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THE MODE OF A VIBRATING HOMOGENEOUS PLATE IS DECREASING ON THE DOMAIN WITH A HOLE

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ABSTRACT. In this note, by some numerical results we show that the principal eigenvalues of the Dirichlet and Bi-harmonic boundary value problems on the tow dimensional domain with a hole are decreasing when the hole moves until it touches the outer boundary of the domain.

1. INTRODUCTION

The vibration of a homogeneous plate, a plate with constant density, having the shape $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega$, is modeled by the following boundary value problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.1}$$

The parameter λ is the eigenvalue (mode) and u the corresponding eigenfunction (bending).

Let $D(h) := B \setminus B_h$, where B stands for the unit ball in $\mathbb{R}^2$, centered at the origin, and B_h is the ball centered at $(h, 0) \in \mathbb{R}^2$ with radius $a < 1$. The mathematical formulation of the eigenvalue problem (1.1) corresponding to $D(h)$ is

$$-\Delta u_h = \lambda(h) u_h \quad \text{in} \quad D(h), \quad \text{and} \quad u_h = 0 \quad \text{on} \quad \partial D(h). \tag{1.2}$$

The principal eigenpair corresponding to (1.2) is denoted $(\lambda(h), u_h) \in \mathbb{R}^+ \times H_0^1(\Omega)$. The following Theorem has been proved in [2]:

Theorem 1.1. *The function $\lambda : [0, 1-a] \to \mathbb{R}^+$ is decreasing.*

We show that this theorem is valid for Bi-harmonic eigenvalue problems and collect some numerical results.

2. BI-HARMONIC EQUATION

It's well known that λ^2 is an eigenvalue for the following Bi-harmonic eigenvalue problem with Navier boundary conditions

$$\Delta^2 u = \lambda^2 u \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

if and only if λ is an eigenvalue for the Dirichlet problem (1.1), see [1]. Therefore, we'll have the following corollary.

Corollary 2.1. *If $\Omega = D(h)$, then the Theorem 1.1 is valid for the principal eigenvalue of (2.1).*

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Now we use the PDETool of MATLAB's software to calculate $\lambda(h)$ for different values of h, when $\Omega = D(h)$ in (2.1). Let $-\Delta u = \lambda v$. Thus with this substitution the problem (2.1) converts to the following system

$$-\Delta u = \lambda v \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega, \tag{2.2}$$

$$-\Delta v = \lambda u \text{ in } \Omega, \text{ and } v = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Let

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{and } W = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Hence the equations (2.2) and (2.3) can be written as the following

$$-\text{div}(C * \nabla W) + A * W = \lambda \ D * W.$$

Now by applying PDETool of MATLAB we have the following examples.

Example 1. Let $\Omega = D(h) = B \setminus B_h$ and $a = 0.3$, thus

h	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\lambda(h)$	19.4848	17.0155	14.3666	12.3177	10.7402	9.5125	8.5468	7.7814

Figure 1 shows the graph of the eigenfunction corresponding to $\lambda(0.5)$.

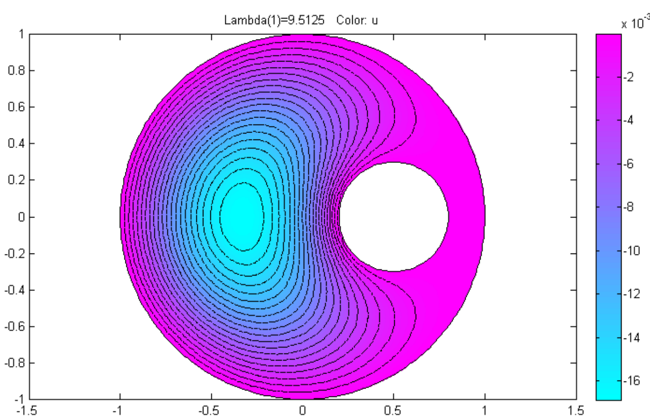

FIGURE 1. Graph of $u_{0.5}(x)$ on $\Omega = B \setminus B_{0.5}$.

3. Numerical results on special domains

In this section, by some numerical results, we show taht, in the prbblem (2.1) λ is decreasing for some tow dimensional symmetric domain with a hole, when the hole moves (along a radius) until it touches the outer boundary of the domain. The analytic proof of these results is an open problem.

Example 2. Let $\Omega = E \setminus E_h$, that $E : x^2 + (\frac{y}{0.5})^2 \leq 1$ and $E_h : (\frac{x-h}{0.5})^2 + (\frac{y}{0.25})^2 \leq 1$, thus

h	0	0.1	0.2	0.3	0.4	0.5
$\lambda(h)$	61.7928	46.935	37.5802	31.267	26.8152	23.5544

In Figure 2 you can see the graph of the principal eigenfunction for $\lambda(0.3)$.

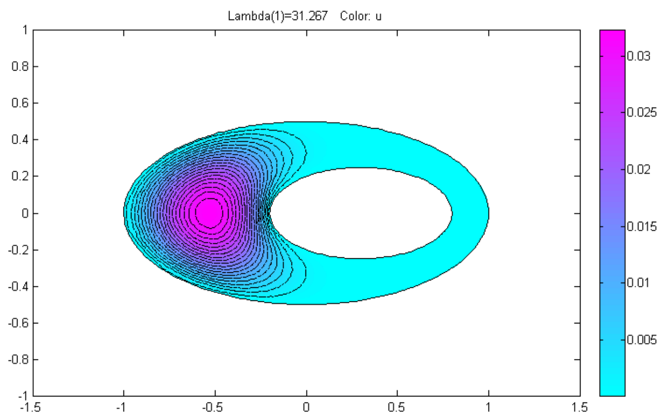

FIGURE 2. Graph of $u_{0.3}(x)$ on $\Omega = E \setminus E_{0.3}$.

Example 3. Let $\Omega = E \setminus E_h$, that $E : x^2 + (\frac{y}{0.5})^2 \leq 1$ and $E_h : (\frac{x-h}{0.2})^2 + (\frac{y}{0.3})^2 \leq 1$, hence

h	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\lambda(h)$	32.6276	27.8041	24.2876	21.6588	19.6534	18.1096	16.9165	16.003

Example 4. Let $\Omega = S \setminus B_h$, that $S = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$ and $B_h : (x-h)^2 + y^2 \leq 0.09$. The values of $\lambda(h)$ are in the following table. Figure 3 shows the graph of $u_{0.5}(x)$.

h	0	0.1	0.2	0.3	0.4	0.5
$\lambda(h)$	15.0883	13.596	11.7832	10.3031	9.1231	8.1797

Remark 3.1. From Examples 1-4 we deduce that $\lambda(0) = \sup_h \lambda(h)$ and on any radius the minimum of $\lambda(h)$ occurs when the hole touches the outer boundary of Ω. These results have been proved for Dirichlet problem on $\Omega = D(h)$.

The following example shows that $\lambda(h)$ is decreasing however Ω is not symmetric.

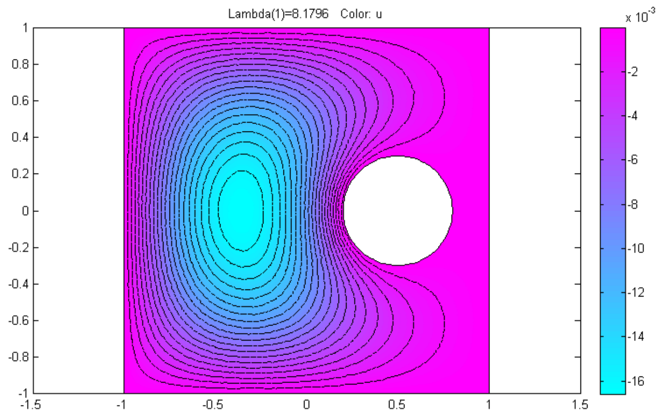

FIGURE 3. The Graph of $u_{0.5}(x)$ on $\Omega = S \setminus B_{0.5}$.

Example 5. Let $\Omega = B \setminus T_h$, that $B : x^2 + y^2 \leq 1$ and T_h is a triangle with vertices $(h + 0.3, 0)$, $(h, -0.2)$ and $(h, 0.4)$, thus

h	0	0.1	0.2	0.3	0.4	0.5
λ	12.9091	11.3613	10.0438	8.9797	8.13	7.4527

The Figure 4 shows the graph of $u_0(x)$.

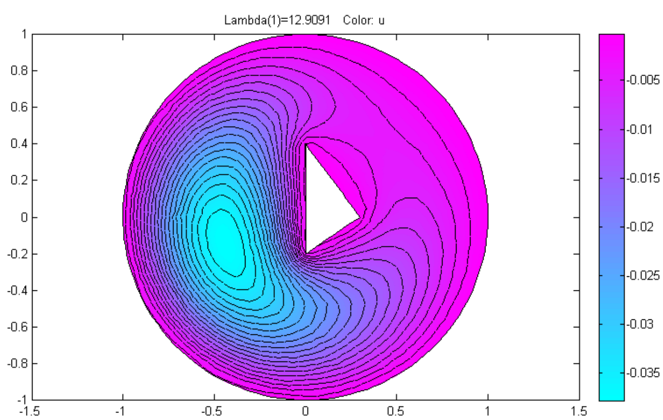

FIGURE 4. The Graph of $u_0(x)$ on $\Omega = B \setminus T_0$.

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