized optimization problem. The comparison of the present paper with some of the related works in the literature is given in Table 1.

Table 1: A comparison of the present model with some related works

Article	Penalty function	Variational problem	Generalized invexity
Aghezzaf and Khazafi [1]	none	Multiobjective	B-invexity
Antczak [5]	Exact penalty	none	Invexity
Antczak [6]	l_1 exact penalty	none	r-invexity
Present paper	l_1 exact penalty	yes	$(p, r) - \rho - (\eta, \theta)$-invexity

The paper is organized as follows. Notation and preliminaries are given in the following section. Section 3 provides sufficient optimality conditions for variational problem, while Section 4 describes the duality results for Mond-Weir type dual variational problem. In Section 5, the l_1 exact exponential penalty function method for variational problem has been discussed. Finally, the paper is concluded in Section 6 with some remarks and future research directions.

2 Notations and Preliminaries

Invex function is one of the most important generalized convex functions. This invexity concept has inspired a great deal of subsequent work which has greatly expanded the role of invexity in optimization. The $(p, r) - \rho - (\eta, \theta)$-invex functions generalize the class of (p, r)-invex functions (see [2]) and $\rho-(\eta, \theta)$-invex functions (see [27]). We will provide the definition of $(p, r)-\rho-(\eta, \theta)$-invex function, throughout the paper it will be used very frequently.

Definition 2.1. ([19]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and p, r be arbitrary real numbers, $\rho \in \mathbb{R}$. The function f is said to be $(p, r) - \rho - (\eta, \theta)$-invex with respect to

$\eta,\ \theta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ at u, if any one of the following conditions holds

$$\frac{1}{r}(e^{r(f(x)-f(u))}-1)\geq\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-\mathbf{1})+\rho\|\theta(x,u)\|^2 \qquad \text{for } p\neq 0,\quad r\neq 0,$$
$$\frac{1}{r}(e^{r(f(x)-f(u))}-1)\geq\nabla f(u)\eta(x,u)+\rho\|\theta(x,u)\|^2 \qquad \text{for } p=0,\quad r\neq 0,$$
$$f(x)-f(u)\geq\frac{1}{p}\nabla f(u)(e^{p\eta(x,u)}-\mathbf{1})+\rho\|\theta(x,u)\|^2 \qquad \text{for } p\neq 0,\quad r=0,$$
$$f(x)-f(u)\geq\nabla f(u)\eta(x,u)+\rho\|\theta(x,u)\|^2 \qquad \text{for } p=0,\quad r=0.$$

Remarks:
1. It should be pointed out that the exponentials appearing on the right-hand sides of the above inequalities are understood to be taken componentwise and $\mathbf{1}=(1,1,...,1)\in\mathbb{R}^n$.
2. In case $p\neq 0,\ r=0$, the functions are called $(p,0)-\rho-(\eta,\theta)$-invex with respect to $\eta,\ \theta$.
3. In case $p=0,\ r\neq 0$, the functions are called $(0,r)-\rho-(\eta,\theta)$-invex with respect to $\eta,\ \theta$ (or shortly $r-\rho-(\eta,\theta)$-invex with respect to $\eta,\ \theta$).
4. In case $p=0,\ r=0$, the functions are called $(0,0)-\rho-(\eta,\theta)$-invex with respect to $\eta,\ \theta$ (or shortly $\rho-(\eta,\theta)$-invex with respect to $\eta,\ \theta$).
5. All theorems in this paper will be proved only in the case when $p\neq 0,\ r\neq 0$ (other cases can be dealt with likewise since the only changes arise from form of inequality). Moreover, without loss of generality, we shall assume that $r>0$ (in the cases when $r<0$, the direction, some of the inequalities in the proofs of the given theorems should be changed to the opposite one).
The following examples 2.1 and 2.2 show that $(p,r)-\rho-(\eta,\theta)$-invexity generalizes both (p,r)-invexity and $\rho-(\eta,\theta)$-invexity.

Example 2.1. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x_1,x_2)=\log^2(x_1)-\log^2(x_2)$ and $X=\{(x_1,x_2)\in\mathbb{R}^2:x_1>0,\ x_2>0\}$.
Define, $\theta_1(x,u)=\begin{cases}1, & \text{if } u_1=1,\\ 0, & \text{if } u_1\neq 1.\end{cases}$ $\qquad \theta_2(x,u)=\begin{cases}1, & \text{if } u_2=1,\\ 0, & \text{if } u_2\neq 1.\end{cases}$

$\eta_1(x,u)=\begin{cases}0, & \text{if } u_1=1,\\ -\frac{u_1}{\log u_1}, & \text{if } u_1\neq 1.\end{cases}$ $\qquad \eta_2(x,u)=\begin{cases}0, & u_2=1,\\ -\frac{u_2}{\log u_2}, & \text{if } u_2\neq 1.\end{cases}$

Here $x=(x_1,x_2)$, $\theta=(\theta_1,\theta_2)$ and $\eta=(\eta_1,\eta_2)$. Now $\frac{1}{r}e^{rf(x)}=\frac{1}{r}e^{r(\log^2(x_1)-\log^2(x_2))}$ and $\frac{1}{r}e^{rf(u)}[1+r\nabla f(u)\eta(x,u)]=\frac{1}{r}$, if we take $u=(1,1)$. Then the inequality

$$e^{r(\log^2(x_1)-\log^2(x_2))}\geq 1 \tag{2.1}$$

is not true. If we take $x_1=1,\ x_2=\dfrac{1}{e}$ and $r=1$, then from the above inequality we get $\dfrac{1}{e}\geq 1$, which is not possible. Therefore the function f is not (p,r)-invex function with respect to η at $u=(1,1)$ for any real numbers p and r. If $p=0,\ r=1$, and $\rho=-1/2$, then

$$\frac{1}{r}(e^{r(f(x)-f(u))}-1)=e^{(\log^2(x_1)-\log^2(x_2))}-1 \tag{2.2}$$

and $\nabla f(u)\eta(x,u)+\rho\|\theta(x,u)\|^2=-1$ at the point $u=(1,1)$. The inequality

$$e^{(\log^2(x_1)-\log^2(x_2))}-1\geq -1 \tag{2.3}$$

is always true. From the Definition 2.1 it is clear that, f is $(0,1)-\rho-(\eta,\theta)$-invex function with respect to η and θ at the point $u=(1,1)$.

Example 2.2. The above Example 2.1 also shows that f is $(0,1)-\rho-(\eta,\theta)$-invex but not $\rho-(\eta,\theta)$-invex.
Now $f(x)-f(u)=\log^2(x_1)-\log^2(x_2)$ (where $x=(x_1,x_2),\ f(u)=0$, at the point $u=(1,1)$) and $\nabla f(u)\eta(x,u)+\rho\|\theta(x,u)\|^2=-1$ at $u=(1,1)$. If we take $\rho=-1/2$, then the inequality

$$\log^2(x_1)-\log^2(x_2)\geq -1 \tag{2.4}$$

is not true. In (2.4) take $x_1=1,\ x_2=\frac{1}{e^2}$, and $\theta(x,u)$, as we defined in previous Example 2.1 we get $-4\geq -1$, which is wrong. Therefore f is not $\rho-(\eta,\theta)$-invex function, but $(0,1)-\rho-(\eta,\theta)$-invex function with respect to η and θ.
Hence examples 2.1 and 2.2 ensure that, our $(p,r)-\rho-(\eta,\theta)$-invexity is more general than (p,r)-invexity, as well as $\rho-(\eta,\theta)$-invexity.

Variational Problem:
Consider the function $f(t,x(t),\dot{x}(t))$ where t is an independent variable, $x:I\to\mathbb{R}^n$ is a function of t and $\dot{x}$ denotes the derivative of x with respect to t. The symbol z^T stands for the transpose of a vector z. Denote the first partial derivatives of f with respect to $x(t)$ and $\dot{x}(t)$ by f_x and $f_{\dot{x}}$, respectively, that is,

$$f_x=\begin{pmatrix}\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial f}{\partial x_n}\end{pmatrix}^T, \qquad f_{\dot{x}}=\begin{pmatrix}\frac{\partial f}{\partial \dot{x}_1} & \frac{\partial f}{\partial \dot{x}_2} & \cdot & \cdot & \cdot & \frac{\partial f}{\partial \dot{x}_n}\end{pmatrix}^T.$$

Let $C(I,\mathbb{R}^n)$ denote the space of piecewise smooth functions with norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differential operator D is given by

$$u=Dx\Longleftrightarrow x(t)=x(a)+\int_a^t u(s)ds.$$

A variational problem is to transfer the state vector from an initial state $x(a)=\alpha$ to a final state $x(b)=\beta$ so as to minimize a functional, subject to constraints on the state variables. Mathematically, we formulate the variational problem as:

$$\textbf{(VP)} \qquad \min:\int_a^b f(t,x,\dot{x})dt$$

subject to

$$x(a)=\alpha,\ x(b)=\beta, \tag{2.5}$$

$$g(t,x,\dot{x})\geq 0, \tag{2.6}$$

where, $I=[a,b]$ and $f:I\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, $g:I\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^p$ are continuously differentiable functions.
The set of feasible solutions of **(VP)** is defined by
$K=\{x\in C(I,\mathbb{R}^n):x(a)=\alpha,\ x(b)=\beta,\ g(t,x,\dot{x})\geq 0, t\in I\}$.
Several authors studied variational problems under some convexity and generalized convexity assumptions. Here we use more generalized invexity, called $(p,r)-\rho-(\eta,\theta)$-invexity. We will give the definition of $(p,r)-\rho-(\eta,\theta)$-invexity, which is used throughout this paper.

Definition 2.2. The functional $F(x,\dot{x})=\int_a^b f(t,x,\dot{x})dt$ is said to be $(p,r)-\rho-(\eta,\theta)$-invex at $\overline{x}$ on $[a,b]$ with respect to $\eta,\ \theta$, if there exist vector functions $\eta,\ \theta\in\mathbb{R}^n$, with $\eta=0$ at $t=a$,

$t=b$, and $\rho\in\mathbb{R}$, if any one of the following conditions holds

$$\frac{1}{r}[e^{r\{\int_a^b(f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt\}}-1]\geq\int_a^b[f_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta(t,x,\overline{x})}-1)$$
$$+f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})e^{p\eta(t,x,\overline{x})}\frac{d\eta}{dt}+\rho\|\theta(t,x,\overline{x})\|^2]dt,\quad p\neq 0,\ r\neq 0,$$
$$\frac{1}{r}[e^{r\{\int_a^b(f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt\}}-1]\geq\int_a^b[f_x(t,\overline{x},\dot{\overline{x}})\eta(t,x,\overline{x})+f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})\frac{d\eta}{dt}$$
$$+\rho\|\theta(t,x,\overline{x})\|^2]dt,\quad p=0,\ r\neq 0,$$
$$\int_a^b(f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt\geq\int_a^b[f_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta(t,x,\overline{x})}-1)$$
$$+f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})e^{p\eta(t,x,\overline{x})}\frac{d\eta}{dt}+\rho\|\theta(t,x,\overline{x})\|^2]dt,\quad p\neq 0,\ r=0,$$
$$\int_a^b(f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt\geq\int_a^b[f_x(t,\overline{x},\dot{\overline{x}})\eta(t,x,\overline{x})+f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})\frac{d\eta}{dt}$$
$$+\rho\|\theta(t,x,\overline{x})\|^2]dt,\quad p=0,\ r=0.$$

Remark 2.1. (i) All the theorems will be proved only in the case when $p\neq 0,\ r\neq 0$ (other cases can be dealt with likewise).

(ii) Without loss of generality, let $r>0$ (in the case when $r<0$, the proof is analogous; one should change only the direction of some inequalities below to the opposite one).

Example 2.3. Let $f:[0,1]\times[0,1]\times[0,1]\longrightarrow\mathbb{R}$ be given by

$$f(t,x,\dot{x})=-x^2(t)t.$$

The function $\int_0^1 f(t,x,\dot{x})dt$ is not $-1-(\eta,\theta)$-invex with respect to $\eta:[0,1]\times[0,1]\times[0,1]\longrightarrow\mathbb{R}$ and $\theta:[0,1]\times[0,1]\times[0,1]\longrightarrow\mathbb{R}$ defined by

$$\eta\,(t,x,\overline{x})=x(t)+\overline{x}(t),$$
$$\theta\,(t,x,\overline{x})=\sqrt{2t}.$$

Now if we take $p=1,\ r=1$ and $\rho=-1$, then

$$\frac{1}{r}[e^{r\int_0^1[f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}})]dt}-1]=e^{\int_0^1(\overline{x}^2-x^2)tdt}-1\ \text{and}$$
$$\int_0^1[f_x(t,\overline{x},\dot{\overline{x}}))\frac{1}{p}(e^{p\eta}-1)+\rho\|\theta(t,x,\overline{x})\|^2]dt=\int_0^1-2\overline{x}t(e^{x+\overline{x}}-1)dt-1,$$

the inequality

$$e^{\int_0^1(\overline{x}^2-x^2)tdt}-1\geq\int_0^1-2\overline{x}t(e^{x+\overline{x}}-1)dt-1$$

is always true. Therefore, the function $\int_0^1 f(t,x,\dot{x})dt$ is $(1,1)-(-1)-(\eta,\theta)$-invex with respect to same $\eta,\ \theta$.

3 Sufficiency for Optimality

The following necessary conditions for the existence of an extremal for **(VP)** were first developed by Valentine [26].

Proposition 3.1. ([26]) If $\overline{x}$ is an optimal solution of **(VP)** and if it is normal, then there exist $\lambda_0 \in \mathbb{R}$, and a piecewise smooth function $\overline{\mu} : I \to \mathbb{R}^m$, such that the function

$$F = \lambda_0 f - \overline{\mu}^T g \tag{3.7}$$

satisfies

$$F_x = \frac{d}{dt}(F_{\dot{x}}), \tag{3.8}$$

$$\overline{\mu}_i g_i = 0, \ i = 1, 2, ..., m, \tag{3.9}$$

$$\overline{\mu}(t) \geq 0, \ t \in I. \tag{3.10}$$

Since the optimal solution $\overline{x}$ is normal, i.e., λ_0 is non-zero, so that without loss of generality, we can take $\lambda_0 = 1$.

We establish that the necessary optimality conditions are also sufficient for optimality if it satisfies some $(p, r) - \rho - (\eta, \theta)$-invexity assumptions.

Theorem 3.1. Let $\overline{x}$ be a feasible solution of **(VP)**. Assume that there exists a piecewise smooth function $\overline{\mu} : I \to \mathbb{R}^p$ such that, for all $t \in I$,

$$f_x(t, \overline{x}, \dot{\overline{x}}) - \overline{\mu}^T g_x(t, \overline{x}, \dot{\overline{x}}) = \frac{d}{dt}[f_{\dot{x}}(t, \overline{x}, \dot{\overline{x}}) - \overline{\mu}^T g_{\dot{x}}(t, \overline{x}, \dot{\overline{x}})], \tag{3.11}$$

$$\overline{\mu}^T g(t, \overline{x}, \dot{\overline{x}}) = 0, \tag{3.12}$$

$$\overline{\mu}(t) \geq 0 \tag{3.13}$$

hold. Further suppose that $\int_a^b f dt$ and $\int_a^b -\overline{\mu}^T g dt$ are $(p, r) - \rho_1 - (\eta, \theta)$-invex and $(p, -r) - \rho_2 - (\eta, \theta)$-invex, respectively at $\overline{x}$ over K with respect to η, θ, and $(\rho_1 + \rho_2) \geq 0$, then $\overline{x}$ is an optimal solution of **(VP)**.

Proof. Since $\int_a^b f(t, x, \dot{x}) dt$ is $(p, r) - \rho_1 - (\eta, \theta)$-invex at $\overline{x}$, then

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x}) - f(t,\overline{x},\dot{\overline{x}}))dt]} - 1) \geq \int_a^b [f_x(t, \overline{x}, \dot{\overline{x}}) \frac{1}{p}(e^{p\eta} - 1)$$
$$+ f_{\dot{x}}(t, \overline{x}, \dot{\overline{x}}) e^{p\eta} \frac{d\eta}{dt} + \rho_1 \|\theta(t, x, \overline{x})\|^2] dt, \ \forall x \in K. \tag{3.14}$$

As $\int_a^b -\overline{\mu}^T g(t, x, \dot{x}) dt$ is $(p, -r) - \rho_2 - (\eta, \theta)$-invex at $\overline{x}$ over K, then we reach the inequality,

$$-\frac{1}{r}(e^{r[\int_a^b \{\overline{\mu}^T (g(t,x,\dot{x}) - g(t,\overline{x},\dot{\overline{x}}))\}dt]} - 1) \geq \int_a^b [-\overline{\mu}^T g_x(t, \overline{x}, \dot{\overline{x}}) \times$$
$$\frac{1}{p}(e^{p\eta(t,x,\dot{\overline{x}})} - 1) - \overline{\mu}^T g_{\dot{x}}(t, \overline{x}, \dot{\overline{x}}) e^{p\eta} \frac{d\eta}{dt} + \rho_2 \|\theta(t, x, \overline{x})\|^2] dt, \ \forall x \in K. \tag{3.15}$$

Since x is an arbitrary feasible solution of **(VP)**, $\overline{\mu}(t) \geq 0$, then from the equation (3.12) and the inequality (3.15), we have

$$\int_a^b [-\overline{\mu}^T g_x(t, \overline{x}, \dot{\overline{x}}) \frac{1}{p}(e^{p\eta} - 1)$$
$$-\overline{\mu}^T g_{\dot{x}}(t, \overline{x}, \dot{\overline{x}}) e^{p\eta} \frac{d\eta}{dt} + \rho_2 \|\theta(t, x, \overline{x})\|^2] dt \leq 0. \tag{3.16}$$

Adding equations (3.14) and (3.16), we obtain

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt]}-1) \geq \int_a^b [\frac{1}{p}(e^{p\eta}-1)\times\{f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})\}$$

$$+e^{p\eta}\frac{d\eta}{dt}\times\{f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})\}+(\rho_1+\rho_2)\int_a^b \|\theta(t,x,\overline{x})\|^2]dt. \qquad (3.17)$$

Now, pre-multiplying equation (3.11) by $\frac{1}{p}(e^{p\eta}-1)$ and integrating, we get

$$\int_a^b (e^{p\eta}-1)[\overline{\lambda}^T f_x(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})]dt$$

$$=\int_a^b (e^{p\eta}-1)\frac{d}{dt}[\overline{\lambda}^T f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})]dt. \qquad (3.18)$$

After integrating by parts of $\int_a^b (e^{p\eta}-1)\frac{d}{dt}[\overline{\lambda}^T f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})]dt$ and using $\eta=0$ at $t=a$ and $t=b$, the equality (3.18) becomes

$$\int_a^b [(e^{p\eta}-1)\{\overline{\lambda}^T f_x(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})\}+$$

$$e^{p\eta}\frac{d\eta}{dt}\{\overline{\lambda}^T f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})\}]dt=0. \qquad (3.19)$$

From equations (3.17) and (3.19), it follows that

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt]}-1) \geq (\rho_1+\rho_2)\int_a^b \|\theta(t,x,\overline{x})\|^2 dt.$$

By the assumption $(\rho_1+\rho_2)\geq 0$, we have

$$\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,\overline{x},\dot{\overline{x}})dt,\ \forall\, x\in K.$$

Hence, $\overline{x}$ is an optimal solution of **(VP)**, and the proof is completed. □

4 Mond-Weir Type Dual Variational Problem

Mond and Husain [17] studied Mond-Weir type dual under pseudo-invexity and quasi-invexity assumptions. To weaken pseudo-invexity and quasi-invexity assumptions, we consider the following Mond-Weir type dual **(MWVD)** for variational problem **(VP)** and establish the duality results under $(p,r)-\rho-(\eta,\theta)$-invexity assumptions.

(MWVD) $\quad \max : \int_a^b f(t,x,\dot{x})dt$

subject to

$$x(a)=\alpha,\ x(b)=\beta, \qquad (4.1)$$

$$f_x(t,x,\dot{x})-\mu(t)^T g_x(t,x,\dot{x})=\frac{d}{dt}[f_{\dot{x}}(t,x,\dot{x})-\mu(t)^T g_{\dot{x}}(t,x,\dot{x})], \qquad (4.2)$$

$$\int_a^b \mu(t)^T g_{\dot{x}}(t,x,\dot{x})dt \leq 0, \qquad (4.3)$$

$$\mu(t)\geq 0. \qquad (4.4)$$

Let H be the set of all feasible solutions of **(MWVD)**.

Theorem 4.1. (Weak Duality). Let x and $(\overline{x}, \overline{\mu})$ be the feasible solutions of the primal problem **(VP)**, and the dual problem **(MWVD)**, respectively. If $\int_a^b f dt$ and $\int_a^b -\overline{\mu}^T g dt$ are $(p,r)-\rho_1-(\eta,\theta)$-invex and $(p,-r)-\rho_2-(\eta,\theta)$-invex, respectively at $\overline{x}$ on H with respect to η, θ and $(\rho_1+\rho_2)\geq 0$. Then inf **(VP)** $\geq$ sup **(MWVD)**.

Proof. Since $\int_a^b f dt$ is $(p,r)-\rho_1-(\eta,\theta)$-invex at $\overline{x}$ on H with respect to η, θ, we reach the inequality

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}})dt]}-1) \geq \int_a^b [f_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1) + f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})e^{p\eta}\frac{d\eta}{dt} + \rho_1\|\theta(t,x,\overline{x})\|^2]dt. \tag{4.5}$$

By the equation (4.2), we get

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt]}-1) \geq \int_a^b [\{\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}}) + \frac{d}{dt}(f_{\dot{x}}(t,\overline{x},\dot{\overline{x}}) - \overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}}))\} \times\frac{1}{p}(e^{p\eta}-1)]dt + \int_a^b [f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})e^{p\eta}\frac{d\eta}{dt} + \rho_1\|\theta(t,x,\overline{x})\|^2]dt. \tag{4.6}$$

After integration by parts of $\int_a^b \{\frac{1}{p}(e^{p\eta}-1)\frac{d}{dt}(f_{\dot{x}}(t,\overline{x},\dot{\overline{x}}) - \overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}}))\}dt$ and taking $\eta=0$ at $t=a$ and $t=b$, from inequality (4.6) we obtain

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt]}-1) \geq \int_a^b [\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1) + \overline{\mu}^T e^{p\eta}\frac{d\eta}{dt} g_{\dot{x}}(t,\overline{x},\dot{\overline{x}}) + \rho_1\|\theta(t,x,\overline{x})\|^2]dt. \tag{4.7}$$

Now, since $-\int_a^b \overline{\mu}^T g(t,x,\dot{x})dt$ is $(p,-r)-\rho_2-(\eta,\theta)$-invex, therefore we reach

$$-\frac{1}{r}[e^{r[\int_a^b \{\overline{\mu}^T(g(t,x,\dot{x})-g(t,\overline{x},\dot{\overline{x}}))\}dt]}-1] \geq \int_a^b [-\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1) - e^{p\eta}\frac{d\eta}{dt}\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}}) + \rho_2\|\theta(t,x,\overline{x})\|^2]dt. \tag{4.8}$$

Adding inequalities (4.7) and (4.8), we have

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt]}-1) \geq \frac{1}{r}[e^{r[\int_a^b \{\overline{\mu}^T(g(t,x,\dot{x})-g(t,\overline{x},\dot{\overline{x}}))\}dt]}-1] + (\rho_1+\rho_2)\int_a^b \|\theta(t,x,\overline{x})\|^2 dt.$$

By the assumption $(\rho_1+\rho_2)\geq 0$, we have

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt]}-1) \geq \frac{1}{r}(e^{r[\int_a^b \{\overline{\mu}^T(g(t,x,\dot{x})-g(t,\overline{x},\dot{\overline{x}}))\}dt]}-1). \tag{4.9}$$

Since x is feasible for **(VP)** and $\overline{\mu}(t)\geq 0$, then $\overline{\mu}^T g(t,x,\dot{x})\geq 0$. Hence,

$$\int_a^b f(t,x,\dot{x})dt \geq \int_a^b \{f(t,\overline{x},\dot{\overline{x}}) - \overline{\mu}^T g(t,\overline{x},\dot{\overline{x}})\}dt.$$

Therefore, inf **(VP)** $\geq$ sup **(MWVD)**. ☐

Theorem 4.2. (Strong Duality). If $\overline{x}$ is a solution of the primal problem **(VP)**, and it is normal (see [16]), then there exists $\overline{\mu}(t)$ such that $(\overline{x},\overline{\mu})$ is a feasible solution of **(MWVD)**. If the hypotheses of the Weak Duality Theorem 4.1 hold, then $(\overline{x},\overline{\mu})$ is an optimal solution of the dual problem **(MWVD)**, and the corresponding objective values are equal.

Proof. Since $\overline{x}$ is a solution of the primal problem **(VP)** and it is normal (see [21]), then from Proposition 3.1, there exists a piecewise smooth function $\overline{\mu}: I \to \mathbb{R}^p$ such that $(\overline{x},\overline{\mu})$ satisfies

$$f_x(t,\overline{x},\dot{\overline{x}}) - \overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}}) = \frac{d}{dt}\{f_{\dot{x}}(t,\overline{x},\dot{\overline{x}}) - \overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})\}, \tag{4.10}$$

$$\overline{\mu}^T g(t,\overline{x},\dot{\overline{x}})dt = 0, \tag{4.11}$$

$$\overline{\mu}(t) \geq 0,\ t \in I. \tag{4.12}$$

Thus, from the above relations, it follows that $(\overline{x},\overline{\mu})$ is a feasible solution of the dual problem **(MWVD)**. Since all the hypotheses of the Weak Duality Theorem 4.1 hold, $(\overline{x},\overline{\mu})$ is an optimal solution of **(MWVD)** and the optimal values of **(VP)** and **(MWVD)** are same. □

Theorem 4.3. (Strict Converse Duality). Let $\overline{x}$ be an optimal solution of **(VP)**, and which also be normal (see [21]). If $(\overline{u},\overline{\mu})$ is an optimal solution of the dual problem **(MWVD)** and moreover, if $\int_a^b f dt$ and $\int_a^b -\mu^T g dt$ are strictly $(p,r)-\rho_1-(\eta,\theta)$-invex and strictly $(p,-r)-\rho_2-(\eta,\theta)$-invex, respectively at $\overline{u}$ on H with $(\rho_1+\rho_2)\geq 0$, then $\overline{x}=\overline{u}$, that is $\overline{u}$ is an optimal solution of **(VP)**.

Proof. We will prove the theorem by the method of contradiction. Suppose that $\overline{x} \neq \overline{u}$. Since $\overline{x}$ is an optimal solution of **(VP)** and it is normal, there exists a piecewise smooth function $\overline{\mu}: I \to \mathbb{R}^p$. Then $(\overline{x},\overline{\mu})$ is an optimal solution of **(MWVD)**. As $(\overline{u},\overline{\mu})$ is also an optimal solution of **(MWVD)**, it follows that

$$\int_a^b f(t,\overline{x},\dot{\overline{x}})dt = \int_a^b f(t,\overline{u},\dot{\overline{u}})dt. \tag{4.13}$$

Since $\int_a^b f dt$ is strictly $(p,r)-\rho_1-(\eta,\theta)$-invex at $\overline{u}$ on H, the strict inequality

$$\frac{1}{r}(e^{r[\int_a^b (f(t,x,\dot{x})-f(t,\overline{u},\dot{\overline{u}})dt]}-1) > \int_a^b [f_x(t,\overline{u},\dot{\overline{u}})\frac{1}{p}(e^{p\eta}-1) + f_{\dot{x}}(t,\overline{u},\dot{\overline{u}})e^{p\eta}\frac{d\eta}{dt} + \rho_1\|\theta(t,x,\overline{u})\|^2]dt \tag{4.14}$$

holds $\forall x \in H$. As $\overline{x} \in H$, the inequality (4.14) reduces to

$$\frac{1}{r}(e^{r[\int_a^b (f(t,\overline{x},\dot{\overline{x}})-f(t,\overline{u},\dot{\overline{u}})dt]}-1) > \int_a^b [f_x(t,\overline{u},\dot{\overline{u}})\frac{1}{p}(e^{p\eta}-1) + f_{\dot{x}}(t,\overline{u},\dot{\overline{u}})e^{p\eta}\frac{d\eta}{dt} + \rho_1\|\theta(t,\overline{x},\overline{u})\|^2]dt. \tag{4.15}$$

Using the equation (4.2) in (4.15) and integrating by parts, we obtain

$$\frac{1}{r}(e^{r[\int_a^b (f(t,\overline{x},\dot{\overline{x}})-f(t,\overline{u},\dot{\overline{u}})dt]}-1) > \int_a^b [\overline{\mu}^T\{\frac{1}{p}(e^{p\eta}-1)g_x(t,\overline{u},\dot{\overline{u}}) + e^{p\eta}\frac{d\eta}{dt}g_{\dot{x}}(t,\overline{u},\dot{\overline{u}})\} + \rho_1\|\theta(t,\overline{x},\overline{u})\|^2]dt. \tag{4.16}$$

From (4.13) and (4.16), we have

$$\int_a^b [\overline{\mu}^T\{\frac{1}{p}(e^{p\eta}-1)g_x(t,\overline{u},\dot{\overline{u}})+e^{p\eta}\frac{d\eta}{dt}g_{\dot{x}}(t,\overline{u},\dot{\overline{u}})\}+\rho_1\|\theta(t,\overline{x},\overline{u})\|^2]dt<0. \tag{4.17}$$

By the strict $(p,-r)-\rho_2-(\eta,\theta)$-invexity assumption of $\int_a^b -\mu^T g dt$, we reach

$$-\frac{1}{r}[e^{r[\int_a^b\{\overline{\mu}^T(g(t,x,\dot{x})-g(t,\overline{u},\dot{\overline{u}}))\}dt]}-1]>\int_a^b[-\overline{\mu}^T g_x(t,\overline{u},\dot{\overline{u}})\frac{1}{p}(e^{p\eta}-1)$$
$$-e^{p\eta}\frac{d\eta}{dt}\overline{\mu}^T g_{\dot{x}}(t,\overline{u},\dot{\overline{u}})+\rho_2\|\theta(t,x,\overline{u})\|^2]dt,\ \forall x\in H. \tag{4.18}$$

As $\overline{x}\in H$, the inequality (4.18) reduces to

$$-\frac{1}{r}[e^{r[\int_a^b\{\overline{\mu}^T(g(t,\overline{x},\dot{\overline{x}})-g(t,\overline{u},\dot{\overline{u}}))\}dt]}-1]>\int_a^b[-\overline{\mu}^T g_x(t,\overline{u},\dot{\overline{u}})\frac{1}{p}(e^{p\eta}-1)$$
$$-e^{p\eta}\frac{d\eta}{dt}\overline{\mu}^T g_{\dot{x}}(t,\overline{u},\dot{\overline{u}})+\rho_2\|\theta(t,\overline{x},\overline{u})\|^2]dt. \tag{4.19}$$

Since $\overline{x}$ and $(\overline{u},\overline{\mu})$ are the optimal solutions of **(VP)** and **(MWVD)**, respectively we get

$$\int_a^b\{\overline{\mu}^T(g(t,\overline{x},\dot{\overline{x}})-g(t,\overline{u},\dot{\overline{u}}))\}dt\geq 0. \tag{4.20}$$

From (4.19) and (4.20), we have

$$\int_a^b[-\overline{\mu}^T g_x(t,\overline{u},\dot{\overline{u}})\frac{1}{p}(e^{p\eta}-1)-e^{p\eta}\frac{d\eta}{dt}\overline{\mu}^T g_{\dot{x}}(t,\overline{u},\dot{\overline{u}})+\rho_2\|\theta(t,\overline{x},\overline{u})\|^2]dt<0. \tag{4.21}$$

Adding (4.17) and (4.21), we obtain

$$(\rho_1+\rho_2)\int_a^b\|\theta(t,x,\overline{u})\|^2dt<0,$$

which contradicts the fact that $(\rho_1+\rho_2)\geq 0$. Therefore, $\overline{x}=\overline{u}$, that is, $\overline{u}$ is an optimal solution of **(VP)**. □

5 l_1 exact exponential penalty in variational problem

In this section, the l_1 exact exponential penalty method has been used to convert a constrained variational problem **(VP)** into an unconstrained one.
We obtain the following unconstrained variational problem corresponding to **(VP)**:

$$(\mathbf{VP_r(c)})\quad \min\ P(x(t),c)=\left[\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt}+c\left\{\sum_{i=1}^{p}\frac{1}{r}\left(1-e^{r\int_a^b g_+^i(t,x,\dot{x})dt}\right)\right\}\right], \tag{5.1}$$

where $\frac{1}{r}\left(1-e^{r\int_a^b g_+^i(t,x,\dot{x})dt}\right)$ is defined by

$$\frac{1}{r}\left(1-e^{r\int_a^b g_+^i(t,x,\dot{x})dt}\right)=\begin{cases}0, & \text{if } g^i(t,x,\dot{x})\geq 0,\\ \frac{1}{r}\left(1-e^{r\int_a^b g^i(t,x,\dot{x})dt}\right), & \text{if } g^i(t,x,\dot{x})<0,\end{cases} \tag{5.2}$$

where $i=1,2,...,p$ and $c>0$ is the penalty parameter.
Now we will show that an optimal solution of the original variational problem **(VP)** is also an optimal solution of the unconstrained variational problem with the l_1 exact exponential penalty function method under the suitable $(p,r)-\rho-(\eta,\theta)$-invexity assumptions.
Further, denote the set of active constraints at point $\overline{x}\in X$ by $I(\overline{x})$, which is defined by

$$I(\overline{x})=\{i\in I: g^i(t,\overline{x},\dot{\overline{x}})=0.\}$$

Theorem 5.1. Let $\overline{x}$ be an optimal solution of **(VP)**. If $\int_a^b f dt,\ -\int_a^b g^i dt,\ i\in I(\overline{x})$, are $(p,r)-\rho-(\eta,\theta)$-invex, $(p,-r)-\hat{\rho}_i-(\eta,\theta)$-invex, respectively at $\overline{x}$ with respect to same $\eta,\ \theta$ with $(\rho+\sum_{i=1}^p \overline{\mu}_i\hat{\rho}_i)\geq 0$, then $\overline{x}$ is an optimal solution of $(\mathbf{VP_r(c)})$ with $c\geq e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}$ max $\{\overline{\mu}_i,\ i\in I=\{1,2,...,p\}\}$.

Proof. Since $\int_a^b f dt$ is $(p,r)-\rho-(\eta,\theta)$-invex at $\overline{x}$, we obtain

$$\frac{1}{r}[e^{r\{\int_a^b(f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt\}}-1]\geq\int_a^b[f_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)$$
$$+f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})e^{p\eta}\frac{d\eta}{dt}+\rho\|\theta(t,x,\overline{x})\|^2]dt. \tag{5.3}$$

By necessary optimality condition (Proposition 3.1), we have

$$\frac{1}{r}[e^{r\{\int_a^b(f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt\}}-1]\geq\int_a^b[\{\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})+\frac{d}{dt}(f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}}))\}$$
$$\times\frac{1}{p}(e^{p\eta}-1)+f_{\dot{x}}e^{p\eta}\frac{d\eta}{dt}+\rho\|\theta(t,x,\overline{x})\|^2]dt.$$
$$=\int_a^b[\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)+\overline{\mu}^T e^{p\eta}\frac{d\eta}{dt}g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})+\rho\|\theta(t,x,\overline{x})\|^2]dt. \tag{5.4}$$

(After integration by parts of $\int_a^b\{\frac{1}{p}(e^{p\eta}-1)\frac{d}{dt}(f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\mu}^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}}))\}dt$ and using $\eta=0$, at $t=a$ and $t=b$).
By the $(p,-r)-\hat{\rho}_i-(\eta,\theta)$-invexity assumption of $-\int_a^b g^i dt$, for $i=1,2,...,p$ at $\overline{x}$ with respect to $\eta,\ \theta$, we have the inequality

$$-\frac{1}{r}(e^{r\int_a^b(g^i(t,x,\dot{x})-g^i(t,\overline{x},\dot{\overline{x}}))dt}-1)\geq\int_a^b[-g_x^i(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)$$
$$-g_{\dot{x}}^i(t,\overline{x},\dot{\overline{x}})e^{p\eta}\frac{d\eta}{dt}+\hat{\rho}_i\|\theta(t,x,\overline{x})\|^2]dt,\ i\in I(\overline{x}), \tag{5.5}$$

Multiplying (5.5) by $\overline{\mu}_i\geq 0,\ i\in I(\overline{x})$, for $i=1,2,...,p$, respectively we get

$$-\frac{1}{r}\overline{\mu}_i(e^{r\int_a^b(g^i(t,x,\dot{x})-g^i(t,\overline{x},\dot{\overline{x}}))dt}-1)\geq\int_a^b[-\overline{\mu}_i g_x^i(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)$$
$$-\overline{\mu}_i g_u^i(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)+\overline{\mu}_i\hat{\rho}\|\theta(t,x,\overline{x})\|^2]dt,\ i\in I(\overline{x}). \tag{5.6}$$

Summing over $i=1,2,...,p$ in the inequality (5.6) we obtain

$$-\frac{1}{r}\sum_{i=1}^p\overline{\mu}_i(e^{r\int_a^b(g^i(t,x,\dot{x})-g^i(t,\overline{x},\dot{\overline{x}}))dt}-1)\geq\int_a^b[-\overline{\mu}^T g_x(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)$$
$$-\overline{\mu}^T g_u(t,\overline{x},\dot{\overline{x}})\frac{1}{p}(e^{p\eta}-1)+\sum_{i=1}^p\overline{\mu}_i\hat{\rho}\|\theta(t,x,\overline{x})\|^2]dt. \tag{5.7}$$

Adding the inequalities (5.4) and (5.7), we have

$$\frac{1}{r}(e^{r\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt}-1)-\frac{1}{r}\sum_{i=1}^{p}\overline{\mu}_i(e^{r\int_a^b (g^i(t,x,\dot{x})-g^i(t,\overline{x},\dot{\overline{x}}))dt}-1)$$
$$\geq (\rho+\sum_{i=1}^{p}\overline{\mu}_i\hat{\rho}_i)\int_a^b \|\theta(t,x,\overline{x})\|^2 dt. \tag{5.8}$$

As $(\rho+\sum_{i=1}^{p}\overline{\mu}_i\hat{\rho}_i)\geq 0$, from (5.8), we obtain

$$\frac{1}{r}(e^{r\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt}-1)-\frac{1}{r}\sum_{i=1}^{p}\overline{\mu}_i(e^{r\int_a^b (g^i(t,x,\dot{x})-g^i(t,\overline{x},\dot{\overline{x}}))dt}-1)\geq 0. \tag{5.9}$$

As $\overline{x}$ is an optimal solution of **(VP)**, by (3.9) (Proposition 3.1) we obtain

$$\frac{1}{r}(e^{r\int_a^b (f(t,x,\dot{x})-f(t,\overline{x},\dot{\overline{x}}))dt}-1)-\frac{1}{r}\sum_{i\in I(\overline{x})}\overline{\mu}_i(e^{r\int_a^b g^i(t,x,\dot{x})dt}-1)\geq 0.$$

Hence, for all $x\in X$,

$$\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt}+\frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}\sum_{i\in I(\overline{x})}\overline{\mu}_i(1-e^{r\int_a^b g^i(t,x,\dot{x})dt})\geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}. \tag{5.10}$$

Thus by (5.2), it follows that

$$\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt}+\frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}\sum_{i\in I(\overline{x})}\overline{\mu}_i(1-e^{r\int_a^b g^i_+(t,x,\dot{x})dt})\geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}. \tag{5.11}$$

Since $c\geq e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}\max\{\overline{\mu}_i, i\in I\}$, therefore for all $x\in X$, we have

$$\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt}+c\left\{\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g^i_+(t,x,\dot{x})dt})\right\}\geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}.$$

As $\overline{x}$ is an optimal solution of **(VP)**, together with (5.2), we obtain the inequality

$$\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt}+c\left\{\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b {g^i}_+(t,x,\dot{x})dt})\right\}$$
$$\geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}+c\left\{\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b {g^i}_+(t,\overline{x},\dot{\overline{x}})dt})\right\}$$

which holds for all $x\in X$. By the definition of the exponential penalized control problem $(\mathbf{VP_r(c)})$, it follows that the inequality

$$P(x(t),c)\geq P(\overline{x}(t),c)$$

holds for all $x\in X$. This implies that $\overline{x}$ is an optimal solution of the l_1 exponential penalized variational problem $(\mathbf{VP_r(c)})$. □

To prove converse of the Theorem 5.1, we need the following proposition.

Proposition 5.1. Let $\overline{x}$ be an optimal solution of the l_1 exponential penalized variational problem $(\mathbf{VP_r(c)})$. Then the inequality

$$\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,\overline{x},\dot{\overline{x}})dt$$

holds for all $x \in K$.

Proof. Since $\overline{x}$ is an optimal solution of the l_1 exponential penalized variational problem $(\mathbf{VP_r(c)})$, then the inequality

$$P(x(t),c) \geq P(\overline{x}(t),c)$$

holds for all $x \in X$. By the definition of the l_1 exponential penalized variational problem $(\mathbf{VP_r(c)})$, it follows that the inequality

$$\begin{aligned}&\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt} + c\left\{\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g^i{}_+(t,x,\dot{x})dt})\right\}\\ &\geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt} + c\left\{\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g^i{}_+(t,\overline{x},\dot{\overline{x}})dt})\right\} \end{aligned} \tag{5.12}$$

holds for all $x \in X$. Thus, for any $x \in K$,

$$\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt} \geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt} + c\left\{\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g^i{}_+(t,\overline{x},\dot{\overline{x}})dt})\right\}. \tag{5.13}$$

Therefore, by (5.2), for any $x \in K$,

$$\frac{1}{r}e^{r\int_a^b f(t,x,\dot{x})dt} \geq \frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}. \tag{5.14}$$

Hence,

$$\int_a^b f(t,x,\dot{x})dt \geq \int_a^b f(t,\overline{x},\dot{\overline{x}})dt, \ \forall x \in K.$$

□

Now, we prove converse of the Theorem 5.1 under the suitable $(p,r)-\rho-(\eta,\theta)$-invexity assumptions imposed on the functions involved in the original variational problem $(\mathbf{VP})$.

Theorem 5.2. Let $\overline{x}$ be an optimal solution of $(\mathbf{VP_r(\overline{c})})$ (where c is replaced by $\overline{c}$ in the equation (5.1)) and let the penalty parameter $\overline{c}$ be sufficiently large number $(\overline{c} > e^{r\int_a^b f(t,\tilde{x},\dot{\tilde{x}})dt} \max\{\tilde{\mu_i},\ i \in I\}$, where $\tilde{x}$ is a feasible solution of $(\mathbf{VP})$ and satisfies the necessary optimality conditions (3.8)-(3.10), with Lagrange multiplier $\tilde{\mu_i}$). Furthermore, if $\int_a^b f dt$, $-\int_a^b g^i dt,\ i \in I(\overline{x})$ are $(p,r)-\rho-(\eta,\theta)$-invex and $(p,-r)-\hat{\rho}_i-(\eta,\theta)$-invex, respectively at $\overline{x}$ with respect to same $\eta,\ \theta$ with $(\rho+\sum_{i=1}^{p}\tilde{\mu_i}\hat{\rho}_i) \geq 0$, and also if the set of all feasible solutions K of $(\mathbf{VP})$ is compact, then $\overline{x}$ is an optimal solution of $(\mathbf{VP})$.

Proof. To prove that $\overline{x}$ is an optimal solution of the original variational problem $(\mathbf{VP})$, first to show that $\overline{x}$ is a feasible solution of the original variational problem $(\mathbf{VP})$. We prove this by the method of contradiction. Suppose that $\overline{x}$ is not a feasible solution of $(\mathbf{VP})$. Since the set of all feasible solutions of $(\mathbf{VP})$ is compact, then $\int_a^b f(t,x,\dot{x})dt$ admits its minimum. Assume that the minimum attains at $\tilde{x}$. Therefore, the original variational problem $(\mathbf{VP})$ has an optimal solution at $\tilde{x}$. Since $\int_a^b f dt$ and $-\int_a^b g^i dt,\ i\in I(\overline{x})$ are $(p,r)-\rho-(\eta,\theta)$-invex and $(p,-r)-\hat{\rho}_i-(\eta,\theta)$-invex respectively, at $\overline{x}$ with respect to same $\eta,\ \theta$, therefore the inequalities

$$\frac{1}{r}(e^{r\int_a^b(f(t,x,\dot{x})-f(t,\tilde{x},\dot{\tilde{x}}))dt}-1) \geq \int_a^b[\frac{1}{p}f_x(t,\tilde{x},\dot{\tilde{x}})(e^{p\eta}-1) + f_{\dot{x}}(t,\tilde{x},\dot{\tilde{x}})e^{p\eta}\frac{d\eta}{dt}+\rho\|\theta(t,x,\tilde{x})\|^2]dt, \tag{5.15}$$

$$-\frac{1}{r}(e^{r\int_a^b(g^i(t,x,\dot{x})-g^i(t,\tilde{x},\dot{\tilde{x}})dt)}-1) \geq \int_a^b[-g^i_x(t,\tilde{x},\dot{\tilde{x}})\frac{1}{p}(e^{p\eta}-1) -g^i_{\dot{x}}(t,\tilde{x},\dot{\tilde{x}})e^{p\eta}\frac{d\eta}{dt}+\hat{\rho}_i\|\theta(t,x,\tilde{x})\|^2]dt,\ i\in I(\overline{x}), \tag{5.16}$$

hold for all $x\in X$. Inequalities (5.15) and (5.16) are also satisfied at $\overline{x}$. Thus,

$$\frac{1}{r}(e^{r\int_a^b(f(t,\overline{x},\dot{\overline{x}})-f(t,\tilde{x},\dot{\tilde{x}}))dt}-1) \geq \int_a^b[\frac{1}{p}f_x(t,\tilde{x},\dot{\tilde{x}})(e^{p\eta}-1) + f_{\dot{x}}(t,\tilde{x},\dot{\tilde{x}})e^{p\eta}\frac{d\eta}{dt}+\rho\|\theta(t,\overline{x},\tilde{x})\|^2]dt, \tag{5.17}$$

$$-\frac{1}{r}(e^{r\int_a^b(g^i(t,\overline{x},\dot{\overline{x}})-g^i(t,\tilde{x},\dot{\tilde{x}}))dt}-1) \geq \int_a^b[-g^i_x(t,\tilde{x},\dot{\tilde{x}})\frac{1}{p}(e^{p\eta}-1) -g^i_{\dot{x}}(t,\tilde{x},\dot{\tilde{x}})e^{p\eta}\frac{d\eta}{dt}+\hat{\rho}_i\|\theta(t,\overline{x},\tilde{x})\|^2]dt,\ i\in I(\overline{x}). \tag{5.18}$$

Applying necessary optimality condition (Proposition 3.1) and integration by parts in the inequality (5.17), we have

$$\frac{1}{r}(e^{r\int_a^b(f(t,\overline{x},\dot{\overline{x}})-f(t,\tilde{x},\dot{\tilde{x}}))dt}-1) \geq \int_a^b[\tilde{\mu}^T g_x(t,\tilde{x},\dot{\tilde{x}})\frac{1}{p}(e^{p\eta}-1) +\tilde{\mu}^T e^{p\eta}\frac{d\eta}{dt}g_{\dot{x}}(t,\tilde{x},\dot{\tilde{x}})+\rho\|\theta(t,\overline{x},\tilde{x})\|^2]dt. \tag{5.19}$$

Multiplying (5.5) by $\tilde{\mu}_i\geq 0,\ i\in I(\overline{x})$ and summing over $i=1,2,...,p$, we obtain

$$-\frac{1}{r}\sum_{i=1}^{p}\tilde{\mu}_i(e^{r\int_a^b(g^i(t,\overline{x},\dot{\overline{x}})-g^i(t,\tilde{x},\dot{\tilde{x}}))dt}-1) \geq \int_a^b[-\tilde{\mu}^T g_x(t,\tilde{x},\dot{\tilde{x}})\frac{1}{p}(e^{p\eta}-1) -\tilde{\mu}^T g_{\dot{x}}(t,\tilde{x},\dot{\tilde{x}})e^{p\eta}\frac{d\eta}{dt}+\sum_{i=1}^{p}\tilde{\mu}_i\hat{\rho}_i\|\theta(t,\overline{x},\tilde{x})\|^2]dt,\ i\in I(\overline{x}). \tag{5.20}$$

Adding the inequalities (5.19) and (5.20), we obtain

$$\frac{1}{r}(e^{r\int_a^b(f(t,\overline{x},\dot{\overline{x}})-f(t,\tilde{x},\dot{\tilde{x}}))dt}-1)-\frac{1}{r}\sum_{i\in I(\overline{x})}\tilde{\mu}_i(e^{r\int_a^b(g^i(t,\overline{x},\dot{\overline{x}})-g^i(t,\tilde{x},\dot{\tilde{x}}))dt}-1) \geq (\rho+\sum_{i=1}^{p}\mu_i\hat{\rho}_i)\int_a^b\|\theta(t,\overline{x},\tilde{x})\|^2dt. \tag{5.21}$$

By assumption $(\rho+\sum_{i=1}^{p}\mu_i\hat{\rho}_i)\geq 0$, the inequality reduces to

$$\frac{1}{r}(e^{r\int_a^b(f(t,\overline{x},\dot{\overline{x}})-f(t,\tilde{x},\dot{\tilde{x}}))dt}-1)+\frac{1}{r}\sum_{i\in I(\overline{x})}\tilde{\mu}_i(1-e^{r\int_a^b(g^i(t,\overline{x},\dot{\overline{x}})-g^i(t,\tilde{x},\dot{\tilde{x}}))dt})\geq 0. \tag{5.22}$$

Now using the necessary optimality conditions together with the feasibility of $\tilde{x}$ in the original variational problem **(VP)**, we obtain

$$\frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}+\frac{1}{r}e^{r\int_a^b f(t,\tilde{x},\dot{\tilde{x}})dt}\sum_{i=1}^{p}\tilde{\mu}_i(1-e^{r\int_a^b g_+^i(t,\overline{x},\dot{\overline{x}})dt})\geq\frac{1}{r}e^{r\int_a^b f(t,\tilde{x},\dot{\tilde{x}})dt}. \tag{5.23}$$

By assumption $\overline{c}>e^{r\int_a^b f(t,\tilde{x},\tilde{u})dt}\max\{\tilde{\mu_i},\ i\in I\}$, from (5.24) we have

$$\frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}+\overline{c}\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g_+^i(t,\overline{x},\dot{\overline{x}})dt})$$
$$>\frac{1}{r}e^{r\int_a^b f(t,\tilde{x},\dot{\tilde{x}})dt}. \tag{5.24}$$

Using the feasibility of $\tilde{x}$ in the original variational problem **(VP)**, together with (5.2), we obtain

$$\frac{1}{r}e^{r\int_a^b f(t,\overline{x},\dot{\overline{x}})dt}+\overline{c}\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g_+^i(t,\overline{x},\dot{\overline{x}})dt})$$
$$>\frac{1}{r}e^{r\int_a^b f(t,\tilde{x},\dot{\tilde{x}})dt}+\overline{c}\sum_{i=1}^{p}\frac{1}{r}(1-e^{r\int_a^b g_+^i(t,\tilde{x},\dot{\tilde{x}})dt}). \tag{5.25}$$

Hence, by the definition of the l_1 exponential penalized variational problem $(\mathbf{VP_r}(\overline{\mathbf{c}}))$, we obtain $P(\overline{x}(t),\overline{c})>P(\tilde{x}(t),\overline{c})$, which contradicts the fact that $\overline{x}$ is an optimal solution of the l_1 exponential penalized variational problem $(\mathbf{VP_r}(\overline{\mathbf{c}}))$. Thus, $\overline{x}$ is a feasible solution of **(VP)**. From the Proposition 5.1, the inequality

$$\int_a^b f(t,x,\dot{x})dt\geq\int_a^b f(t,\overline{x},\dot{\overline{x}})dt$$

holds for all $x\in K$. We conclude that $\overline{x}$ is an optimal solution of the original variational problem **(VP)**. □

6 Conclusion

In this paper, we have established the sufficient optimality conditions and several duality results for variational problems under $(p,r)-\rho-(\eta,\theta)$-invexity assumptions. We have defined $(p,r)-\rho-(\eta,\theta)$-invexity for functional and studied the duality results for Mond-Weir type dual problem. Example 2.3 ensures that the concept of $(p,r)-\rho-(\eta,\theta)$-invexity for functional is more general than $\rho-(\eta,\theta)$-invexity for functional. We also convert constrained variational problem into unconstrained problem by the help of l_1 exponential penalty function method. We characterize the solutions of the variational problem **(VP)** in terms of the minimizers of the unconstrained variational problem with the l_1 exact exponential penalty function. Thus, we establish the equivalence between an optimal solution of the original variational problem **(VP)** and a minimizer of its associated exponential penalized variational problem with the l_1 exact exponential penalty function. We prove this result under the suitable $(p,r)-\rho-(\eta,\theta)$-invexity assumptions imposed on the functions constituting the original variational problem **(VP)**. In the Section 5 we introduce a new type of unconstrained variational problem with the l_1 exact exponential penalty function method. To the best of our knowledge, it is completely new and not available in the existing literature. There is a rich scope to study this type of problems in non-smooth case. One can also formulate a fractional analogue of our model and study duality results and also establish the sufficient optimality conditions of the fractional variational problems.

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